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TAIL GINI FUNCTIONAL UNDER ASYMPTOTIC INDEPENDENCE

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Abstract: Tail Gini functional is a measure of tail risk variability for systemic risks, and has many applications in banking, finance and insurance. Meanwhile, there is growing attention on asymptotic independent pairs in quantitative risk management. This paper addresses the estimation of the tail Gini functional under asymptotic independence. We first estimate the tail Gini functional at an intermediate level and then extrapolate it to the extreme tails. The asymptotic normalities of both the intermediate and extreme estimators are established. The simulation study shows that our estimator performs comparatively well in view of both bias and variance. The application to measure the tail variability of weekly loss of individual stocks given the occurrence of extreme events in the market index in Hong Kong Stock Exchange provides meaningful results, and leads to new insights in risk management.

Key words and phrases: Tail variability, Systemic risk, Asymptotic independence.

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1. Introduction

Measuring tail risk is crucial in many different fields such as banking, finance and insurance. The most popular tail-based risk measures in the literature are Value at Risk (VaR) and Expected Shortfall (ES). See Patton, Ziegel and Chen (2019), Li and Wang (2023) and Hoga and Demetrescu (2023) for some recent discussions on VaR and ES in quantitative risk management. Due to their nature, VaR and ES do not capture the variability of the risk random variable beyond the high quantile. To incorporate variability in tail risk analysis, Furman, Wang and Zitikis (2017) extends the (classic) Gini mean difference to tail Gini functional. The tail Gini functional for X is defined by

$$\mathrm{TG}_p(X) = \frac{4}{p}\,\mathrm{Cov}\left\{X, F_1(X) \mid F_1(X) > 1-p\right\},$$

where F_1 is the cumulative distribution function of random variable X and $p > 0$ is a sufficient small value. Obviously, the tail Gini functional $\mathrm{TG}_p(X)$ is designed to measure the variability of X on the upper tail region and quantifies the risk of X solely. In finance, X may be the loss of an individual asset and tail Gini functional for X indicates the risk measure of tail variability of that asset.

However, in practice, regulators are concerned not only with measuring risks for individual asset or entity, but also with measuring individual risks given the impact from systemic variables. A series of studies have delved into modeling and measuring systemic risks, including Cai, Einmahl, de Haan and Zhou (2015), Adrian and Brunnermeier (2016), Acharya, Pedersen, Philippon, and Richardson (2017), and so on. To extend tail Gini functional for systemic risk analysis, we consider tail Gini functional for bivariate random vector (X, Y), introduced by Hou and Wang (2021) as

$$\mathrm{TG}_p(X;Y) = \frac{4}{p}\operatorname{Cov}\left\{X, F_2(Y) \mid F_2(Y) > 1-p\right\},$$

where F_2 is the marginal cumulative distribution function of Y. Here, Y could be a systemic variable indicating the loss of a financial system. By conditioning on $F_2(Y) > 1-p$, we are focusing on the the tail variability of X under the tail scenarios of the systemic variable Y. In this way, $\mathrm{TG}_p(X;Y)$ is a systemic tail variability measure incorporating both the marginal risk severity of X and the tail structure of (X, Y).

Under the assumption that X is in the Fréchet domain of attraction with the extreme value index $\gamma_1 \in (0,1)$, Hou and Wang (2021) obtains the asymptotic limit of $\mathrm{TG}_p(X;Y)$, that is

$$\lim_{p\to 0}\frac{\mathrm{TG}_p(X;Y)}{Q_1(1-p)} = \theta_0 \in [0,\infty),$$

where Q_1 is the quantile function of X. The condition $0 < \gamma_1 < 1$ guarantees that $\mathbf{E}[X]$ exists. Based on the above limit, Hou and Wang (2021) proposes an estimator for $\mathrm{TG}_p(X;Y)$ with $\theta_0 > 0$. Note that the corresponding $\theta_0 > 0$ holds only if X and Y are asymptotically dependent, i.e. the tail copula is non-degenerate. If X and Y are asymptotically independent, we have $\theta_0 = 0$. In this paper, we will study the estimation of $\mathrm{TG}_p(X;Y)$ under asymptotic independence. We refer to Ledford and Tawn (1996) for the concepts of asymptotic independence and asymptotic dependence.

Although most research articles in bivariate extreme value framework deal with asymptotic dependence, there is increasing evidence that weaker dependence actually exists in bivariate tail region in many applications, for example, significant wave height (Wadsworth and Tawn, 2012), spatial precipitation (Le, Davison, Engelke, Leonard and Westra, 2018) and daily stock prices (Lehtomma and Resnick, 2020). Asymptotic independence is therefore the more appropriate model for such applications. In the field of quantitative risk management, there is also growing attention on risk measures for asymptotically independent pairs (see Kulik and Soulier, 2015; Das and Fasen-Hartmann , 2018; Cai and Musta, 2020; Sun and Chen, 2023, etc).

Ledford and Tawn (1996) proposes the coefficient of tail dependence, named η, to measure the severity asymptotic independence. Assume that there exists an $\eta \in (0, 1]$ such that the following limit exists and is positive for all $(x, y) \in (0, \infty)^2$:

$$\lim_{p \to 0} p^{-\frac{1}{\eta}} \mathbf{P}\left\{1 - F_1(X) < px, 1 - F_2(Y) < py\right\} =: \tau(x, y) > 0. \tag{1.1}$$

For either $x = 0$ or $y = 0$, we let $\tau(x, y) = 0$. One important property of τ is that τ is a homogeneous function of order $1/\eta$, i.e., $\tau(ax, ay) = a^{1/\eta}\tau(x, y)$ for $a > 0$. The coefficient of tail dependence η describes the strength of extremal dependence in the bivariate tail. If $\eta = 1$, we say that X and Y are asymptotically dependent. If $0 < \eta < 1$, we say that X and Y are asymptotically independent. Moreover, if $1/2 < \eta < 1$, X and Y are called asymptotically independent but positively associated; if $0 < \eta < 1/2$, X and Y are called asymptotically independent but negatively associated. When X and Y are independent, then $\eta = 1/2$. For more details on the interpretation of η, see Ledford and Tawn (1996).

It is the goal of this paper to estimate $\mathrm{TG}_p(X; Y)$ under asymptotic independence but positive association ($1/2 < \eta < 1$). To our best knowledge, there is no literature addressing the estimation problem for tail-based measure of variability for asymptotically independent structures. Our work is also of great significance since we consider not only positive loss variables but also real loss variables in the context of asymptotic independence. For the case of asymptotic independence but negative association ($0 < \eta < 1/2$), it is much more technically challenging and we leave it for future research. Meanwhile, $\eta > 1/2$ is more common in real cases, see Table 3 in Section 4.

The rest of the paper is organized as follows. Section 2 studies the asymptotic normality of the proposed estimator for $\mathrm{TG}_p(X;Y)$. The performance of our proposed estimator is illustrated by a simulation study in Section 3, and a real application to Hong Kong Stock Exchange is given in Section 4. The proofs of the main theorems are provided in Section 5. Additional proofs are given in the Supplementary Material.

Throughout the paper, the notation $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$. A Lebesgue measurable function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called regularly varying (at infinity) with index $\alpha \in \mathbb{R}$, if $\lim_{t\to\infty} f(tx)/f(t) = x^\alpha$, for $x > 0$.

2. Main results

Let (X, Y) be a pair of random loss variables. We propose estimators of the tail Gini functional $\mathrm{TG}_p(X;Y)$ by a two-step approach. More specifically, we first estimate $\mathrm{TG}_p(X;Y)$ at intermediate level $p = k/n$, where $k = k(n) \to \infty$ and $k/n \to 0$ as $n \to \infty$, then extrapolate these estimators to extreme level $p = p(n) \to 0$ and $np = O(1)$ as $n \to \infty$.

