Statistica Sinica Preprint No: SS-2022-0215	
Title	Particle-based, Rapid Incremental Smoother Meets Particle Gibbs
Manuscript ID	SS-2022-0215
URL	http://www.stat.sinica.edu.tw/statistica/
DOI	10.5705/ss.202022.0215
Complete List of Authors	Gabriel Cardoso, Eric Moulines and Jimmy Olsson
Corresponding Authors	Gabriel Cardoso
E-mails	gvictorinocardoso@gmail.com

Statistica Sinica

Particle-based, Rapid Incremental Smoother Meets Particle Gibbs

Gabriel Cardoso, Eric Moulines, and Jimmy Olsson

Centre de Mathématiques Appliquées, Ecole polytechnique, UMR 7642, Palaiseau, France

Electrophysiology and Heart Modeling Institute (IHU-Liryc), Pessac, France

Department of Mathematics, KTH Royal Institute of Technology, Stockholm, Sweden

Abstract: The particle-based rapid incremental smoother (`PARIS`) is a sequential Monte Carlo technique that allows for efficient online approximations of expectations of additive functionals under Feynman–Kac path distributions. Under weak assumptions, the algorithm has linear computational complexity and limited memory requirements. It also comes with a number of nonasymptotic bounds and convergence results. However, being based on self-normalized importance sampling, the `PARIS` estimator is biased. This bias is inversely proportional to the number of particles, but has been found to grow linearly with the time horizon, under appropriate mixing conditions. In this work, we propose the Parisian particle Gibbs (`PPG`) sampler, which has essentially the same complexity as that of the `PARIS`, but significantly reduces the bias for a given computational complexity at the cost of a modest increase in the variance. This method is a wrapper, in the sense that it uses the `PARIS` algorithm in the inner loop of the particle Gibbs algorithm to form a bias-reduced version of the targeted quantities. We substantiate the `PPG` algorithm with theoretical results, including new bounds on the bias and variance, as well as deviation inequalities. We illustrate our theoretical results using numerical experiments that support our claims.

Key words and phrases: sequential Monte Carlo, particle Gibbs, bias reduction, smoothing of additive functionals, state space smoothing, particle filters.

1. Introduction

Feynman–Kac formulae play a key role in many models used in statistics, physics, and many other fields; see [10, 11, 8], and the references therein. Let $\{(\mathsf{X}_n, \mathcal{X}_n)\}_{n\in\mathbb{N}}$ be a sequence of measurable spaces and define, for every $n \in \mathbb{N}$, $\mathsf{X}_{0:n} := \prod_{m=0}^{n} \mathsf{X}_m$ and $\mathcal{X}_{0:n} := \bigotimes_{m=0}^{n} \mathcal{X}_m$. For a sequence $\{M_n\}_{n\in\mathbb{N}}$ of Markov kernels $M_n : \mathsf{X}_n \times \mathcal{X}_{n+1} \to [0,1]$, an initial distribution $\eta_0 \in \mathsf{M}_1(\mathcal{X}_0)$, and a sequence $\{g_n\}_{n\in\mathbb{N}}$ of bounded measurable potential functions $g_n : \mathsf{X}_n \to \mathbb{R}_+$, a sequence $\{\eta_{0:n}\}_{n\in\mathbb{N}}$ of *Feynman–Kac path measures* is defined by

$$\eta_{0:n} : \mathcal{X}_{0:n} \ni A \mapsto \frac{\gamma_{0:n}(A)}{\gamma_{0:n}(\mathsf{X}_{0:n})}, \quad n \in \mathbb{N}, \tag{1.1}$$

where

$$\gamma_{0:n} : \mathcal{X}_{0:n} \ni A \mapsto \int \mathbb{1}_A(x_{0:n})\, \eta_0(\mathrm{d}x_0) \prod_{m=0}^{n-1} Q_m(x_m, \mathrm{d}x_{m+1}), \tag{1.2}$$

with

$$Q_m : \mathsf{X}_m \times \mathcal{X}_{m+1} \ni (x, A) \mapsto g_m(x) M_m(x, A) \tag{1.3}$$

being unnormalized kernels. By convention, $\eta_{0:0} := \eta_0$. Note that each $\eta_{0:n}$ is a probability measure, whereas $\gamma_{0:n}$ is not normalized. For every $n \in \mathbb{N}^*$, we also define the marginal distribution $\eta_n : \mathcal{X}_n \ni A \mapsto \eta_{0:n}(\mathsf{X}_{0:n-1} \times A)$. In the context of nonlinear filtering in *general state-space hidden Markov models*(HMMs), $\eta_{0:n}$ and η_n are, the *joint smoothing* and *filter distribution*, respectively, at time n; see [10, 7, 8].

doi:10.5705/ss.202022.0215

For most problems of practical interest, the Feynman–Kac path or marginal measures are intractable, and so is any expectation associated with the same. As a result, considerable research has been devoted to developing Monte Carlo, or *particle*, approximations of such measures. A *particle filter* approximates the marginal distribution flow $\{\eta_n\}_{n\in\mathbb{N}}$ by a sequence of occupation measures, associated with a swarm of *particles* $\{\xi_n^i\}_{i=1}^N$, $n \in \mathbb{N}$, where each particle ξ_n^i is a random draw in X_n. Particle filters revolve around two operations: a *selection step*, which duplicates or sorts out particles with large or small importance weights, respectively, and a *mutation step*, which randomly evolves the selected particles in the state space. An alternating and iterative application of selection and mutation results in a swarm of N particles that are both serially and spatially dependent. Feynman–Kac path models can also be interpreted as laws associated with a certain type of Markovian backward dynamics; this interpretation is useful, for example, for the smoothing problem in nonlinear filtering [15, 12]. Several convergence results have been established for particle filters, as the number N of particles tends to infinity; see for example, [10, 16, 11, 8]. In addition, a number of nonasymptotic results have been obtained for these methods, including bounds on their bias and L_p error, as well as exponential concentration inequalities and propagation of chaos estimates. Extensions to the backward interpretation can also be found in [15, 12].

In this work, we focus on the problem of recursively computing smoothed expectations

$$\eta_{0:n} h_n = \int h_n(x_{0:n})\, \eta_{0:n}(\mathrm{d}x_{0:n}), \quad n \in \mathbb{N},$$

where we introducethe vector notation $x_{0:n} = (x_0, \ldots, x_n) \in \mathsf{X}_{0:n} \coloneqq \mathsf{X}_0 \times \cdots \times \mathsf{X}_n$ for *additive functionals* h_n of the form

$$h_n(x_{0:n}) \coloneqq \sum_{m=0}^{n-1} \tilde{h}_m(x_{m:m+1}), \quad x_{0:n} \in \mathsf{X}_{0:n}. \tag{1.4}$$

In nonlinear filtering problems, such expectations appear in the context of maximum-likelihood parameter estimation, for instance, when computing the *score function* (the gradient of the log-likelihood function) or the *expectation–maximization* (EM) surrogate; see [4, 2, 26, 5, 27]. In [24], the authors propose an efficient *particle-based rapid incremental smoother* (`PARIS`), with linear computational complexity in the number of particles under weak assumptions and limited memory requirements, that samples on-the-fly from the backward dynamics induced by the particle filter. An interesting feature is that it requires two or more backward draws per particle to cope with the degeneracy of the sampled trajectories and remain numerically stable in the long run, with an asymptotic variance that grows only linearly with time.

In this paper, we propose a method to reduce the bias of the `PARIS` estimator of $\eta_{0:n} h_n$. The idea is to mix the `PARIS` with a version of the *particle Gibbs* algorithm with backward sampling [3, 23, 9, 14, 13] by introducing a conditional `PARIS` algorithm. This leads to the *Parisian particle Gibbs* (`PPG`) *algorithm*, from which we derive an upper bound on the bias that decreases inversely proportionally to the number of particles and exponentially fast with the iteration index (under assumptions guaranteeing that the particle Gibbs sampler is uniformly ergodic).

The remainder of the paper is structured as follows. In 2 we discuss the Feynman–Kac model, along with its backward interpretation, and introduce the particle Gibbs sampler. Our presentation is inspired by [14], but differs in that it avoids the use of quotient spaces of [14] and the extension of the distribution to the particle ancestral indices of [3]. In 3, we introduce the `PARIS` algorithm and its conditional version, and show how it can be coupled with the particle Gibbs method with backward sampling, yielding the `PPG` algorithm. In 4, we present the central result of this study, namely, an upper bound on the bias of the `PPG` estimator as a function of the number of particles and the iteration index of the Gibbs algorithm. In addition, we provide an upper bound on the mean-squared error (MSE). In 5, we provide numerical experiment to

Statistica Sinica: Preprint
doi:10.5705/ss.202022.0215

illustrate our results. In 6, we present the most important and original proofs. Finally, the supplementary material contain pseudocode and additional technical proofs, respectively.