Below we present our assumptions on the tail distribution of X and the tail dependence between X and Y. Here, any constants, functions, or

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conditions introduced in an assumption are implicitly assumed to hold in the sequel. Define

$$\tau_p(x,y) := p^{-1/\eta}\mathbf{P}\left\{1 - F_1(X) < px, 1 - F_2(Y) < py\right\}.$$

Throughout the paper, we assume that (1.1) holds, which means that $\lim_{p\to 0}\tau_p(x,y) = \tau(x,y)$, for all $(x,y) \in (0,\infty)^2$.

Assumption 1. There exist $\gamma_1 > 1/\eta - 1$ and a regularly varying function A_1 with index $\rho_1 < \frac{1}{2} - \frac{1}{2\eta}$ such that

$$\sup_{x>1}\left| x^{-\gamma_1}\frac{Q_1\{1-1/(tx)\}}{Q_1(1-1/t)} - 1\right| = O\left\{A_1(t)\right\}, \quad \text{as } t \to \infty.$$

Assumption 2. There exists $\delta > 0$ such that uniformly for all $y \in [0,1]$,

$$\left|\int_0^1 \tau(x,y)dx^{-(2+\delta)\gamma_1}\right| < \infty, \quad \text{and} \quad \left|\int_1^\infty \tau(x,y)^2 dx^{-\gamma_1}\right| < \infty.$$

Assumption 3. There exist $\beta_1 > (2+\delta)\gamma_1$ and $\xi > \max\left\{(1-\eta)/2\eta^2, 1/(1+\gamma_1 - 1/\eta)\right\}$ such that as $p \to 0$,

$$\sup_{0<x\le 1, 0<y\le 1}\left|\tau_p(x,y) - \tau(x,y)\right| x^{-\beta_1} = O\left(p^\xi\right).$$

Assumption 4. There exists $0 < \beta_2 < \xi/\left(1-\gamma_1\right)$ such that as $p \to 0$,

$$\sup_{1<x<\infty, 0<y\le 1}\left|\tau_p(x,y) - \tau(x,y)\right| x^{-\beta_2} = O\left(p^\xi\right).$$

Assumption 1 is a second order condition for the distribution of X, which is commonly assumed in extreme value theory. We refer readers

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to Section 2 in de Haan and Ferreira (2006) for the explanation of this assumption. Assumption 2 is a technical condition which imposes some integrality condition on the function τ. Assumption 2 and the monotonicity of $\tau(x, y)$ imply that for $\rho \in \{1, 2, 2+\delta\}$, uniformly for all $y \in [0, 1]$,

$$\sup_{y\in[0,1]} \left| \int_0^\infty \tau(x, y) dx^{-\rho\gamma_1} \right| < \infty.$$

We will deal with such integral throughout the proofs. Assumptions 3 and 4 are second order strengthenings of relation (1.1).

2.1 Positive loss

In this subsection we assume the random loss X is positive. To estimate $\mathrm{TG}_p(X; Y)$, we assume that $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent copies of (X, Y). A natural nonparametric estimator of $\mathrm{TG}_{k/n}(X; Y)$ at intermediate level $p = k/n \to 0$ is

$$\hat{\theta}_{k/n} = \frac{4n}{k^2(k-1)} \sum_{i<j} (X_i - X_j)\{F_{n2}(Y_i) - F_{n2}(Y_j)\} I(Y_i, Y_j > Y_{n-k,n}),$$

where $Y_{1,n} \le Y_{2,n} \le \cdots \le Y_{n,n}$ are the order statistics of $\{Y_1, Y_2, \cdots, Y_n\}$, and $F_{n2}(y) = \frac{1}{n+1} \sum_{i=1}^n I(Y_i \le y)$ is the empirical distribution function based on $Y_1, \ldots, Y_n$. Moreover, we choose the intermediate sequence k as follows.

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Assumption 5. As $n \to \infty$, $k/n^{1-\eta} \to \infty$, $k/n^a \to 0$, where $1-\eta < a < a_0$ and

$$a_0 = \min\left(1 - \frac{\eta}{1+\eta\gamma_1}, 1 + \frac{\eta}{1-2\eta-2\eta\gamma_1}, 1 + \frac{1}{-\frac{1}{\eta} - 2\xi + 2\beta_2(1-\gamma_1)}, 1 + \frac{1}{2\rho_1 - 1}, \frac{2\eta\xi}{2\eta\xi+1}\right).$$

Assumption 5 imposes both lower and upper bounds for the choice of k. We can prove that $a_0 > 1-\eta$, so the conditions are compatible. The upper bound of k is a typical constraint in extreme value theory literature to control the bias of the estimators, for example see Cai and Musta (2020). The lower bound is used to guarantee the convergence rate $\sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}}$ goes to infinity in Proposition 1 below.

Let $W(\cdot)$ be a mean zero Gaussian process on $[0,1]$ with covariance structure

$$\mathbf{E}\left\{W(y_1)\,W(y_2)\right\} = -\int_0^\infty \tau(x, y_1 \wedge y_2)\, dx^{-2\gamma_1}, \quad y_1, y_2 \in [0,1].$$

The following proposition shows the asymptotic normality of the estimator $\hat{\theta}_{k/n}$ for $\mathrm{TG}_{k/n}(X;Y)$.

Proposition 1. *Let $\{(X_i, Y_i)\}_{i=1}^n$ be independent copies of (X, Y). Under the condition that $X > 0$ and Assumptions 1-5, it follows that*

$$\sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}}\left\{\frac{\hat{\theta}_{k/n}}{\mathrm{TG}_{k/n}(X;Y)} - 1\right\} \xrightarrow{d} \Phi := -\frac{4}{\phi_0}\left\{\int_0^1 W(y)dy + \frac{1}{2}W(1)\right\},$$

where

$$\phi_0 = \frac{2(1+\gamma_1 - 1/\eta)}{1-\gamma_1+1/\eta}\int_0^\infty \tau\left(x^{-\frac{1}{\gamma_1}}, 1\right) dx.$$

Note that if $\eta = 1$, i.e. asymptotic dependence, the convergence rate $\sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}}$ becomes $\sqrt{k}$. When $1/2 < \eta < 1$, i.e. asymptotic independence but positive association, the convergence rate is lower than $\sqrt{k}$.

Proposition 1 states equivalently that

$$\sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}} \log\left\{\frac{\hat{\theta}_{k/n}}{\mathrm{TG}_{k/n}(X;Y)}\right\} \xrightarrow{d} \Phi. \tag{2.1}$$

The log-ratio of the estimator to the true risk measure has a centered normal limit. In the simulation below, we compare the sample quantiles of log-ratios with the normal quantiles to demonstrate its asymptotic property.

Now we consider the estimation of $\mathrm{TG}_p(X;Y)$ at extreme level $p \to 0$ such that $np = O(1)$. Sun and Chen (2023) shows that, for $0 < \eta \leq 1$, as $n \to \infty$,

$$\lim_{p \to 0} \frac{\mathrm{TG}_p(X;Y)}{p^{\frac{1}{\eta}-1} Q_1(1-p)} = \phi_0. \tag{2.2}$$

By (2.2), we have that, as $n \to \infty$,

$$\mathrm{TG}_p(X;Y) \sim \frac{Q_1(1-p)}{Q_1(1-k/n)}\left(\frac{k}{np}\right)^{1-1/\eta} \theta_{k/n} \sim \left(\frac{k}{np}\right)^{1-1/\eta+\gamma_1} \mathrm{TG}_{k/n}(X;Y).$$

Thus, we estimate $\mathrm{TG}_p(X;Y)$ by

$$\hat{\theta}_p = \left(\frac{k}{np}\right)^{1-1/\hat{\eta}+\hat{\gamma}_1} \hat{\theta}_{k/n}, \tag{2.3}$$

where $\hat{\eta}$ and $\hat{\gamma}_1$ are some suitable estimators for η and γ_1, respectively.

Let k_1 and k_2 be two intermediate sequences for the estimators $\hat{\eta}$ and $\hat{\gamma}_1$, respectively, i.e. $k_1 = k_1(n) \to \infty$, $k_1/n \to 0$, $k_2 = k_2(n) \to \infty$, $k_2/n \to 0$, as $n \to \infty$. We estimate γ_1 by the Hill estimator

$$\hat{\gamma}_1 = \frac{1}{k_1} \sum_{i=1}^{k_1} \log X_{n-i+1,n} - \log X_{n-k_1,n},$$

where $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$ are the order statistics of $\{X_1, X_2, \cdots, X_n\}$, and estimate η by the estimator proposed by Draisma, Drees, Ferreira and de Haan (2004)

$$\hat{\eta} = \frac{1}{k_2} \sum_{i=1}^{k_2} \log T_{n-i+1,n} - \log T_{n-k_2,n},$$

where $T_{1,n} \leq T_{2,n} \leq \cdots \leq T_{n,n}$ are the order statistics of the non-independent but identically distributed sequence $\{T_1, T_2, \cdots, T_n\}$ with

$$T_i := \frac{1}{\{1 - F_{n1}(X_i)\} \vee \{1 - F_{n2}(Y_i)\}}, \quad i = 1, 2, \ldots, n,$$

and $F_{n1}(x) = \frac{1}{n+1} \sum_{i=1}^{n} I\left(X_i \leq x\right)$ is the empirical distribution function based on $X_1, \ldots, X_n$.

Note that the intermediate sequences k_1 and k_2 might be different from k. In the rest of this paper, we choose suitable k_1 and k_2 such that

$$\sqrt{k}\left(\hat{\gamma}_1 - \gamma_1\right) = O_{\mathbf{P}}(1), \quad \sqrt{k}\left(\hat{\eta} - \eta\right) = O_{\mathbf{P}}(1). \tag{2.4}$$

Condition (2.4) can be achieved by choosing k_1 and k_2 at the same order as k, combining with some mild conditions. We refer to Theorem 3.2.5 and

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Theorem 7.6.1 in de Haan and Ferreira (2006) for the asymptotic behaviors of $\hat{\gamma}_1$ and $\hat{\eta}$, respectively.

To derive the the asymptotic normality of $\hat{\theta}_p$ at extreme level p, we require the following condition on both lower bound and upper bound for the speed of $p \to 0$.

Assumption 6. $\lim_{n\to\infty}(n/k)^{1/2-1/(2\eta)} \log d_n = 0$, where $d_n = k/(np) \geq 1$.

Theorem 1. *Assume the same assumptions as in Proposition 1. Suppose* (2.4) *and Assumption 6 hold. Then, as* $n \to \infty$,

$$\sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}}\left\{\frac{\hat{\theta}_p}{\mathrm{TG}_p(X;Y)}-1\right\} \xrightarrow{d} \Phi,$$

where Φ *is the same as in Proposition 1.*

2.2 General Loss

In this subsection, we extend the results in Section 2.1 to the case when the random loss X is real. Denote $X^+ = \max(X,0)$ and $X^- = \min(X,0)$, so $X = X^+ + X^-$ and $\mathrm{TG}_p(X;Y) = \mathrm{TG}_p(X^+;Y) + \mathrm{TG}_p(X^-;Y)$.

For a real random loss X, we need to modify the estimator at intermediate level $p = k/n$ in (2.3) as

$$\hat{\theta}_{k/n} := \frac{4n}{k^2(k-1)}\sum_{i<j}(X_i - X_j)\left(F_{n2}(Y_i) - F_{n2}(Y_j)\right) I\left(X_i, X_j > 0, Y_i, Y_j > Y_{n-k,n}\right). \tag{2.5}$$

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Notice this is indeed the same estimator in the case of positive random loss X. Therefore, we do not use another symbol to represent the estimator for the sake of simplicity.

Under asymptotic dependence, the results for general loss could be easily derived under some mild conditions on the negative part of the general loss, see Hou and Wang (2021). But in the case of asymptotic independence, there is greater probability for X to take negative values given large values of Y. This is totally different from the case of asymptotic dependence. It means that the tail variability of a general loss X may not be negligible as in the case of asymptotic dependence when the level p goes to zero. In order to render $\mathrm{TG}_p(X^-;Y)$ ignorable, we need additional conditions.

Assumption 7. There exists $\zeta > 1$ such that $\mathbf{E}\left|X^-\right|^\zeta < \infty$.

Assumption 8. $1 - 1/\eta > \xi - \beta_2$ and $\sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}} p^{1-\frac{1}{\zeta}-\frac{1}{\eta}+\gamma_1} \to 0$.

Remark 1. Assumption 7 imposes the condition on the left tail of X. Assumption 8 imposes an upper bound for p. We note that this upper bound holds as long as $p = O\left(n^b\right)$ with $b < \frac{1-\eta-a}{2\eta(1+\gamma_1-1/\eta)}$, with a specified in Assumption 5, and that it is compatible with the lower bound of p stated in Assumption 6.

Now we can apply extrapolation techniques to define $\hat{\theta}_p$ at extreme level

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p based on the same representation (2.3) with using $\hat{\theta}_{k/n}$ in (2.5) instead. The asymptotic normality of $\hat{\theta}_p$ is also guaranteed.

Theorem 2. *Let* $\{(X_i, Y_i)\}_{i=1}^n$ *be independent copies of* (X, Y). *Under the condition that* X *is real, Assumptions 1-8 and Condition* (2.4), *it follows that*

$$\sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}}\left\{\frac{\hat{\theta}_p}{\mathrm{TG}_p(X;Y)}-1\right\}\xrightarrow{d}\Phi,$$

where Φ *is the same as in Theorem 1.*

3. Simulation

In this section, we study the finite sample performance of our estimator $\hat{\theta}_p$ by simulation. We simulate the data from the following two models in Cai and Musta (2020). Let $a_1, a_2 \in (0, 1)$.

Model 1. Let Z_1, Z_2, and Z_3 be independent Pareto random variables with parameters a_1, a_2, and a_1, respectively. Here, a Pareto distribution with parameter $a > 0$ means that the probability density function is $f(x) = a^{-1}x^{-1/a-1}$ for $x > 1$. Define

$$(X, Y) = B\,(Z_1, Z_3) + (1 - B)\,(Z_2, Z_2)\,,$$

where B is a Bernoulli(1/2) random variable independent of Z_i's. For this model, we have $\gamma_1 = a_1$, $\rho_1 = 1-a_1/a_2, \eta = a_2/a_1$, and $\tau(x, y) = 2^{a_1/a_2-1}(x\wedge$

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$y)^{a_1/a_2}$. We consider four settings of (a_1, a_2), see Table 1. Here we can take $\xi = 2a_1/a_2 - 1, \delta = (2 - a_2^4)/a_2 - 1/a_1 - 2, \beta_1 = a_1(2 - a_2^3)/a_2 - 1, \beta_2 = (2a_1 - a_2^2)/(a_2(1 - a_1)) - 1/(1 - a_1)$, and we note that Model 1 satisfies Assumptions 1 to 6.

Model 2. Define

$$(X, Y) = \left(\{1 - \Phi(\widetilde{X})\}^{-a_1}, \widetilde{Y}\right),$$

where $\widetilde{X}$ and $\widetilde{Y}$ are two standard normal random variables with correlation a_2, and Φ is the distribution function of $\widetilde{X}$. Thus, X follows from a Pareto distribution with parameter a_1, and (X, Y) has a Gaussian copula. For this model, $\gamma_1 = a_1, \rho_1 = 0$, $\eta = (1 + a_2)/2$, and $\tau(x, y) = (xy)^{1/(1+a_2)}$. Obviously, $\int_0^\infty \tau\left(x^{-\frac{1}{\gamma_1}}, 1\right) dx = \infty$. Thus Model 2 does not satisfy Assumption 2 and hence Theorem 1 does not hold.