Notation. Let $\mathbb{R}_+ \coloneqq [0, \infty)$, $\mathbb{R}_+^* \coloneqq (0, \infty)$, $\mathbb{N} \coloneqq \{0, 1, 2, \ldots\}$, and $\mathbb{N}^* \coloneqq \{1, 2, 3, \ldots\}$ denote the sets of nonnegative and positive real numbers and the same for integers, respectively. We denote by I_N the $N \times N$ identity matrix. For any quantities $\{a_\ell\}_{\ell=m}^n$, we denote vectors as $a_{m:n} \coloneqq (a_m, \ldots, a_n)$, and for any $(m, n) \in \mathbb{N}^2$ such that $m \leq n$, we let $[\![m, n]\!] \coloneqq \{m, m+1, \ldots, n\}$. For a given measurable space $(\mathsf{X}, \mathcal{X})$, where $\mathcal{X}$ is a countably generated σ-field, we denote by $\mathsf{F}(\mathcal{X})$ the set of bounded $\mathcal{X}/\mathcal{B}(\mathbb{R})$-measurable functions on X. For any $h \in \mathsf{F}(\mathcal{X})$, we let $\|h\|_\infty \coloneqq \sup_{x \in \mathsf{X}} |h(x)|$ and $\operatorname{osc}(h) \coloneqq \sup_{(x,x') \in \mathsf{X}^2} |h(x) - h(x')|$ denote the supremum and oscillator norms, respectively, of h. Let $\mathsf{M}(\mathcal{X})$ be the set of σ-finite measures on $(\mathsf{X}, \mathcal{X})$, and $\mathsf{M}_1(\mathcal{X}) \subset \mathsf{M}(\mathcal{X})$ be the probability measures.

Let $(\mathsf{Y}, \mathcal{Y})$ be another measurable space. A possibly unnormalized transition kernel K on $\mathsf{X} \times \mathcal{Y}$ induces two integral operators, one acting on measurable functions, and the other on measures; specifically, for $h \in \mathsf{F}(\mathcal{X} \otimes \mathcal{Y})$ and $\nu \in \mathsf{M}_1(\mathcal{X})$, define the measurable function

$$Kh : \mathsf{X} \ni x \mapsto \int h(x, y)\, K(x, \mathrm{d}y)$$

and the measure

$$\nu K : \mathcal{Y} \ni A \mapsto \int K(x, A)\, \nu(\mathrm{d}x),$$

whenever these quantities are well defined. Now, let $(\mathsf{Z}, \mathcal{Z})$ be a third measurable space and L be another possibly unnormalized transition kernel on $\mathsf{Y} \times \mathcal{Z}$; we then define, with K as above, two different products of K and L, namely,

$$KL : \mathsf{X} \times \mathcal{Z} \ni (x, A) \mapsto \int L(y, A)\, K(x, \mathrm{d}y)$$

and

$$K \otimes L : \mathsf{X} \times (\mathcal{Y} \otimes \mathcal{Z}) \ni (x, A) \mapsto \iint \mathbb{1}_A(y, z)\, K(x, \mathrm{d}y)\, L(y, \mathrm{d}z),$$

Statistica Sinica: Preprint
doi:10.5705/ss.202022.0215

whenever these are well defined. This also defines the $\otimes$ product of a kernel K on $\mathsf{X} \times \mathcal{Y}$ and a measure ν on $\mathcal{X}$, as well as of a kernel L on $\mathsf{Y} \times \mathcal{X}$ and a measure μ on $\mathcal{Y}$, as the measures

$$\nu \otimes K : \mathcal{X} \otimes \mathcal{Y} \ni A \mapsto \iint \mathbb{1}_A(x, y)\, K(x, \mathrm{d}y)\, \nu(\mathrm{d}x),$$
$$L \otimes \mu : \mathcal{X} \otimes \mathcal{Y} \ni A \mapsto \iint \mathbb{1}_A(x, y)\, L(y, \mathrm{d}x)\, \mu(\mathrm{d}y).$$

2. Particle models

In the next sections, we discuss *many-body Feynman–Kac models*, *backward interpretations*, *conditional dual processes*, and the `PARIS` algorithm. Our presentation follows that of [14] closely, but with a different definition of the many-body extensions. We restate (in 1) a duality formula of [14] relating these concepts. This formula provides a foundation for the particle Gibbs sampler described in 2.3 and subsequent developments.

2.1 Many-body Feynman–Kac models

In the following, we assume that all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The distribution flow $\{\eta_m\}_{m \in \mathbb{N}}$ is intractable, in general, but can be approximated by using random samples $\boldsymbol{\xi}_m = (\xi_m^1, \ldots, \xi_n^N)$, for $m \in \mathbb{N}$, of particles, where $N \in \mathbb{N}^*$ is a fixed Monte Carlo sample size and each particle ξ_m^i is an X_m-valued random variable. Such a particle approximation is based on the recursion $\eta_{m+1} = \Phi_m(\eta_m)$, for $m \in \mathbb{N}$, where Φ_m denotes the mapping

$$\Phi_m : \mathsf{M}_1(\mathcal{X}_m) \ni \eta \mapsto \frac{\eta Q_m}{\eta g_m}, \tag{2.1}$$

taking on values in $\mathsf{M}_1(\mathcal{X}_{m+1})$. In order to describe recursively the evolution of the particle population, let $m \in \mathbb{N}$ and assume that the particles $\boldsymbol{\xi}_m$ form a consistent approximation of η_m, in the sense that $\mu(\boldsymbol{\xi}_m)h$, where $\mu(\boldsymbol{\xi}_m) := N^{-1} \sum_{i=1}^N \delta_{\xi_m^i}$ (with δ_x denoting the Dirac measure located at x) is the occupation measure formed by $\boldsymbol{\xi}_m$, serves as a proxy for $\eta_m h$ for any η_m-integrable test function h. (Under general conditions, $\mu(\boldsymbol{\xi}_m)h$ converges in probability to η_m as $N \to \infty$; see [10, 8], and the references therein.) Then, in order to generate an updated particle sample approximating η_{m+1}, new particles $\boldsymbol{\xi}_{m+1} = (\xi_{m+1}^1, \ldots, \xi_{m+1}^N)$ are drawn conditionally independently given $\boldsymbol{\xi}_m$ according to

$$\xi_{m+1}^i \sim \Phi_m(\mu(\boldsymbol{\xi}_m)) = \sum_{\ell=1}^N \frac{g_m(\xi_m^\ell)}{\sum_{\ell'=1}^N g_m(\xi_m^{\ell'})} M_m(\xi_m^\ell, \cdot), \quad i \in [\![1, N]\!].$$

Because this process of particle updating involves sampling from the mixture distribution $\Phi_m(\mu(\boldsymbol{\xi}_m))$, it can be decomposed into two substeps: *selection* and *mutation*. The selection step randomly chooses the ℓth mixture stratum with probability $g_m(\xi_m^\ell)/\sum_{\ell'=1}^N g_m(\xi_m^{\ell'})$, and the mutation draws a new particle ξ_{m+1}^i from the selected stratum $M_m(\xi_m^\ell, \cdot)$. In [14], the term *many-body Feynman–Kac models* is related to the law of process $\{\boldsymbol{\xi}_m\}_{m\in\mathbb{N}}$. For all $m \in \mathbb{N}$, let $\mathbf{X}_m \coloneqq \mathsf{X}_m^N$ and $\boldsymbol{\mathcal{X}}_m \coloneqq \mathcal{X}_m^{\otimes N}$; then, $\{\boldsymbol{\xi}_m\}_{m\in\mathbb{N}}$ is an inhomogeneous Markov chain on $\{\mathbf{X}_m\}_{m\in\mathbb{N}}$, with transition kernels

$$\boldsymbol{M}_m : \mathbf{X}_m \times \boldsymbol{\mathcal{X}}_{m+1} \ni (\boldsymbol{x}_m, A) \mapsto \Phi_m(\mu(\boldsymbol{x}_m))^{\otimes N}(A)$$

and initial distribution $\boldsymbol{\eta}_0 = \eta_0^{\otimes N}$. Now, denote $\mathbf{X}_{0:n} \coloneqq \prod_{m=0}^n \mathbf{X}_m$ and $\boldsymbol{\mathcal{X}}_{0:n} \coloneqq \bigotimes_{m=0}^n \boldsymbol{\mathcal{X}}_m$. (Here, and in the following, we use a bold symbol to stress that a quantity is related to the many-body process.) The *many-body Feynman–Kac path model* refers to the flows $\{\boldsymbol{\gamma}_m\}_{m\in\mathbb{N}}$ and $\{\boldsymbol{\eta}_m\}_{m\in\mathbb{N}}$ of the unnormalized and normalized probability distributions, respectively, on $\{\boldsymbol{\mathcal{X}}_{0:m}\}_{m\in\mathbb{N}}$ generated by (1.1) and (1.2) for the Markov kernels $\{\boldsymbol{M}_m\}_{m\in\mathbb{N}}$, the initial distribution $\boldsymbol{\eta}_0$, the potential functions

$$\boldsymbol{g}_m : \mathbf{X}_m \ni \boldsymbol{x}_m \mapsto \mu(\boldsymbol{x}_m) g_m = \frac{1}{N} \sum_{i=1}^N g_m(x_m^i), \quad m \in \mathbb{N},$$

and the corresponding unnormalized transition kernels

$$\boldsymbol{Q}_m : \mathbf{X}_m \times \boldsymbol{\mathcal{X}}_{m+1} \ni (\boldsymbol{x}_m, A) \mapsto \boldsymbol{g}_m(\boldsymbol{x}_m) \boldsymbol{M}_m(\boldsymbol{x}_m, A), \quad m \in \mathbb{N}.$$

Finally, note that in the previous construction, the Markov property of the many-body Feynman–Kac model relies on the fact that each potential g_m is a function of a single state x_m only, as is the case in the standard Feynman–Kac model framework [10], and that the evolution of the particles follows the model dynamics given in (2.1) (so-called *bootstrap particle filtering*). In order to extend this to more general models (such as models where the potentials are allowed to depend on two consecutive states [22] or, even more generally, where no structure at all is assumed for the unnormalized kernels (1.3) [18]) and particle dynamics (such as the *auxiliary particle filtering* framework introduced in [25]), we need to form a Markovian many-body process with tractable dynamics by furnishing each particle with an importance weight and an index that records the particle's ancestor in the previous generation. However, to avoid this technicality and to allow for a more clear-cut presentation of the methods and theoretical analysis in the coming sections, we stay within the framework of the standard Feynman–Kac models and bootstrap-type particle filters, even though extensions to more general settings may be possible.