For comparison between the estimators and true values, we evaluate the true value $\mathrm{TG}_p(X; Y)$ by using the true density functions and drawing 200 replications with sample size 1,000,000. The true values are then approximated by the corresponding median of overall 200 replications. Table 1 shows the parameters for the distributions and the approximated true values of the tail Gini functional.

Next, we draw $m = 2000$ replications from each model with sample sizes $n = 1500$ and 5000. For each replication, we compute the proposed

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Table 1: Parameters of five models and the approximated true values of the tail Gini functional.

	(a_1, a_2)	γ_1	η	$-1/\eta + 1 + \gamma_1$	$p = 0.01$	$p = 0.001$
Model 1(a)	(0.35, 0.3)	0.35	6/7	0.183	0.5835	0.8965
Model 1(b)	(0.4, 0.35)	0.4	0.875	0.251	1.0923	1.9283
Model 1(c)	(0.6,0.5)	0.6	5/6	0.1	4.2418	10.9131
Model 1(d)	(0.5, 0.4)	0.5	0.8	0.3	1.3009	2.1104
Model 2	(0.6, 0.9)	0.6	0.95	0.547	24.6808	84.0422

nonparametric estimator $\hat{\theta}_p$ with $p = 0.01$ and 0.001. The proper choice of (k, k_1, k_2), that is, the number of tail observations used in the estimation of $\mathrm{TG}_{k/n}(X;Y)$, γ_1, and η, respectively, is always a delicate problem in the extreme value theory. To investigate how sensitive our result is with respect to the choice of (k, k_1, k_2) and to see the range of suitable (k, k_1, k_2), we compute the scaled mean squared errors (sMSE):

$$\mathrm{sMSE}\,(k, k_1, k_2) = \frac{1}{m}\sum_{i=1}^{m}\left\{\frac{\hat{\theta}_{p,i}\,(k, k_1, k_2)}{\mathrm{TG}_p(X;Y)} - 1\right\}^2.$$

Let $\alpha = k/n, \alpha_1 = k_1/n, \alpha_2 = k_2/n$. Figure 1 shows the results for the five models with $n = 5000$, where the solid lines denote the results for $p = 0.01$ and the dotted lines denote the results for $p = 0.001$. For each

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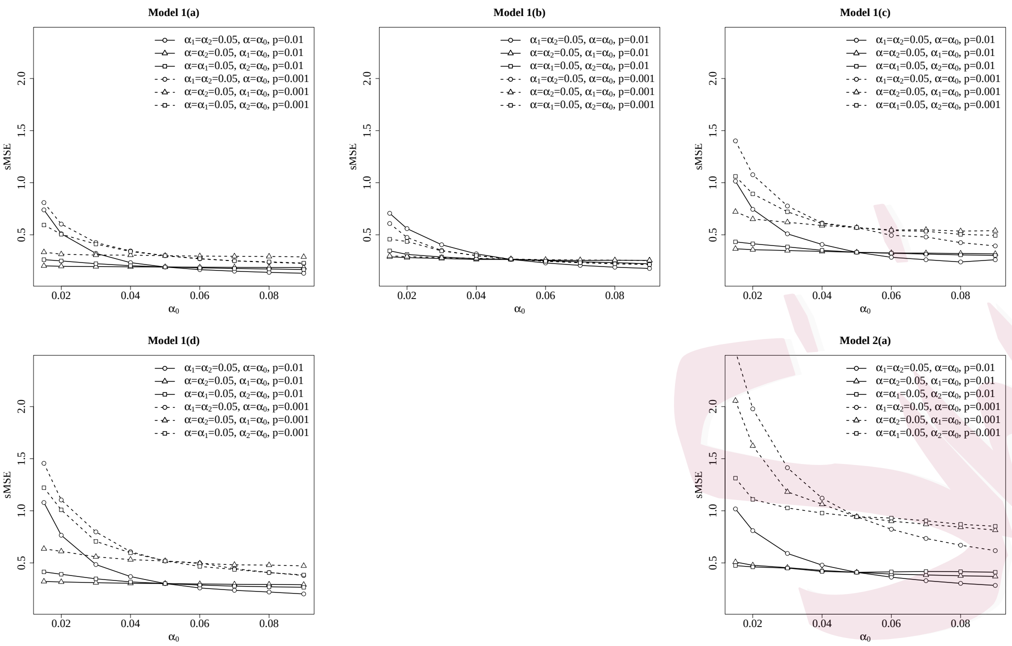

Figure 1: sMSE for different choice of intermediate levels $(\alpha, \alpha_1, \alpha_2)$.

curve, we fix the parameters values of α's to be 0.05 and let the remaining α vary. Figure 1 suggests that sMSE is rather stable for a wide range of α_1 and α_2.

In order to assess the finite sample performance of our estimator, the ratio $\hat{\theta}_p / \operatorname{TG}_p(X;Y)$ is calculated and the corresponding means and standard errors are reported in Table 2. Here we compare our proposed method (denoted by AIE) and the method in Hou and Wang (2021) (denoted by HW). Note that we also compute the estimated value with the naive estimator by utilizing the order statistics. However, we omit the results here

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due to its relatively poor performance. We set $\alpha = 0.09, \alpha_1 = \alpha_2 = 0.05$. We can make the following conclusions from Table 2. First, in each case, the means of the ratios for AIE are closer to one than that of the ratios for HW; the standard errors of the ratios for AIE are smaller than that of the ratios for HW. Both indicate the accuracy of our proposed estimators under asymptotic independence. It is worth noting that AIE tends to underestimate by a small margin while HW tends to overestimate by a larger margin. This is due to the fact that in AIE the $\hat{\eta}$ is usually smaller than the true η. In HW, on the other hand, η is taken as 1 by default. Second, for both AIE and HW, the estimators at extreme level $p = 0.01$ perform better than that at $p = 0.001$ given the sample size n. Third, in each case, if we compare the standard errors for different sample sizes n, it is obvious that those results with a larger sample size have smaller standard errors. Last but not least, Model 2 exhibits the poorest performance in terms of both the means of the ratios and the standard errors, which possibly stems from the fact that Assumption 2 is not satisfied for Model 2.

In addition, we present the boxplots of $\log\left(\hat{\theta}_p/\mathrm{TG}_p(X;Y)\right)$. From the boxplots in Figure 2, we can see that most of the estimated values obtained through our proposed estimator are distributed symmetrically around 1. In contrast, the distribution of the estimator in Hou and Wang (2021) mostly

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Table 2: Means of the ratios of the proposed estimators for the tail Gini functional and the true values for $n = 1500, 5000$ and $p = 0.01, 0.001$ are reported with corresponding standard deviation given in the brackets.

		AIE		HW	
		$n = 1500$	$n = 5000$	$n = 1500$	$n = 5000$
Model 1(a)	$p = 0.01$	0.9136(0.6472)	0.9263(0.3831)	1.4123(0.7436)	1.3955(0.4291)
	$p = 0.001$	0.8087(0.7715)	0.8661(0.4416)	1.8791(1.0801)	1.9696(0.6133)
Model 1(b)	$p = 0.01$	0.8749(0.6171)	0.9028(0.3503)	1.2989(0.7135)	1.3092(0.3940)
	$p = 0.001$	0.8292(0.7900)	0.8583(0.4527)	1.7876(1.1019)	1.7911(0.6174)
Model 1(c)	$p = 0.01$	0.8837(0.8274)	0.9137(0.5278)	1.4439(1.1285)	1.4568(0.7272)
	$p = 0.001$	0.8800(1.3837)	0.7995(0.5506)	2.2123(2.1902)	2.0634(1.0907)
Model 1(d)	$p = 0.01$	0.9444(0.8701)	0.9528(0.4914)	1.6712(1.2342)	1.6627(0.7273)
	$p = 0.001$	0.9591(1.0914)	0.9641(0.6230)	2.9303(2.2103)	2.9681(1.3097)
Model 2	$p = 0.01$	0.8809(0.9172)	0.9541(3.7531)	1.1608(1.2054)	1.2149(4.4512)
	$p = 0.001$	0.8865(1.1556)	0.8536(0.9595)	1.5029(1.6746)	1.3889(1.5865)

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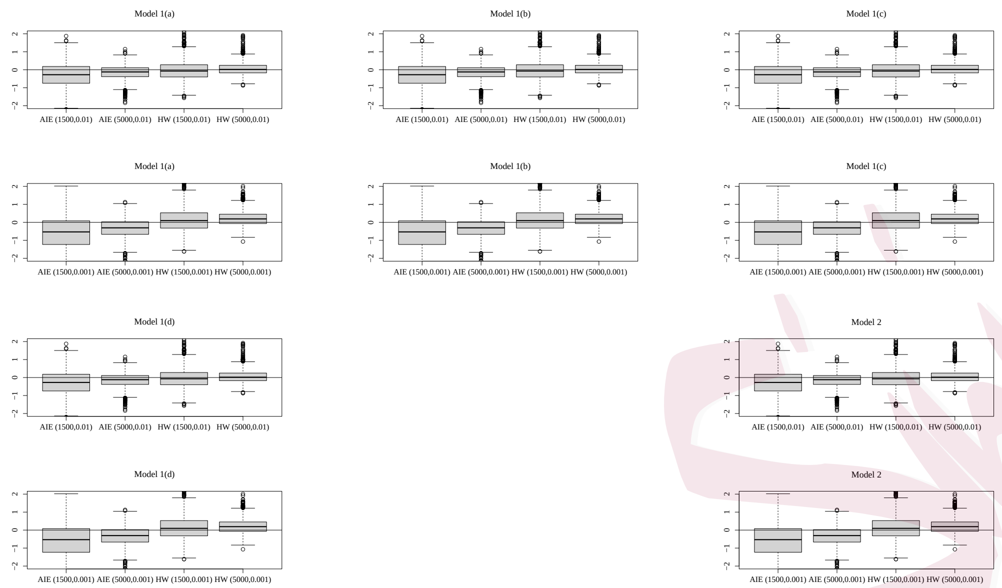

Figure 2: Boxplots of log ratios $\log(\hat{\theta}_p/\mathrm{TG}_p(X;Y))$ with (n, p).

deviates from 1 because it fails to take η into consideration and overestimates the tail Gini functional. Moreover, when the sample size n increases, the body of the box becomes narrower, which shows the convergence of risk measures in probability as shown in Theorems 1.

To show the asymptotic property of our proposed estimators, we also compare the sample quantiles of log-ratios at all levels with the quantiles of the theoretical limit distribution Φ by using QQ plots. We generate samples of Φ by simulation, based on its definition in Proposition 1. Figure 3 shows that most of the scatters line up on the red straight line, which indicates no big difference from a normal distribution.

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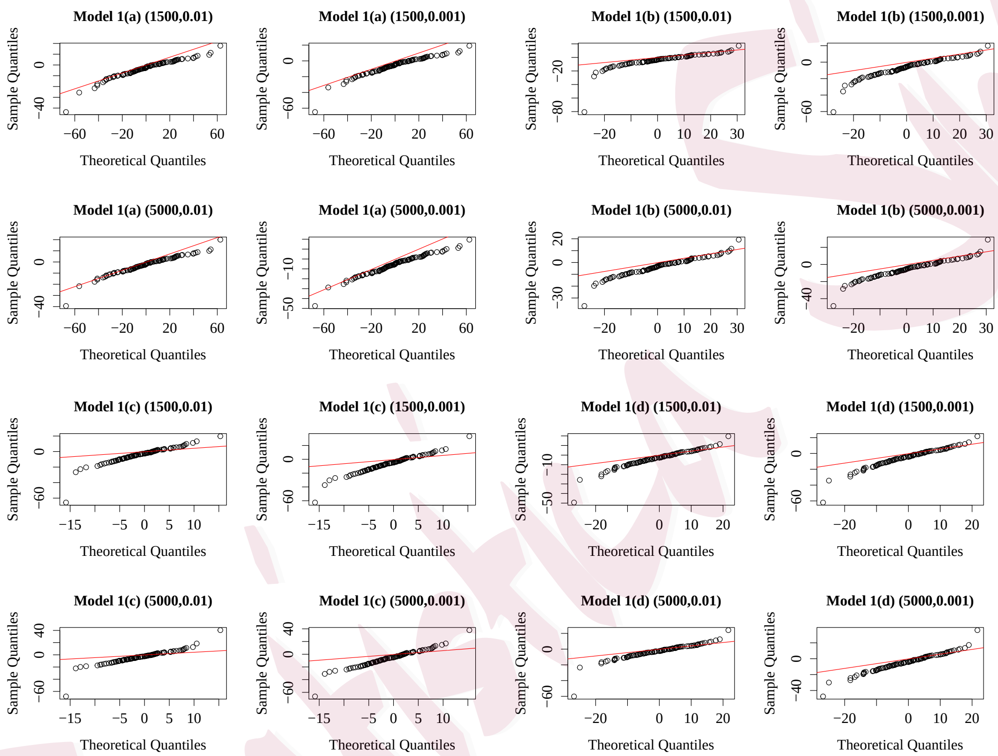

Figure 3: QQ plots of log ratios for AIE estimators for (n, p)

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4. Application

In this section, we employ our estimator for the tail Gini functional on a dataset encompassing the daily stock price of the Hang Seng Index (HSI) and 17 companies from December 20, 1992 to December 30, 2022. Our aim is to examine the impact on individual stocks during the occurrence of an exceptionally high-risk event associated with the systemic variable, which is HSI in this application.

Following Hou and Wang (2021), we use "loss" to represent the percentage of the negative weekly returns. Upon calculation, we have $n = 1565$ observations of losses for HSI and the 17 firms. Table 3 shows the tickers, full names and the summary statistics of the losses for the 17 firms plus HSI. Figure 4 shows the boxplots of all losses.

Before studying the effect of the extreme loss of HSI, the systemic variable Y, on the losses of individual stocks, denoted by $X_i, i = 1, \cdots, 17$, we would like to check the asymptotic independence assumption for each pair of $(X_i, Y), i = 1, \cdots, 17$. Here, we apply the Tail Quotient Correlation Coefficient (TQCC)-based test in Zhang, Zhang and Cui (2017) to test the null hypothesis of asymptotic independence, which corresponds to the case $\eta \in (0, 1)$.

To conduct the test, we first fit generalized extreme value distribution

Table 3: Summary statistics, TQCC, p-values and estimated values.