2.2 Backward interpretation of Feynman–Kac path flows

Suppose that each kernel Q_n, for $n \in \mathbb{N}$, defined in (1.3), has a transition density q_n with respect to some dominating measure $\lambda_{n+1} \in \mathsf{M}(\mathcal{X}_{n+1})$. Then, for $n \in \mathbb{N}$ and $\eta \in \mathsf{M}_1(\mathcal{X}_n)$, we define the *backward kernel*

$$\overleftarrow{Q}_{n,\eta} : \mathsf{X}_{n+1} \times \mathcal{X}_n \ni (x_{n+1}, A) \mapsto \frac{\int \mathbb{1}_A(x_n) q_n(x_n, x_{n+1})\, \eta(\mathrm{d}x_n)}{\int q_n(x_n', x_{n+1})\, \eta(\mathrm{d}x_n')}. \tag{2.2}$$

Now, for $n \in \mathbb{N}^*$, denoting

$$B_n : \mathsf{X}_n \times \mathcal{X}_{0:n-1} \ni (x_n, A) \mapsto \int \cdots \int \mathbb{1}_A(x_{0:n-1}) \prod_{m=0}^{n-1} \overleftarrow{Q}_{m,\eta_m}(x_{m+1}, \mathrm{d}x_m), \tag{2.3}$$

we may state the following—now classical—*backward decomposition* of the Feynman–Kac path measures, a result that plays a pivotal role in the following.

Proposition 1. *For every $n \in \mathbb{N}^*$, it holds that $\gamma_{0:n} = \gamma_n \otimes B_n$ and $\eta_{0:n} = \eta_n \otimes B_n$.*

Although the decomposition in 1 is well known (see, *e.g.*, [12, 14]), we provide a proof in 6.1 for completeness. Using backward decomposition, we can obtain a particle approximation of a given Feynman–Kac path measure $\eta_{0:n}$ by first sampling, in an initial forward pass, particle clouds $\{\boldsymbol{\xi}_m\}_{m=0}^n$ from $\boldsymbol{\eta}_0 \otimes \boldsymbol{M}_0 \otimes \cdots \otimes \boldsymbol{M}_{n-1}$. Then, in a subsequent backward pass, we sample N conditionally independent paths $\{\tilde{\xi}_{0:n}^i\}_{i=1}^N$ from $\mathbb{B}_n(\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_n, \cdot)$, where

$$\mathbb{B}_n : \mathbf{X}_{0:n} \times \mathcal{X}_{0:n} \ni (\boldsymbol{x}_{0:n}, A) \mapsto \int \cdots \int \mathbb{1}_A(x_{0:n}) \left(\prod_{m=0}^{n-1} \overleftarrow{Q}_{m,\mu(\boldsymbol{x}_m)}(x_{m+1}, \mathrm{d}x_m) \right) \mu(\boldsymbol{x}_n)(\mathrm{d}x_n) \tag{2.4}$$

is a Markov kernel describing the time-reversed dynamics induced by the particle approximations generated in the forward pass. (Here, and in the following, we use blackboard notation to denote kernels related to many-body path spaces.) Finally, $\mu(\{\tilde{\xi}_{0:n}^i\}_{i=1}^N)h$ is returned as an estimator of $\eta_{0:n}h$ for any $\eta_{0:n}$-integrable test function h. This algorithm is referred to as the *forward-filtering backward-simulation* (`FFBSi`) *algorithm* in the literature, and was introduced in [19]; see also [6, 15]. More precisely, given the forward particles $\{\boldsymbol{\xi}_m\}_{m=0}^n$, each path $\tilde{\xi}_{0:n}^i$ is generated by first drawing $\tilde{\xi}_n^i$ uniformly from among the particles $\boldsymbol{\xi}_n$ in the previous generation, and then drawing, recursively,

$$\tilde{\xi}_m^i \sim \overleftarrow{Q}_{m,\mu(\boldsymbol{\xi}_m)}(\tilde{\xi}_{m+1}^i, \cdot) = \sum_{j=1}^N \frac{q_m(\xi_m^j, \tilde{\xi}_{m+1}^i)}{\sum_{\ell=1}^N q_m(\xi_m^\ell, \tilde{\xi}_{m+1}^i)} \delta_{\xi_m^j}; \tag{2.5}$$

that is, given $\tilde{\xi}_{m+1}^i$, $\tilde{\xi}_m^i$ is picked at random from among $\boldsymbol{\xi}_m$ based on weights proportional to $\{q_m(\xi_m^j, \tilde{\xi}_{m+1}^i)\}_{j=1}^N$. Note that in this basic formulation of the `FFBSi` algorithm, each backward-sampling operation (2.5) requires the computation of the normalising constant $\sum_{\ell=1}^N q_m(\xi_m^\ell, \tilde{\xi}_{m+1}^i)$, which implies an overall quadratic complexity of the algorithm. Still, this heavy computational burden can be eased by using an effective accept–reject technique, as discussed in 2.4.

2.3 Conditional dual processes and particle Gibbs

The *dual process* associated with a given Feynman–Kac model (1.1–1.2) and a given trajectory $\{z_n\}_{n\in\mathbb{N}}$, where $z_n \in \mathsf{X}_n$ for every $n \in \mathbb{N}$, is defined as the canonical Markov chain with kernels

$$\boldsymbol{M}_n\langle z_{n+1}\rangle : \mathbf{X}_n \times \boldsymbol{\mathcal{X}}_{n+1} \ni$$
$$(\boldsymbol{x}_n, A) \mapsto \frac{1}{N}\sum_{i=0}^{N-1}\left(\Phi_n(\mu(\boldsymbol{x}_n))^{\otimes i} \otimes \delta_{z_{n+1}} \otimes \Phi_n(\mu(\boldsymbol{x}_n))^{\otimes(N-i-1)}\right)(A), \quad (2.6)$$

for $n \in \mathbb{N}$, and initial distribution

$$\boldsymbol{\eta}_0\langle z_0\rangle \coloneqq \frac{1}{N}\sum_{i=0}^{N-1}\left(\eta_0^{\otimes i} \otimes \delta_{z_0} \otimes \eta_0^{\otimes(N-i-1)}\right). \quad (2.7)$$

As is clear from (2.6–2.7), given $\{z_n\}_{n\in\mathbb{N}}$, a realization $\{\boldsymbol{\xi}_n\}_{n\in\mathbb{N}}$ of the dual process is generated as follows. At time zero, the process is initialized by inserting z_0 at a randomly selected position in the vector $\boldsymbol{\xi}_0$, while drawing independently the remaining elements in the same vector from η_0. After this, the process proceeds in a Markovian manner by, given $\boldsymbol{\xi}_n$, inserting z_{n+1} at a randomly selected position in $\boldsymbol{\xi}_{n+1}$, while drawing independently the remaining elements from $\Phi_n(\mu(\boldsymbol{\xi}_n))$.

In order to describe compactly the law of the conditional dual process, we define the Markov kernel

$$\mathbb{C}_n : \mathsf{X}_{0:n} \times \boldsymbol{\mathcal{X}}_{0:n} \ni (z_{0:n}, A) \mapsto \boldsymbol{\eta}_0\langle z_0\rangle \otimes \boldsymbol{M}_0\langle z_1\rangle \otimes \cdots \otimes \boldsymbol{M}_{n-1}\langle z_n\rangle(A).$$

The following result elegantly combines the underlying model (1.1–1.2), the many-body Feynman–Kac model, the backward decomposition, and the conditional dual process.

Statistica Sinica: Preprint
doi:10.5705/ss.202022.0215

Theorem 1 ([14]). *For all $n \in \mathbb{N}$, it holds that*

$$\mathbb{B}_n \otimes \boldsymbol{\gamma}_{0:n} = \gamma_{0:n} \otimes \mathbb{C}_n. \tag{2.8}$$

In [14], each state $\boldsymbol{\xi}_n$ of the many-body process maps an outcome ω of the sample space Ω onto an *unordered set* of N elements in X_n. However, we have chosen to let each $\boldsymbol{\xi}_n$ take values in the standard *product space* X_n^N, for two reasons. First, the construction of [14] requires sophisticated measure-theoretic arguments to endow such unordered sets with suitable σ-fields and appropriate measures. Second, we see no need to ignore the index order of the particles, as long as the Markovian dynamics (2.6–2.7) of the conditional dual process are symmetrized over the particle cloud. Therefore, in 6.2, we include our own proof of duality (2.8) for completeness. Note that the measure (2.8) on $\mathcal{X}_{0:n} \otimes \boldsymbol{\mathcal{X}}_{0:n}$ is unnormalized, but because the kernels $\mathbb{B}_n$ and $\mathbb{C}_n$ are both Markov, normalizing the identity with $\gamma_{0:n}(\mathsf{X}_{0:n}) = \boldsymbol{\gamma}_{0:n}(\boldsymbol{\mathsf{X}}_{0:n})$ immediately yields

$$\mathbb{B}_n \otimes \boldsymbol{\eta}_{0:n} = \eta_{0:n} \otimes \mathbb{C}_n. \tag{2.9}$$

Because the two sides of (2.9) provide the full conditionals, it is natural to take a data-augmentation approach, and sample the target (2.9) using a two-stage deterministic-scan Gibbs sampler [3, 9]. Specifically, assume we generate a state $(\boldsymbol{\xi}_{0:n}[\ell], \zeta_{0:n}[\ell])$ comprising a dual process with an associated path on the basis of $\ell \in \mathbb{N}$ iterations of the sampler. Then, we generate the next state $(\boldsymbol{\xi}_{0:n}[\ell+1], \zeta_{0:n}[\ell+1])$ in a Markovian fashion by first sampling $\boldsymbol{\xi}_{0:n}[\ell+1] \sim \mathbb{C}_n(\zeta_{0:n}[\ell], \cdot)$, and then sampling $\zeta_{0:n}[\ell+1] \sim \mathbb{B}_n(\boldsymbol{\xi}_{0:n}[\ell+1], \cdot)$. After arbitrary initialization (and the discard of possible burn-in), this procedure produces a Markov trajectory $\{(\boldsymbol{\xi}_{0:n}[\ell], \zeta_{0:n}[\ell])\}_{\ell \in \mathbb{N}}$, and under weak additional technical conditions, this Markov chain admits (2.9) as its unique invariant distribution. In such a case, the Markov chain is ergodic [17, Chapter 5], and the marginal distribution of the conditioning path $\zeta_{0:n}[\ell]$ converges to the target distribution $\eta_{0:n}$. Therefore, for every $h \in \mathsf{F}(\mathcal{X}_{0:n})$, it holds that $\lim_{L \to \infty} L^{-1} \sum_{\ell=1}^{L} h(\zeta_{0:n}[\ell]) =$

$\eta_{0:n}h$, $\mathbb{P}$-a.s.. This algorithm is given in the discussion in [29] of the original particle Gibbs paper [3]; however, the justification of [29], involving an extension of the law targeted by the particle Gibbs sampler to the ancestral indices of particles, differs somewhat from the one presented here.

2.4 The PARIS algorithm

In the following, we assume that we are given a sequence $\{h_n\}_{n\in\mathbb{N}}$ of additive state functionals of type (1.4). Interestingly, as noted in [5, 12], the backward decomposition allows, when applied to additive state functionals, a forward recursion for the expectations $\{\eta_{0:n}h_n\}_{n\in\mathbb{N}}$. More specifically, using the forward decomposition $h_{n+1}(x_{0:n+1}) = h_n(x_{0:n}) + \tilde{h}_n(x_n, x_{n+1})$ and the backward kernel B_{n+1} defined in (2.3), we may write, for $x_{n+1} \in \mathsf{X}_{n+1}$,

$$\begin{aligned} B_{n+1}h_{n+1}(x_{n+1}) &= \int \overleftarrow{Q}_{n,\eta_n}(x_{n+1}, \mathrm{d}x_n) \int \left(h_n(x_{0:n}) + \tilde{h}_n(x_n, x_{n+1})\right) B_n(x_n, \mathrm{d}x_{0:n-1}) \\ &= \overleftarrow{Q}_{n,\eta_n}(B_n h_n + \tilde{h}_n)(x_{n+1}), \end{aligned} \tag{2.10}$$

which, by 1, implies that

$$\eta_{0:n+1}h_{n+1} = \eta_{n+1}\overleftarrow{Q}_{n,\eta_n}(B_n h_n + \tilde{h}_n). \tag{2.11}$$

The marginal flow $\{\eta_n\}_{n\in\mathbb{N}}$ can be expressed recursively using the mappings $\{\Phi_n\}_{n\in\mathbb{N}}$. Thus, (2.11) provides, in principle, a basis for an online computation of $\{\eta_{0:n}h_n\}_{n\in\mathbb{N}}$. Because the marginals are generally intractable, following [12], we plug particle approximations $\mu(\boldsymbol{\xi}_{n+1})$ and $\overleftarrow{Q}_{n,\mu(\boldsymbol{\xi}_n)}$ (see (2.5)) of η_{n+1} and $\overleftarrow{Q}_{n,\mu(\eta_n)}$, respectively, into the recursion (2.11). More precisely, we proceed recursively, and assume that at time n, we have a sample $\{(\xi_n^i, \beta_n^i)\}_{i=1}^N$ of particles with associated statistics, where each statistic β_n^i serves as an approximation of $B_n h_n(\xi_n^i)$. Then evolving the particle cloud according to $\boldsymbol{\xi}_{n+1} \sim \boldsymbol{M}_n(\boldsymbol{\xi}_n, \cdot)$ and updating the statistics using (2.10), with $\overleftarrow{Q}_{n,\eta_n}$ replaced by $\overleftarrow{Q}_{n,\mu(\boldsymbol{\xi}_n)}$, yields the particle-wise recursion

$$\beta_{n+1}^i = \sum_{\ell=1}^N \frac{q_n(\xi_n^\ell, \xi_{n+1}^i)}{\sum_{\ell'=1}^N q_n(\xi_n^{\ell'}, \xi_{n+1}^i)} \left(\beta_n^\ell + \tilde{h}_n(\xi_n^\ell, \xi_{n+1}^i) \right), \quad i \in [\![1, N]\!], \tag{2.12}$$

and, finally, the estimator

$$\mu(\boldsymbol{\beta}_n)(\mathrm{id}) = \frac{1}{N} \sum_{i=1}^N \beta_n^i \tag{2.13}$$

of $\eta_{0:n} h_n$, where we set $\boldsymbol{\beta}_n \coloneqq (\beta_n^1, \ldots, \beta_n^N)$, for $i \in [\![1, N]\!]$, and id is the identity mapping. The procedure is initialized by simply letting $\beta_0^i = 0$, for all $i \in [\![1, N]\!]$. Note that (2.13) provides a particle interpretation of the backward decomposition in 1. This algorithm is a special case of the *forward-filtering backward-smoothing* (`FFBSm`) *algorithm* (see [2, 19, 15, 28]) for additive functionals satisfying (1.4). It allows for online processing of the sequence $\{\eta_{0:n} h_n\}_{n \in \mathbb{N}}$, but also has the appealing property that only the current particles $\boldsymbol{\xi}_n$ and statistics $\boldsymbol{\beta}_n$ need to be stored in memory. However, because each update (2.12) requires a summation of N terms, the scheme has an overall *quadratic* complexity in the number of particles, leading to a computational bottleneck in applications to complex models that require large particle sample sizes N.

To avoid the computational burden of this forward-only implementation of `FFBSm`, the `PARIS` algorithm [24] updates the statistics $\boldsymbol{\beta}_n$ by replacing each sum (2.12) with the Monte Carlo estimate

$$\beta_{n+1}^i = \frac{1}{M} \sum_{j=1}^M \left(\tilde{\beta}_n^{i,j} + \tilde{h}_n(\tilde{\xi}_n^{i,j}, \xi_{n+1}^i) \right), \quad i \in [\![1, N]\!], \tag{2.14}$$

where $\{(\tilde{\xi}_n^{i,j}, \tilde{\beta}_n^{i,j})\}_{j=1}^M$ are drawn randomly from among $\{(\xi_n^i, \beta_n^i)\}_{i=1}^N$ with replacement, by assigning $(\tilde{\xi}_n^{i,j}, \tilde{\beta}_n^{i,j})$ the value of $(\xi_n^\ell, \beta_n^\ell)$ with probability $q_n(\xi_n^\ell, \xi_{n+1}^i) / \sum_{\ell'=1}^N q_n(\xi_n^{\ell'}, \xi_{n+1}^i)$, and the Monte Carlo sample size $M \in \mathbb{N}^*$ is much smaller than N (say, less than five). Formally,

$$\{(\tilde{\xi}_n^{i,j}, \tilde{\beta}_n^{i,j})\}_{j=1}^M \sim \left(\sum_{\ell=1}^N \frac{q_n(\xi_n^\ell, \xi_{n+1}^i)}{\sum_{\ell'=1}^N q_n(\xi_n^{\ell'}, \xi_{n+1}^i)} \delta_{(\xi_n^\ell, \beta_n^\ell)} \right)^{\otimes M}, \quad i \in [\![1, N]\!].$$

The resulting procedure, summarized in 1, allows for online processing with constant memory requirements, because it only needs to store the current particle cloud and the estimated auxiliary statistics at each iteration. Moreover, when the Markov transition densities of the model can be uniformly bounded, that is, there exists, for every $n \in \mathbb{N}$, an upper bound $\bar{\sigma}_n > 0$ such that for all $(x_n, x_{n+1}) \in \mathsf{X}_n \times \mathsf{X}_{n+1}$, $m_n(x_n, x_{n+1}) \leq \bar{\sigma}_n$ (a weak assumption satisfied for most models of interest), then we can generate a sample $(\tilde{\xi}_n^{i,j}, \beta_n^{i,j})$ by drawing, with replacement and until acceptance, candidates $(\tilde{\xi}_n^{i,*}, \tilde{\beta}_n^{i,*})$ from $\{(\xi_n^i, \beta_n^i)\}_{i=1}^N$ based on the normalized particle weights $\{g_n(\xi_n^\ell)/\sum_{\ell'} g_n(\xi_n^{\ell'})\}_{\ell=1}^N$ (obtained as a by-product in the generation of $\boldsymbol{\xi}_{n+1}$), and accepting the same with probability $m_n(\tilde{\xi}_n^{i,*}, \xi_{n+1}^i)/\bar{\sigma}_n$. Because this sampling procedure bypasses the calculation of the normalizing constant $\sum_{\ell'=1}^N q_n(\xi_n^{\ell'}, \xi_{n+1}^i)$ of the targeted categorical distribution, it yields an overall $\mathcal{O}(MN)$ complexity of the algorithm; see [15] for details.