Ticker	Firms	Mean (%)	SD (%)	TQCC	p-value	$\hat{\gamma}_1$	$\hat{\eta}$	$\hat{\theta}_{0.01}$	$\hat{\theta}_{0.001}$
HSI	Heng Seng Index	-0.1647	3.1101	–	–	–	–	–	–
X0001.HK	CKH Holdings	-0.1482	4.2117	0.0001	0.9997	0.3744	0.8597	3.4121	5.5498
X0002.HK	CLP Holdings	-0.1278	2.5857	0.0000	1.0000	0.4099	0.8214	1.2826	1.9974
X0003.HK	HK & China GAS	-0.0103	3.2390	0.0000	1.0000	0.5285	0.7501	1.9425	3.0459
X0004.HK	Wharf Holdings	-0.2317	4.9306	0.0000	1.0000	0.3904	0.8649	3.7878	6.4946
X0005.HK	HSBC Holdings	-0.4176	9.6040	0.1043	0.0000	–	–	–	–
X0006.HK	Power Assets	-0.1408	2.6953	0.0000	1.0000	0.4203	0.7591	0.9901	1.2547
X0010.HK	Hang Lung Group	-0.0650	4.2654	0.0003	0.9985	0.3483	0.8590	2.8533	4.3601
X0011.HK	Hang Seng Bank	-0.0844	3.2972	0.0207	0.1063	0.4536	0.8824	3.4732	7.2628
X0012.HK	Henderson Land	-0.1587	4.6614	0.0014	0.9630	0.4031	0.8767	3.2079	5.7107
X0016.HK	SHK PPT	-0.1871	4.4221	0.0003	0.9970	0.3993	0.8210	2.8891	4.3858
X0017.HK	New World Dev	-0.3594	9.0423	0.3364	0.0000	–	–	–	–
X0019.HK	Swire Pacific A	-0.1326	4.3048	0.0001	0.9997	0.4527	0.8805	4.7142	9.7794
X0023.HK	Bank of E Asia	-0.0510	4.2934	0.0003	0.9985	0.4229	0.7898	2.8040	4.0235
X0083.HK	Sino Land	-0.4961	11.2818	0.0699	0.0000	–	–	–	–
X0087.HK	Swire Pacific B	-0.1126	3.8678	0.0001	0.9999	0.4538	0.8948	4.6822	10.1563
X0101.HK	Hang Lung PPT	-0.0933	4.3302	0.0016	0.9689	0.3646	0.8846	3.2072	5.4987
X0293.HK	Cathay Pacific	-0.1125	4.1938	0.0048	0.8127	0.3911	0.8371	2.2030	3.4641

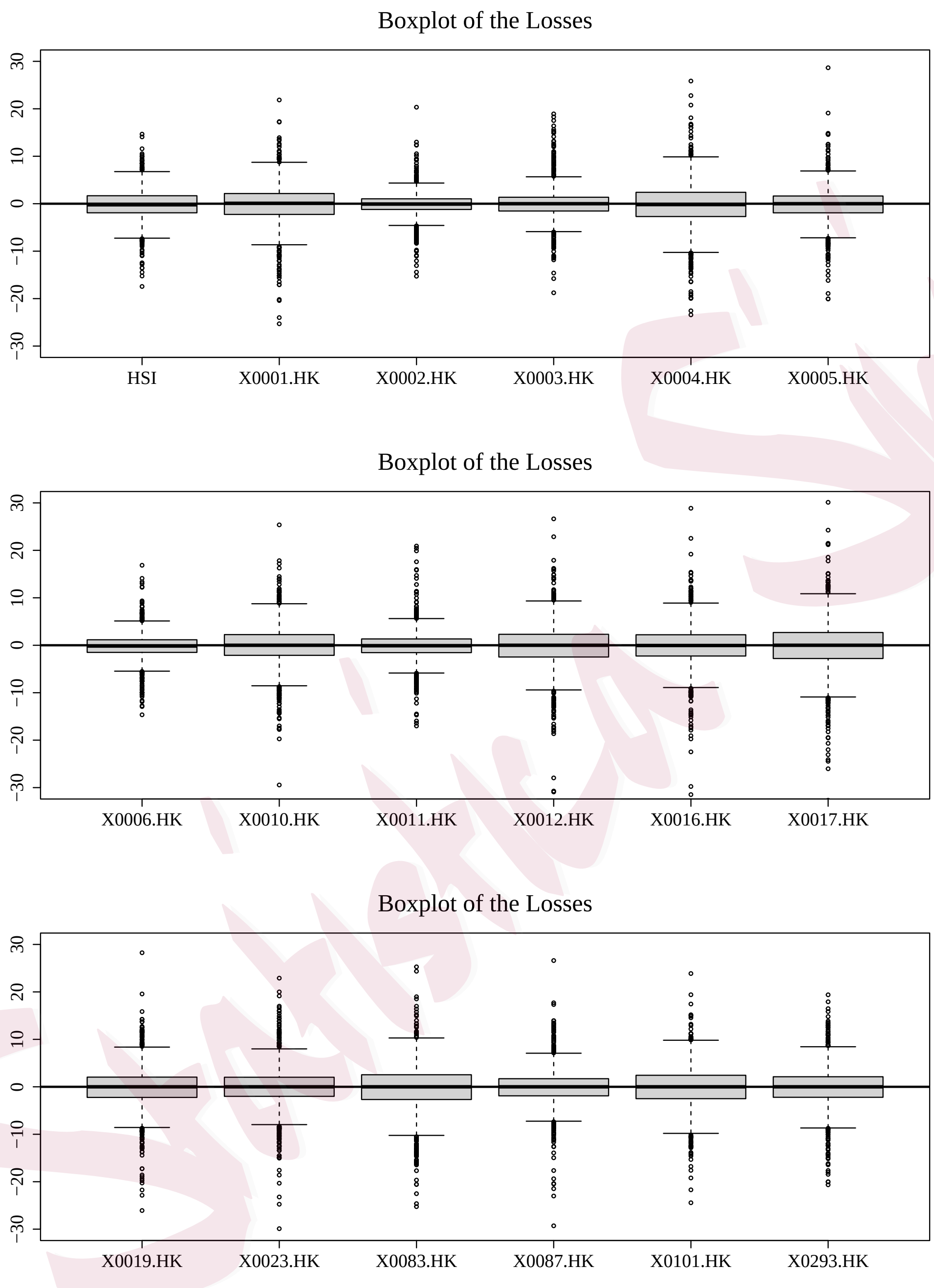

Figure 4: Boxplots of the losses.

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to each series and perform marginal transformations. We apply TQCC to the transformed data, and choose the threshold as the smaller one of the two empirical 95th percentiles. For more details on TQCC-test, we refer to Zhang, Zhang and Cui (2017). The computed TQCC measures and p-values are summarized in Table 3. We exclude stocks with p-values under 0.05 from consideration. Specifically, HSBC Holdings (Ticker: X0005.HK), New World Development (Ticker: X0017.HK), and Sino Land (Ticker: X0083.HK) are removed due to substantial statistical evidence supporting the rejection of asymptotic independence between these stocks and HSI.

Subsequently, we assess the signs of $\hat{\gamma}_1$ and $\hat{\eta}$ for the remaining set of 14 stocks. From Figures S1 and S2 in the supplementary material, we can see that $\hat{\gamma}_1 > 0$ and $\hat{\eta} \in (0.5, 1)$ for each pair of losses across different α_1 and α_2. Choosing $\alpha_1 = \alpha_2 = 0.08$, we obtain the corresponding $\hat{\gamma}_1$ and $\hat{\eta}$ in Table 3. Figure 5 plots the values of AIE estimator $\hat{\theta}_{0.01}$ and $\hat{\theta}_{0.001}$ against α for the 14 stocks, from which we conclude $\alpha = 0.09$ lying in the interval where the estimates are stable. We thus report the corresponding estimators for $p = 0.01$ and $p = 0.001$ in Table 3. It is evident that the values generated by AIE estimators exhibit a consistent pattern. They all remain below 6 when considering the scenario with $p = 0.01$, and similarly, they remain below 15 when dealing with $p = 0.001$. These values are notably smaller

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when contrasted with the findings presented in the reference Hou and Wang (2021). In essence, this implies that asymptotic independence typically aligns with reduced tail variability in relation to extreme events within systemic variables.

5. Proofs

Proof of Theorem 1. Let $M_n = \sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}}$. Recall that $d_n = \frac{k}{np}$, $\hat{\theta}_p = (\frac{k}{np})^{1-1/\hat{\eta}+\hat{\gamma}_1}\hat{\theta}_{k/n}$.