Increasing M improves the accuracy of the algorithm at the cost of additional computational complexity.

As shown in [24], there is a qualitative difference between the cases $M = 1$ and $M \geq 2$, and the latter is required to keep the `PARIS` numerically stable. More precisely, in the latter case, it can be shown that the `PARIS` estimator $\mu(\boldsymbol{\beta}_n)$ satisfies, as N tends to infinity while M is held fixed, a central limit theorem (CLT) at the rate $\sqrt{N}$, with an n-normalized asymptotic variance of order $\mathcal{O}(1 - 1/(M-1))$. As is clear from this bound, using a large M only wastes computational work, and setting M to two or three typically works well in practice.

3. The PPGsampler

We now introduce the `PPG` *algorithm*. For all $n \in \mathbb{N}^*$, let $\mathsf{Y}_n := \mathsf{X}_{0:n} \times \mathbb{R}$ and $\mathcal{Y}_n := \mathcal{X}_{0:n} \otimes \mathcal{B}(\mathbb{R})$. Moreover, let $\mathsf{Y}_0 := \mathsf{X}_0 \times \{0\}$ and $\mathcal{Y}_0 := \mathcal{X}_0 \otimes \{\{0\}, \emptyset\}$. An element of Y_n is always denoted by $y_n = (x_{0:n|n}, b_n)$. The `PPG` sampler includes, as a key ingredient, a *conditional `PARIS` step*, that recursively updates a set of Y_n-valued random variables $\upsilon_n^i := (\xi_{0:n|n}^i, \beta_n^i)$, for $i \in [\![1, N]\!]$. Let $(\boldsymbol{\upsilon}_n)_{n\in\mathbb{N}}$ denote the corresponding many-body process, with each $\boldsymbol{\upsilon}_n :=$

$((\xi^1_{0:n|n}, \beta^1_n), \ldots, (\xi^N_{0:n|n}, \beta^N_n))$ taking on values in the space $\mathbf{Y}_n \coloneqq \mathsf{Y}_n^N$, which we furnish with a σ-field $\boldsymbol{\mathcal{Y}}_n \coloneqq \mathcal{Y}_n^{\otimes N}$. The space $\mathbf{Y}_0$ and the corresponding σ-field $\boldsymbol{\mathcal{Y}}_0$ are defined accordingly. For every $n \in \mathbb{N}$, we write $\boldsymbol{\xi}_{0:n|n} = (\xi^1_{0:n|n}, \ldots, \xi^N_{0:n|n})$ for the collection of paths in $\boldsymbol{\upsilon}_n$, and $\boldsymbol{\xi}_{n|n} = (\xi^1_n, \ldots, \xi^N_n)$ for the collection of end points of the same.

In the following, we let $n \in \mathbb{N}$ be a fixed time horizon, and describe in detail how the `PPG` approximates $\eta_{0:n} h_n$ iteratively. In short, at each iteration ℓ, and given an input conditional path $\zeta_{0:n}[\ell]$, the `PPG` produces a many-body system $\boldsymbol{\upsilon}_n[\ell+1]$ by using a series of conditional `PARIS` operations. Then, an updated path $\zeta_{0:n}[\ell+1]$, which serves as input at the next iteration, is generated by picking one of the paths $\boldsymbol{\xi}_{0:n|n}[\ell+1]$ in $\boldsymbol{\upsilon}_n[\ell+1]$ at random. At each iteration, the produced statistics $\boldsymbol{\beta}_n$ (in $\boldsymbol{\upsilon}_n$) provide an approximation of $\eta_{0:n} h_n$, according to (2.13).

More precisely, given a path $\zeta_{0:n}[\ell]$, the conditional `PARIS` operations are executed as follows. In the initial step, $\boldsymbol{\xi}_{0|0}[\ell+1]$ are drawn from $\boldsymbol{\eta}_0\langle\zeta_0[\ell]\rangle$ defined in (2.7), and $\upsilon^i_0[\ell+1] \leftarrow (\xi^i_{0|0}[\ell+1], 0)$, for all $i \in [\![1, N]\!]$; then, recursively, for $m \in [\![0, n]\!]$, assuming access to $\boldsymbol{\upsilon}_m[\ell+1]$, we

(1) generate an updated particle cloud $\boldsymbol{\xi}_{m+1}[\ell+1] \sim \boldsymbol{M}_m\langle\zeta_{m+1}[\ell]\rangle(\boldsymbol{\xi}_{m|m}[\ell+1], \cdot)$,

(2) pick at random, for each $i \in [\![1, N]\!]$, an ancestor path with associated statistics $(\tilde{\xi}^{i,1}_{0:m}[\ell+1], \tilde{\beta}^{i,1}_m[\ell+1])$ from among $\boldsymbol{\upsilon}_m[\ell+1]$ by drawing

$$(\tilde{\xi}^{i,1}_{0:m}[\ell+1], \tilde{\beta}^{i,1}_m[\ell+1]) \sim \sum_{s=1}^N \frac{q_m(\xi^s_{m|m}[\ell+1], \xi^i_{m+1}[\ell+1])}{\sum_{s'=1}^N q_m(\xi^{s'}_{m|m}[\ell+1], \xi^i_{m+1}[\ell+1])} \delta_{\upsilon^s_m[\ell+1]},$$

(3) pick at random, for each $i \in [\![1, N]\!]$, with replacement, $M-1$ ancestor particles and associated statistics $\{(\tilde{\xi}^{i,j}_m[\ell+1], \tilde{\beta}^{i,j}_m[\ell+1])\}_{j=2}^M$ at random from $\{(\xi^s_{m|m}[\ell+1], \beta^s_m[\ell+$

$1])\}_{s=1}^{N}$ according to

$$\{(\tilde{\xi}_m^{i,j}[\ell+1], \tilde{\beta}_m^{i,j}[\ell+1])\}_{j=2}^{M} \sim \left(\sum_{s=1}^{N} \frac{q_m(\xi_{m|m}^{s}[\ell+1], \xi_{m+1}^{i}[\ell+1])}{\sum_{s'=1}^{N} q_m(\xi_{m|m}^{s'}[\ell+1], \xi_{m+1}^{i}[\ell+1])} \delta_{(\xi_{m|m}^{s}[\ell+1], \beta_m^{s}[\ell+1])} \right)^{\otimes(M-1)},$$

(4) set, for all $i \in [\![1, N]\!]$, $\xi_{0:m+1|m+1}^{i}[\ell+1] \leftarrow (\tilde{\xi}_{0:m}^{i,1}[\ell+1], \xi_{m+1}^{i}[\ell+1])$ and $\upsilon_{m+1}^{i}[\ell+1] \leftarrow (\xi_{0:m+1|m+1}^{i}[\ell+1], \beta_{m+1}^{i}[\ell+1])$, where

$$\beta_{m+1}^{i}[\ell+1] \leftarrow M^{-1} \sum_{j=1}^{M} \left(\tilde{\beta}_m^{i,j}[\ell+1] + \tilde{h}_m(\tilde{\xi}_m^{i,j}[\ell+1], \xi_{m+1}^{i}[\ell+1]) \right).$$

This conditional `PARIS` procedure is summarized in pseudocode in 2 in S2.

In addition to recursively propagating the statistics $\{\boldsymbol{\beta}_m[\ell+1]\}_{m=0}^{n}$ to form the final estimator, this scheme also recursively propagates the trajectories $\{\boldsymbol{\xi}_{0:m|m}[\ell+1]\}_{m=0}^{n}$ used as a pool of candidates for the updated conditional path $\zeta_{0:n}[\ell+1]$. Once we have the set $\boldsymbol{\upsilon}_n[\ell+1]$ of trajectories and associated statistics formed using n recursive conditional `PARIS` updates, we draw an updated path $\zeta_{0:n}[\ell+1]$ from $\mu(\boldsymbol{\xi}_{0:n|n}[\ell+1])$ (*i.e.*, uniformly among the elements of $\boldsymbol{\xi}_{0:n|n}[\ell+1]$). As a result, the updated conditional path $\zeta_{0:n}[\ell+1]$ and the statistics $\boldsymbol{\beta}_n[\ell+1]$ are statistically intertwined conditionally on the conditional dual particle process underpinning the algorithm. The main reason for this is to avoid computational waste. By letting the updated conditional path $\zeta_{0:n}[\ell+1]$ be formed by reusing the backward samples from those generated to form the statistics $\boldsymbol{\beta}_n[\ell+1]$ included in the estimator, our procedure optimizes available computational resources. The full `PPG` is summarized in pseudocode in 3 in S2.

The following Markov kernels play an instrumental role in the following. For a given path $\{z_m\}_{m\in\mathbb{N}}$, the conditional `PARIS` update in 2 defines an inhomogeneous Markov chain on the spaces $\{(\mathbf{Y}_m, \boldsymbol{\mathcal{Y}}_m)\}_{m\in\mathbb{N}}$ with kernels

$$\mathbf{Y}_m \times \boldsymbol{\mathcal{Y}}_{m+1} \ni (\boldsymbol{y}_m, A) \mapsto \int \boldsymbol{M}_m\langle z_{m+1}\rangle(\boldsymbol{x}_{m|m}, \mathrm{d}\boldsymbol{x}_{m+1})\, \boldsymbol{S}_m(\boldsymbol{y}_m, \boldsymbol{x}_{m+1}, A), \quad m \in \mathbb{N},$$

where

$$
\begin{aligned}
&\boldsymbol{S}_m : \mathbf{Y}_m \times \mathbf{X}_{m+1} \times \boldsymbol{\mathcal{Y}}_{m+1} \ni (\boldsymbol{y}_m, \boldsymbol{x}_{m+1}, A) \qquad (3.1)\\
&\mapsto \int \cdots \int \mathbb{1}_A \left(\left\{ \left((\tilde{x}^{i,1}_{0:m}, x^i_{m+1}), \frac{1}{M} \sum_{j=1}^{M} \left(\tilde{b}^{i,j}_m + \tilde{h}_m(\tilde{x}^{i,j}_m, x^i_{m+1}) \right) \right) \right\}_{i=1}^{N} \right) \\
&\quad \times \prod_{i=1}^{N} \left(\sum_{\ell=1}^{N} \frac{q_m(x^\ell_{m|m}, x^i_{m+1})}{\sum_{\ell'=1}^{N} q_m(x^{\ell'}_{m|m}, x^i_{m+1})} \delta_{y^\ell_m}(\mathrm{d}(\tilde{x}^{i,1}_{0:m}, \tilde{b}^{i,1}_m)) \right. \\
&\quad \times \left. \left(\sum_{\ell=1}^{N} \frac{q_m(x^\ell_{m|m}, x^i_{m+1})}{\sum_{\ell'=1}^{N} q_m(x^{\ell'}_{m|m}, x^i_{m+1})} \delta_{(x^\ell_{m|m}, b^\ell_m)} \right)^{\otimes(M-1)} (\mathrm{d}(\tilde{x}^{i,2}_m, \tilde{b}^{i,2}_m, \ldots, \tilde{x}^{i,M}_m, \tilde{b}^{i,M}_m)) \right).
\end{aligned}
$$

In addition, we introduce the joint law

$$
\begin{aligned}
&\mathbb{S}_n : \mathbf{X}_{0:n} \times \boldsymbol{\mathcal{Y}}_n \ni (\boldsymbol{x}_{0:n}, A) \\
&\qquad \mapsto \int \cdots \int \mathbb{1}_A(\boldsymbol{y}_n)\, \boldsymbol{S}_0(\mathbf{J}\boldsymbol{x}_0, \boldsymbol{x}_1, \mathrm{d}\boldsymbol{y}_1) \prod_{m=1}^{n-1} \boldsymbol{S}_m(\boldsymbol{y}_m, \boldsymbol{x}_{m+1}, \mathrm{d}\boldsymbol{y}_{m+1}), \qquad (3.2)
\end{aligned}
$$

where we define $\mathbf{J} := \mathrm{I}_N \otimes (0,1)^\intercal$.

The kernel $\mathbb{S}_n$ can be viewed as a *superincumbent sampling kernel* that describes the distribution of the output $\boldsymbol{\upsilon}_n$ generated by a sequence of PARIS iterations when the many-body process $\{\boldsymbol{\xi}_m\}_{m=0}^{n}$ associated with the underlying particle filter is given. This allows us to describe the PPG alternatively as follows: given $\zeta_{0:n}[\ell]$, draw $\boldsymbol{\xi}_{0:n}[\ell+1] \sim \mathbb{C}_n(\zeta_{0:n}[\ell], \cdot)$; then, draw $\boldsymbol{\upsilon}_n[\ell+1] \sim \mathbb{S}_n(\boldsymbol{\xi}_{0:n}[\ell+1], \cdot)$ and pick a trajectory $\zeta_{0:n}[\ell+1]$ from $\boldsymbol{\xi}_{0:n|n}[\ell+1]$ at random. The following proposition, establishes that the conditional distribution of $\zeta_{0:n}[\ell+1]$ given $\boldsymbol{\xi}_{0:n}[\ell+1]$ coincides, as expected, with the particle-induced backward dynamics $\mathbb{B}_n$.

Proposition 2. *For all $n \in \mathbb{N}^*$, $N \in \mathbb{N}^*$, $\boldsymbol{x}_{0:n} \in \mathbf{X}_{0:n}$, and $h \in \mathsf{F}(\mathcal{X}_{0:n})$,*

$$\int \mathbb{S}_n(\boldsymbol{x}_{0:n}, \mathrm{d}\boldsymbol{y}_n)\, \mu(\boldsymbol{x}_{0:n|n}) h = \mathbb{B}_n h(\boldsymbol{x}_{0:n}).$$

Finally, we define the Markov kernel induced by the PPG, as well as the extended probability distribution targeted by the same. For this purpose, we introduce the extended measurable space

$(\mathsf{E}_n, \boldsymbol{\mathcal{E}}_n)$, with

$$\mathsf{E}_n \coloneqq \mathsf{Y}_n \times \mathsf{X}_{0:n}, \quad \boldsymbol{\mathcal{E}}_n \coloneqq \boldsymbol{\mathcal{Y}}_n \otimes \mathcal{X}_{0:n}.$$

The PPG described in 3 defines a Markov chain on $(\mathsf{E}_n, \boldsymbol{\mathcal{E}}_n)$ with the Markov transition kernel

$$\mathbb{K}_n : \mathsf{E}_n \times \boldsymbol{\mathcal{E}}_n \ni (\boldsymbol{y}_n, z_{0:n}, A) \\ \mapsto \iiint \mathbb{1}_A(\tilde{\boldsymbol{y}}_n, \tilde{z}_{0:n}) \, \mathbb{C}_n(z_{0:n}, \mathrm{d}\tilde{\boldsymbol{x}}_{0:n}) \, \mathbb{S}_n(\tilde{\boldsymbol{x}}_{0:n}, \mathrm{d}\tilde{\boldsymbol{y}}_n) \, \mu(\tilde{\boldsymbol{x}}_{0:n|n})(\mathrm{d}\tilde{z}_{0:n}).$$

Note that the values of $\mathbb{K}_n$ defined above do not depend on $\boldsymbol{y}_n$, but only on $(z_{0:n}, A)$. For any given initial distribution $\xi \in \mathsf{M}_1(\mathcal{X}_{0:n})$, let $\mathbb{P}_\xi$ be the distribution of the canonical Markov chain induced by the kernel $\mathbb{K}_n$ and the initial distribution ξ. In the special case where $\xi = \delta_{z_{0:n}}$, for some given path $z_{0:n} \in \mathsf{X}_{0:n}$, we use the short-hand notation $\mathbb{P}_{\delta_{z_{0:n}}} = \mathbb{P}_{z_{0:n}}$. In addition, denote by

$$K_n : \mathsf{X}_{0:n} \times \mathcal{X}_{0:n} \ni (z_{0:n}, A) \mapsto \iiint \mathbb{1}_A(\tilde{z}_{0:n}) \, \mathbb{C}_n(z_{0:n}, \mathrm{d}\tilde{\boldsymbol{x}}_{0:n}) \, \mathbb{S}_n(\tilde{\boldsymbol{x}}_{0:n}, \mathrm{d}\tilde{\boldsymbol{y}}_n) \, \mu(\tilde{\boldsymbol{x}}_{0:n|n})(\mathrm{d}\tilde{z}_{0:n})$$

the path-marginalized version of $\mathbb{K}_n$. By 2, it holds that $K_n = \mathbb{C}_n \mathbb{B}_n$, which shows that K_n coincides with the Markov transition kernel of the backward-sampling-based particle Gibbs sampler discussed in 2.3.

Finally, in order to prepare for the statement of our theoretical results on the PPG, we need to introduce the following Feynman–Kac path model *with a frozen path*. More precisely, for a given path $z_{0:n} \in \mathsf{X}_{0:n}$, define, for every $m \in [\![0, n-1]\!]$, the unnormalized kernel

$$Q_m\langle z_{m+1}\rangle : \mathsf{X}_m \times \mathcal{X}_{m+1} \ni (x_m, A) \mapsto \left(1 - \frac{1}{N}\right) Q_m(x_m, A) + \frac{1}{N} g_m(x_m) \, \delta_{z_{m+1}}(A)$$

and the initial distribution $\eta_0\langle z_0\rangle : \mathcal{X}_0 \ni A \mapsto (1 - 1/N)\eta_0(A) + \delta_{z_0}(A)/N$. Given these quantities, define, for $m \in [\![0, n]\!]$, $\gamma_m\langle z_{0:m}\rangle \coloneqq \eta_0\langle z_0\rangle Q_0\langle z_1\rangle \cdots Q_{m-1}\langle z_m\rangle$, and its normalized counterpart $\eta_m\langle z_{0:m}\rangle \coloneqq \gamma_m\langle z_{0:m}\rangle / \gamma_m\langle z_{0:m}\rangle \mathbb{1}_{\mathsf{X}_{0:m}}$. Finally, we introduce, for $m \in [\![0, n]\!]$, the kernels

$$B_m\langle z_{0:m-1}\rangle : \mathsf{X}_m \times \mathcal{X}_{0:m-1} \ni (x_m, A) \mapsto \int \cdots \int \mathbb{1}_A(x_{0:n-1}) \prod_{m=0}^{n-1} \overleftarrow{Q}_{m,\eta_m\langle z_{0:m}\rangle}(x_{m+1}, \mathrm{d}x_m)$$

and the path model $\eta_{0:m}\langle z_{0:m}\rangle \coloneqq B_m\langle z_{0:m-1}\rangle \otimes \eta_m\langle z_{0:m}\rangle$.