We rewrite

$$\begin{aligned}\frac{\hat{\theta}_p}{\mathrm{TG}_p(X;Y)} &= d_n^{\hat{\gamma}_1-\gamma_1} \times d_n^{\frac{1}{\eta}-\frac{1}{\hat{\eta}}} \times \frac{\hat{\theta}_{k/n}}{\mathrm{TG}_{k/n}(X;Y)} \\ &\quad \times \frac{d_n^{-\frac{1}{\eta}+1}\mathrm{TG}_{k/n}(X;Y)/Q_1(1-k/n)}{\mathrm{TG}_p(X;Y)/Q_1(1-p)} \times \frac{d_n^{\gamma_1}Q_1(1-k/n)}{Q_1(1-p)}. \\ &=: I_1 \times I_2 \times I_3 \times I_4 \times I_5.\end{aligned}$$

By the assumption that $M_n \log d_n = o(\sqrt{k})$ and $\sqrt{k}\,(\hat{\gamma}_1 - \gamma_1) = O_{\mathbf{P}}(1)$, it follows that

$$I_1 - 1 = e^{(\hat{\gamma}_1-\gamma_1)\log d_n} - 1 = (\hat{\gamma}_1-\gamma_1)\log d_n + o_{\mathbf{P}}\left((\hat{\gamma}_1-\gamma_1)\log d_n\right) = O_{\mathbf{P}}\left(\frac{\log d_n}{\sqrt{k}}\right) = o_{\mathbf{P}}\left(\frac{1}{M_n}\right).$$

In the same way, we get $I_2 - 1 = o_{\mathbf{P}}\left(\frac{1}{M_n}\right)$. By Proposition 1 we have

$$I_3 = \frac{\hat{\theta}_{k/n}}{\mathrm{TG}_{k/n}(X;Y)} = 1 + \frac{1}{M_n}\Phi + o_{\mathbf{P}}\left(\frac{1}{M_n}\right).$$

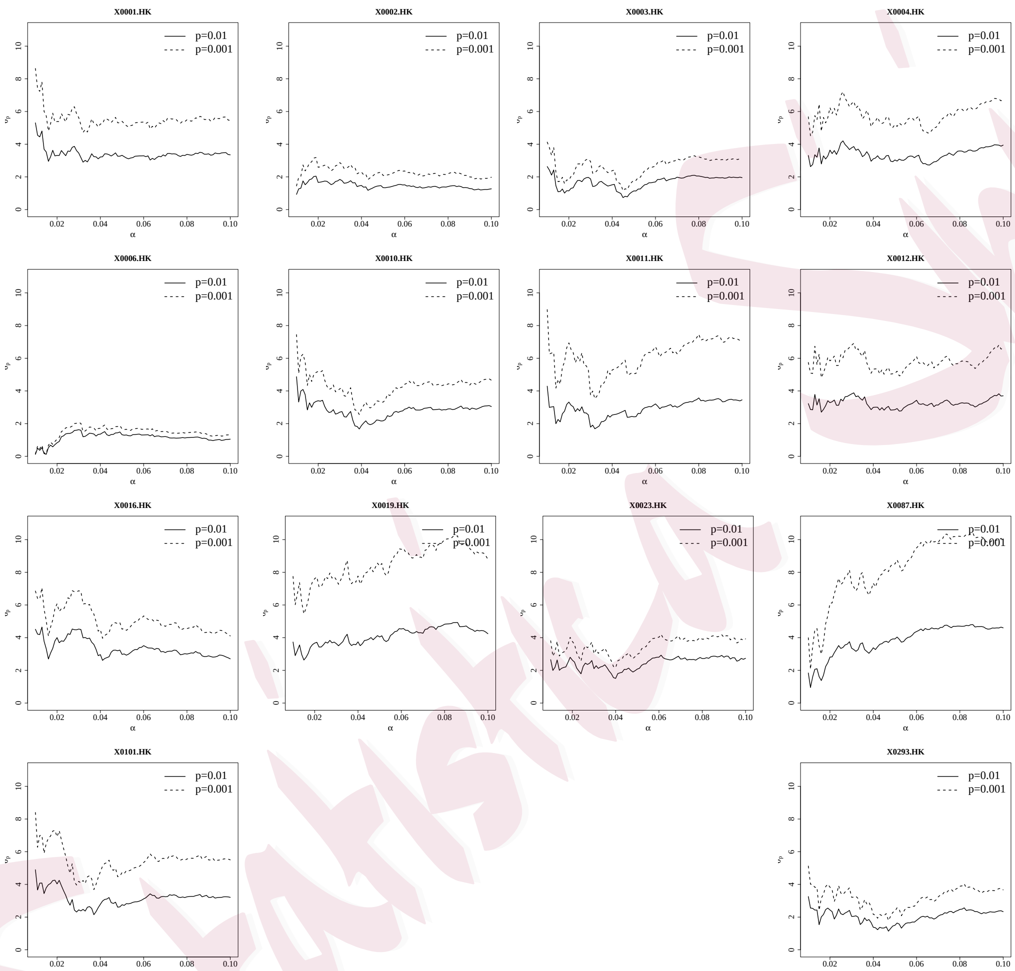

Figure 5: The estimates of $TG_{0.01}$ and $TG_{0.001}$ against α for the 14 stocks.

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For I_4, by (4), (ii) and (iv) in Lemma 1 in supplementary material, we have

$$\begin{aligned}\frac{\mathrm{TG}_{k/n}(X;Y)}{(\frac{k}{n})^{\frac{1}{\eta}-1}Q_1(1-k/n)} =& 4\int_0^\infty \int_0^1 \tau_{k/n}\left(s_{k/n}(x), y\right) dy dx^{-\gamma_1} - 2\int_0^\infty \tau_{k/n}\left(s_{k/n}(x), 1\right) dx^{-\gamma_1} \\ =& \phi_0 + o\left(\frac{1}{M_n}\right).\end{aligned}$$

Similarly, for $p \le k/n$, we have

$$\frac{\mathrm{TG}_p(X;Y)}{p^{\frac{1}{\eta}-1}Q_1(1-p)} = \phi_0 + o\left(\frac{1}{M_n}\right).$$

Thus

$$I_4 = \frac{(k/n)^{1-\frac{1}{\eta}}\,\mathrm{TG}_{k/n}(X;Y)/Q_1(1-k/n)}{p^{1-\frac{1}{\eta}}\,\mathrm{TG}_p(X;Y)/Q_1(1-p)} = \frac{\phi_0 + o\left(\frac{1}{M_n}\right)}{\phi_0 + o\left(\frac{1}{M_n}\right)} = 1 + o\left(\frac{1}{M_n}\right).$$

It follows from Assumptions 1 and 5 that

$$I_5 = \frac{d_n^{\gamma_1} Q_1(1-k/n)}{Q_1(1-p)} = 1 + O\left\{A_1\left(\frac{n}{k}\right)\right\} = 1 + o(\frac{1}{\sqrt{k}}).$$

Hence, we finally obtain

$$\begin{aligned}\frac{\hat{\theta}_p}{\mathrm{TG}_p(X;Y)} =& \left\{1 + o_{\mathbf{P}}\left(\frac{1}{M_n}\right)\right\}^2 \times \left\{1 + \frac{\Phi}{M_n} + o_{\mathbf{P}}\left(\frac{1}{M_n}\right)\right\} \times \left\{1 + o\left(\frac{1}{M_n}\right)\right\} \times \left\{1 + o(\frac{1}{\sqrt{k}})\right\} \\ =& 1 + \frac{\Phi}{M_n} + o_{\mathbf{P}}\left(\frac{1}{M_n}\right),\end{aligned}$$

which implies the statement of Theorem 1. □

Proof of Theorem 2. Write

$$\frac{\hat{\theta}_p}{\mathrm{TG}_p(X;Y)} = \frac{\hat{\theta}_p}{\mathrm{TG}_p(X^+;Y)} \times \frac{\mathrm{TG}_p(X^+;Y)}{\mathrm{TG}_p(X;Y)}.$$

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We first consider $\frac{\hat{\theta}_p}{\mathrm{TG}_p(X^+;Y)}$ and show that it has the same the asymptotic normality as $\frac{\hat{\theta}_p}{\mathrm{TG}_p(X;Y)}$, which is stated in Theorem 1. In other words, we need to check that Assumptions 1 to 6 also hold for (X^+, Y). Note that Assumptions 2, 5 and 6 hold automatically. Thus we only need to show that (X^+, Y) satisfies Assumptions 1, 3 and 4.