4. Main results

4.1 Theoretical results

In this section, we establish our main result, namely, the exponentially contracting bias bound stated in 2. This result is proved under the following strong mixing assumptions, which are standard in the literature (see [10, 16, 11, 14]):

A 4.1 (strong mixing)**.** For every $n \in \mathbb{N}$, there exist $\underline{\tau}_n$, $\bar{\tau}_n$, $\underline{\sigma}_n$, and $\bar{\sigma}_n$ in $\mathbb{R}_+^*$ such that

(i) $\underline{\tau}_n \leq g_n(x_n) \leq \bar{\tau}_n$ for every $x_n \in \mathsf{X}_n$,

(ii) $\underline{\sigma}_n \leq m_n(x_n, x_{n+1}) \leq \bar{\sigma}_n$ for every $(x_n, x_{n+1}) \in \mathsf{X}_{n:n+1}$.

Under 4.1, define, for every $n \in \mathbb{N}$,

$$\rho_n \coloneqq \max_{m \in [\![0,n]\!]} \frac{\bar{\tau}_m \bar{\sigma}_m}{\underline{\tau}_m \underline{\sigma}_m} \tag{4.1}$$

and, for every $n \in \mathbb{N}$ and $N \in \mathbb{N}^*$ such that $N > N_n \coloneqq (1 + 5\rho_n^2 n/2) \vee 2n(1 + 2\rho_n^2)$,

$$\kappa_{N,n} \coloneqq 1 - \frac{1 - (1 + 5n\rho_n^2/2)/N}{1 + 4n(1 + 2\rho_n^2)/N}. \tag{4.2}$$

Note that $\kappa_{N,n} \in (0, 1)$, for all N and n, as above.

Theorem 2. *Assume 4.1. Then, for every $n \in \mathbb{N}$, there exist c_n^{bias}, c_n^{mse}, and c_n^{cov} in $\mathbb{R}_+^*$ such*

Statistica Sinica: Preprint
doi:10.5705/ss.202022.0215

that for every $M \in \mathbb{N}^*$, $\xi \in \mathsf{M}_1(\mathcal{X}_{0:n})$, $\ell \in \mathbb{N}^*$, $s \in \mathbb{N}^*$, and $N \in \mathbb{N}^*$ such that $N > N_n$,

$$|\mathbb{E}_\xi\left[\mu(\boldsymbol{\beta}_n[\ell])(\mathrm{id})\right] - \eta_{0:n}h_n| \leq \mathsf{c}_n^{bias} \left(\sum_{m=0}^{n-1} \|\tilde{h}_m\|_\infty\right) N^{-1}\kappa_{N,n}^{\ell}, \tag{4.3}$$

$$\mathbb{E}_\xi\left[(\mu(\boldsymbol{\beta}_n[\ell])(\mathrm{id}) - \eta_{0:n}h_n)^2\right] \leq \mathsf{c}_n^{mse} \left(\sum_{m=0}^{n-1} \|\tilde{h}_m\|_\infty\right)^2 N^{-1}, \tag{4.4}$$

$$\begin{aligned}|\mathbb{E}_\xi\left[(\mu(\boldsymbol{\beta}_n[\ell])(\mathrm{id}) - \eta_{0:n}h_n)\,(\mu(\boldsymbol{\beta}_n[\ell+s])(\mathrm{id}) \; - \eta_{0:n}h_n)\right]| \\ \leq \mathsf{c}_n^{cov} \left(\sum_{m=0}^{n-1} \|\tilde{h}_m\|_\infty\right)^2 N^{-3/2}\kappa_{N,n}^{s}.\end{aligned} \tag{4.5}$$

The constants c_n^{bias}, c_n^{mse}, and c_n^{cov} are given explicitly in the proof. Because we focus on the dependence on N and the index ℓ, we make no attempt to optimize the dependence of these constants on n in our proofs; nevertheless, we believe that it is possible to prove, under the stated assumptions, that this dependence is linear. The proof of the bound in 2 is based on four key ingredients. The first is the following unbiasedness property of the `PARIS` under the many-body Feynman–Kac path model.

Theorem 3. *For every $n \in \mathbb{N}$, $N \in \mathbb{N}^*$, and $\ell \in \mathbb{N}^*$,*

$$\mathbb{E}_{\eta_{0:n}}\left[\mu(\boldsymbol{\beta}_n[\ell])(\mathrm{id})\right] = \int \eta_{0:n}\mathbb{C}_n\mathbb{S}_n(\mathrm{d}\boldsymbol{b}_n)\,\mu(\boldsymbol{b}_n)(\mathrm{id}) = \int \boldsymbol{\eta}_{0:n}\mathbb{S}_n(\mathrm{d}\boldsymbol{b}_n)\,\mu(\boldsymbol{b}_n)(\mathrm{id}) = \eta_{0:n}h_n.$$

The proof of 3 is found in 6.3. The second is the uniform geometric ergodicity of the particle Gibbs with backward sampling established in [13].

Theorem 4. *Assume 4.1. Then, for every $n \in \mathbb{N}$, $(\mu, \nu) \in \mathsf{M}_1(\mathcal{X}_{0:n})^2$, $\ell \in \mathbb{N}^*$, and $N \in \mathbb{N}^*$ such that $N > N_n$, $\|\mu K_n^\ell - \nu K_n^\ell\|_{\mathrm{TV}} \leq \kappa_{N,n} n N^\ell$, where $\kappa_{N,n}$ is defined in (4.2).*

As a third ingredient, we require the following uniform exponential concentration inequality of the conditional `PARIS` with respect to the frozen-path Feynman–Kac model defined in the previous section.

Theorem 5. *For every $n \in \mathbb{N}$, there exist $\mathsf{c}_n > 0$ and $\mathsf{d}_n > 0$ such that for every $M \in \mathbb{N}^*$, $z_{0:n} \in \mathsf{X}_{0:n}$, $N \in \mathbb{N}^*$, and $\varepsilon > 0$,*

$$\int \mathbb{C}_n \mathbb{S}_n(z_{0:n}, \mathrm{d}\boldsymbol{b}_n) \mathbb{1}\left\{|\mu(\boldsymbol{b}_n)(\mathrm{id}) - \eta_{0:n}\langle z_{0:n}\rangle h_n| \geq \varepsilon\right\} \leq \mathsf{c}_n \exp\left(-\frac{\mathsf{d}_n N \varepsilon^2}{2(\sum_{m=0}^{n-1} \|\tilde{h}_m\|_\infty)^2}\right).$$

The proof of 5 is found in S3.2, and is based on arguments similar to those used in the proofs of [24, Theorem 1] and [15, Theorem 5] in the framework of the conditional dual process. 5 implies, in turn, the following conditional variance bound.

Proposition 3. *For every $n \in \mathbb{N}$, $M \in \mathbb{N}^*$, $z_{0:n} \in \mathsf{X}_{0:n}$, and $N \in \mathbb{N}^*$,*

$$\int \mathbb{C}_n \mathbb{S}_n(z_{0:n}, \mathrm{d}\boldsymbol{b}_n) \left|\mu(\boldsymbol{b}_n)(\mathrm{id}) - \eta_{0:n}\langle z_{0:n}\rangle h_n\right|^2 \leq \frac{\mathsf{c}_n}{\mathsf{d}_n}\left(\sum_{m=0}^{n-1} \|\tilde{h}_m\|_\infty\right)^2 N^{-1}.$$

Using 3, we deduce, in turn, the following bias bound, the proof is postponed to S3.4.

Proposition 4. *For every $n \in \mathbb{N}$, there exists $\bar{\mathsf{c}}_n^{bias} > 0$ such that for every $M \in \mathbb{N}^*$, $z_{0:n} \in \mathsf{X}_{0:n}$, and $N \in \mathbb{N}^*$,*

$$\left|\int \mathbb{C}_n \mathbb{S}_n(z_{0:n}, \mathrm{d}\boldsymbol{b}_n)\, \mu(\boldsymbol{b}_n)(\mathrm{id}) - \eta_{0:n}\langle z_{0:n}\rangle h_n\right| \leq \bar{\mathsf{c}}_n^{bias}\left(\sum_{m=0}^{n-1} \|\tilde{h}_m\|_\infty\right) N^{-1}.$$

A fourth and last ingredient in the proof of 2 is the following bound on the discrepancy between the additive expectations under the original and frozen-path Feynman–Kac models. This bound is established using novel results in [18]. More precisely, because for every $m \in \mathbb{N}$, $(x, z) \in \mathsf{X}_m^2$, $N \in \mathbb{N}^*$, and $h \in \mathsf{F}(\mathcal{X}_{m+1})$, using 4.1,

$$|Q_m\langle z\rangle h(x) - Q_m h(x)| \leq \frac{1}{N}\|g_m\|_\infty \|h\|_\infty \leq \frac{1}{N}\bar{\tau}_m \|h\|_\infty,$$

applying [18, Theorem 4.3] yields the following.

Proposition 5. *Assume 4.1. Then, there exists $\mathsf{c} > 0$ such that for every $n \in \mathbb{N}$, $N \in \mathbb{N}$, and $z_{0:n} \in \mathsf{X}_{0:n}$,*

$$|\eta_{0:n}\langle z_{0:n}\rangle h_n - \eta_{0:n} h_n| \leq \mathsf{c} N^{-1} \sum_{m=0}^{n-1} \|\tilde{h}_m\|_\infty.$$

In addition, we assume $\sup_{n\in\mathbb{N}} \|\tilde{h}_n\|_\infty < \infty$ yields an $\mathcal{O}(n/N)$ bound in 5.