Denote the distribution of X^+as F_1^+, the quantile function of X^+as Q_1^+, and

$$\tau_p^+(x,y) = p^{-1/\eta}\mathbf{P}\left\{1 - F_1^+(X^+) < px, 1 - F_2(Y) < py\right\}, \quad x, y > 0.$$

As X has a continuous distribution, a simple calculation leads to $Q_1^+(1 - p) = Q_1(1-p)I(0 < p \leq \bar{F}_1(0))$, which implies that X^+ satisfies Assumption 1 and that

$$\begin{aligned}
&\mathbf{P}\left\{1 - F_1^+\left(X^+\right) < u, 1 - F_2(Y) < v\right\} \\
&= \begin{cases} \mathbf{P}\left\{1 - F_1\left(X\right) < u, 1 - F_2(Y) < v\right\}, & 0 < u < \bar{F}_1(0), \\ v, & u \geq \bar{F}_1(0). \end{cases}
\end{aligned}$$

Thus,

$$\tau_p^+(x,y) = \begin{cases} \tau_p(x,y), & 0 < x < \bar{F}_1(0)/p, \\ p^{1-\frac{1}{\eta}}y, & x \geq \bar{F}_1(0)/p, \end{cases}$$

which means that Assumption 3 also holds for (X^+, Y).

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For Assumption 4, notice that for suffiently small $p > 0$,

$$\sup_{\substack{1<x<\bar{F}_1(0)/p \\ 0<y\le 1}} \left|\tau_p^+(x,y) - \tau(x,y)\right| x^{-\beta_2} = \sup_{\substack{1<x<\bar{F}_1(0)/p \\ 0<y\le 1}} \left|\tau_p(x,y) - \tau(x,y)\right| x^{-\beta_2} = O\left(p^{\xi}\right).$$

Moreover, by the homogeneity of τ and setting $x = 1/p$, we have, for $0 < y \le 1$

$$p^{1-1/\eta} y - y^{1/\eta}\tau\left(\frac{1}{py}, 1\right) = O(p^{\xi-\beta_2}). \tag{5.1}$$

So for $0 < y \le 1$, we have

$$p^{1-\frac{1}{\eta}} y - y^{1/\eta}\tau\left(\frac{\bar{F}_1(0)}{py}, 1\right) = \begin{cases} O(p^{1-1/\eta}), & 1-1/\eta < \xi - \beta_2, \\ O(p^{\xi-\beta_2}), & 1-1/\eta > \xi-\beta_2. \end{cases}$$

Therefore, for $x \ge \bar{F}_1(0)/p$ and p suffiently small, it follows that

$$\begin{aligned} x^{-\beta_2}\left\{\tau_p^+(x,y) - \tau(x,y)\right\} &= x^{-\beta_2}\left\{p^{1-\frac{1}{\eta}} y - \tau(x,y)\right\} \\ &\le \left\{\frac{\bar{F}_1(0)}{p}\right\}^{-\beta_2}\left\{p^{1-\frac{1}{\eta}} y - \tau(\bar{F}_1(0)/p, y)\right\} \\ &= \left\{\frac{p}{\bar{F}_1(0)}\right\}^{\beta_2}\left\{p^{1-\frac{1}{\eta}} y - y^{1/\eta}\tau\left(\frac{\bar{F}_1(0)}{py}, 1\right)\right\} \\ &= \begin{cases} O(p^{\beta_2+1-1/\eta}), & 1-1/\eta < \xi-\beta_2, \\ O(p^{\xi}), & 1-1/\eta > \xi-\beta_2. \end{cases} \end{aligned}$$

Since $1 - 1/\eta > \xi - \beta_2$ (see Assumption 7), it follows that

$$\sup_{\substack{x\ge\bar{F}_1(0)/p \\ 0\le y\le 1}} \left|\tau_p^+(x,y) - \tau(x,y)\right| x^{-\beta_2} = O\left(p^{\xi}\right),$$

which means that Assumption 4 also holds for $(X^+; Y)$.

As a result, Theorem 1 can be applied and we have

$$\sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}}\left(\frac{\hat{\theta}_p}{\mathrm{TG}_p(X^+;Y)}-1\right)\xrightarrow{d}\Phi.$$

Next we show $\frac{\mathrm{TG}_p(X;Y)}{\mathrm{TG}_p(X^+;Y)}=1+o\left(\frac{1}{M_n}\right)$. Note that

$$\frac{\mathrm{TG}_p(X;Y)}{\mathrm{TG}_p(X^+;Y)}-1=\frac{4}{p}\frac{\mathrm{Cov}\left(X^-,F_2(Y)\mid F_2(Y)>1-p\right)}{\mathrm{TG}_p(X^+;Y)}.$$

Rewrite

$$\begin{aligned}
&\left|\mathrm{Cov}\left(X^-,F_2(Y)\mid F_2(Y)>1-p\right)\right|\\
=&\left|\mathbb{E}\left\{X^-F_2(Y)\mid F_2(Y)>1-p\right\}-\mathbb{E}\left\{X^-\mid F_2(Y)>1-p\right\}\mathbb{E}\left\{F_2(Y)\mid F_2(Y)>1-p\right\}\right|\\
=&\left|\mathbb{E}\left\{X^-F_2(Y)\mid F_2(Y)>1-p\right\}-(1-p/2)\mathbb{E}\left\{X^-\mid F_2(Y)>1-p\right\}\right|\\
=&\left|\mathbb{E}\left[X^-\left\{F_2(Y)-(1-p/2)\right\}\mid F_2(Y)>1-p\right]\right|\\
\le&\mathbb{E}\left\{|X^-||F_2(Y)-(1-p/2)|\mid F_2(Y)>1-p\right\}\\
\le&\frac{p}{2}\mathbb{E}\left\{|X^-|\mid F_2(Y)>1-p\right\}\\
\le&\frac{1}{2}\left(\mathbb{E}\left|X^-\right|^\zeta\right)^{1/\zeta}\left[\mathbb{E}I\left\{X<0,\bar{F}_2(Y)<p\right\}\right]^{1-1/\zeta}\\
=&O(p^{1-1/\zeta}),
\end{aligned}$$

where the last inequality is guaranteed by Hölder's inequality, and we have $\mathbb{E}\left|X^-\right|^\zeta<\infty$ by Assumption 7.

Thus, it follows that

$$
\begin{aligned}
M_n\left(\frac{\mathrm{TG}_p(X;Y)}{\mathrm{TG}_p\left(X^+;Y\right)}-1\right) &= M_n\frac{4}{p}\frac{\mathrm{Cov}\left(X^-,F_2(Y)\mid F_2(Y)>1-p\right)}{\mathrm{TG}_p^+(X;Y)} \\
&= M_n\frac{4}{p}\frac{\mathrm{Cov}\left(X^-,F_2(Y)\mid F_2(Y)>1-p\right)}{O\left(p^{1/\eta-1}Q_1(1-p)\right)} \\
&= O\left(M_n p^{\gamma_1-\frac{1}{\zeta}+1-\frac{1}{\eta}}\right) \\
&= o(1),
\end{aligned}
$$

and the proof is completed.

□

Supplementary Material

The supplementary material contains the proofs of four auxiliary lemmas and Proposition 1 as well as some additional figures for the Simulation and Application.

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