Finally, by combining these ingredients, we are now ready to present a proof of 2.

Proof of 2. Write, using the tower property,

$$\mathbb{E}_\xi \left[\mu(\boldsymbol{\beta}_n\,[\ell])(\mathrm{id})\right] = \mathbb{E}_\xi \left[\mathbb{E}_{\zeta_{0:n}[\ell]} \left[\mu(\boldsymbol{\beta}_n\,[0])(\mathrm{id})\right]\right] = \int \xi K_n^\ell \mathbb{C}_n \mathbb{S}_n(\mathrm{d}\boldsymbol{b}_n)\, \mu(\boldsymbol{b}_n)(\mathrm{id}).$$

Thus, by the unbiasedness property in 3,

$$\begin{aligned}
&\left|\mathbb{E}_\xi \left[\mu(\boldsymbol{\beta}_n\,[\ell])(\mathrm{id})\right] - \eta_{0:n} h_n\right| \\
&\qquad = \left|\int \xi K_n^\ell \mathbb{C}_n \mathbb{S}_n(\mathrm{d}\boldsymbol{b}_n)\, \mu(\boldsymbol{b}_n)(\mathrm{id}) - \int \eta_{0:n} \mathbb{C}_n \mathbb{S}_n(\mathrm{d}\boldsymbol{b}_n)\, \mu(\boldsymbol{b}_n)(\mathrm{id})\right| \\
&\qquad \le \left\|\xi K_n^\ell - \eta_{0:n}\right\|_{\mathrm{TV}} \operatorname{osc}\left(\int \mathbb{C}_n \mathbb{S}_n(\cdot, \mathrm{d}\boldsymbol{b}_n)\, \mu(\boldsymbol{b}_n)(\mathrm{id})\right),
\end{aligned}$$

where, by 4, $\|\xi K_n^\ell - \eta_{0:n}\|_{\mathrm{TV}} \le \kappa_{N,n}^\ell$. Moreover, to derive an upper bound on the oscillation, we consider the decomposition

$$\begin{aligned}
&\operatorname{osc}\left(\int \mathbb{C}_n \mathbb{S}_n(\cdot, \mathrm{d}\boldsymbol{b}_n)\, \mu(\boldsymbol{b}_n)(\mathrm{id})\right) \\
&\qquad \le 2\left(\left\|\int \mathbb{C}_n \mathbb{S}_n(\cdot, \mathrm{d}\boldsymbol{b}_n)\, \mu(\boldsymbol{b}_n)(\mathrm{id}) - \eta_{0:n}\langle\cdot\rangle h_n\right\|_\infty + \|\eta_{0:n}\langle\cdot\rangle h_n - \eta_{0:n} h_n\|_\infty\right),
\end{aligned}$$

where the two terms on the right-hand side can be bounded using 5 and 4, respectively. This completes the proof of (4.3). We now consider the proof of (4.4). Writing

$$\mathbb{E}_\xi \left[\left(\mu(\boldsymbol{\beta}_n[\ell])(\mathrm{id}) - \eta_{0:n} h_n\right)^2\right] = \int \xi K_n^\ell(\mathrm{d}z_{0:n})\, \mathbb{C}_n \mathbb{S}_n(z_{0:n}, \mathrm{d}\boldsymbol{b}_n)\, \left(\mu(\boldsymbol{b}_n)(\mathrm{id}) - \eta_{0:n} h_n\right)^2,$$

we establish (4.4) using 3 and 5. Finally, WE consider (4.5). Using the Markov property, we obtain

$$\begin{aligned}
&\mathbb{E}_\xi \left[\left(\mu(\boldsymbol{\beta}_n[\ell])(\mathrm{id}) - \eta_{0:n} h_n\right)\left(\mu(\boldsymbol{\beta}_n[\ell+s])(\mathrm{id}) \;- \eta_{0:n} h_n\right)\right] \\
&\qquad = \mathbb{E}_\xi \left[\left(\mu(\boldsymbol{\beta}_n[\ell])(\mathrm{id}) - \eta_{0:n} h_n\right)\left(\mathbb{E}_{\zeta_{0:n}[\ell]}[\mu(\boldsymbol{\beta}_n[s])(\mathrm{id})] \;- \eta_{0:n} h_n\right)\right],
\end{aligned}$$

from which we may deduce (4.5) using (4.3) and (4.4). □

4.2 The roll-out PPG estimator

In light of the previous results, it is natural to consider an estimator formed by an average across successive conditional PPG estimators $\{\mu(\boldsymbol{\beta}_n[\ell])\}_{\ell\in\mathbb{N}}$. To mitigate the bias, we remove a "burn-in" period, with length k_0 chosen proportionally to the mixing time of the particle Gibbs chain $\{\zeta_{0:n}[\ell]\}_{\ell\in\mathbb{N}^*}$. This yields the estimator

$$\Pi_{(k_0,k),N}(h_n) = (k-k_0)^{-1}\sum_{\ell=k_0+1}^{k}\mu(\boldsymbol{\beta}_n[\ell])(\mathrm{id}). \tag{4.6}$$

The total number of particles underlying this estimator is $C = (N-1)k$. We denote by $\upsilon = (k-k_0)/k$ the ratio of the number of particles used in the estimator to the total number of sampled particles.

As a final main result, we provide bounds on the bias and the MSE of the estimator (4.6). The proof is postponed to S3.5.

Theorem 6. *Assume 4.1. Then, for every* $n\in\mathbb{N}$, $M\in\mathbb{N}^*$, $\xi\in\mathsf{M}_1(\mathcal{X}_{0:n})$, $\ell\in\mathbb{N}^*$, $s\in\mathbb{N}^*$, *and* $N\in\mathbb{N}^*$ *such that* $N>N_n$,

$$\left|\mathbb{E}_\xi[\Pi_{(k_0,k),N}(h_n)] - \eta_{0:n}h_n\right| \le \mathsf{c}_n^{bias}\left(\sum_{m=0}^{n-1}\|\tilde{h}_m\|_\infty\right)\frac{\kappa_{N,n}^{k_0}}{N(k-k_0)(1-\kappa_{N,n})}, \tag{4.7}$$

$$\mathbb{E}_\xi\left[\left(\Pi_{(k_0,k),N}(h_n)-\eta_{0:n}h_n\right)^2\right] \le \left(\sum_{m=0}^{n-1}\|\tilde{h}_m\|_\infty\right)^2\frac{\mathsf{c}_n^{mse}+2\mathsf{c}_n^{cov}N^{-1/2}(1-\kappa_{N,n})^{-1}}{N(k-k_0)} \tag{4.8}$$

Setting the burn-in k_0 in the roll-out estimator is nontrivial. However, because the estimator converges for *any* choice of k_0, including the trivial choice $k_0=1$, we can view this algorithmic parameter as an opportunity for the user to optimize the implementation of the algorithm. For given (N,k), the choice of k_0 involves a classical trade-off between bias and variance; indeed, for fixed (N,k), the bias upper bound (4.7) decreases with k_0 proportionally to $\kappa_{N,n}^{k_0}/(k-k_0)$ whereas the MSE upper bound (4.8) increases with k_0 proportionally to $1/(k-k_0)$.

These bounds suggest that we should take $k_0 = \lceil k(1 - \ell^{-1}) \rceil$ if we are willing to bound the MSE increase of the roll-out estimator by a factor ℓ with respect to the `PARIS`. However, the bias reduction is not easily quantified, because it depends mainly on the mixing rate $\kappa_{N,n}$ of the `PPG` chain, and we only have access to upper bounds on this rate that are, in general, too conservative.

5. Numerical results

In this section, we evaluate numerically the proposed `PPG` sampler in the context of general state-space HMMs. Given measurable spaces $(\mathsf{X}, \mathcal{X})$ and $(\mathsf{Z}, \mathcal{Z})$, an HMM is a bivariate (possibly inhomogeneous) Markov chain $\{(X_m, Z_m)\}_{m \in \mathbb{N}}$ taking values in the product space $(\mathsf{X} \times \mathsf{Z}, \mathcal{X} \otimes \mathcal{Z})$. In such a model, the process $\{X_n\}_{n \in \mathbb{N}}$, referred to as the *state sequence*, is assumed to be itself a (possibly inhomogeneous) Markov chain, specified by some initial distribution χ and some sequence $\{M_n\}_{n \in \mathbb{N}}$ of Markov kernels. The state sequence is latent and only partially observed through the *observation process* $\{Z_m\}_{m \in \mathbb{N}}$. Conditionally on the state sequence, the observations are assumed to be independent; furthermore, the conditional marginal distribution of each Z_m is assumed to depend only on the corresponding state X_m and to have a density $g_m(X_m, \cdot)$ with respect to some dominating measure. HMMs are used in numerous scientific and engineering disciplines; see [1, 7, 8]. Inference in HMMs typically involves computing conditional distributions of unobserved states, given observations. Of particular interest are the sequence of *filter distributions*, where the filter at time $m \in \mathbb{N}$, denoted as η_m, is defined as the conditional distribution of X_m, given $Z_{0:m} \coloneqq (Z_0, \ldots, Z_m)$, and the *joint-smoothing distributions*, where the joint-smoothing distribution at time m, denoted as $\eta_{0:m}$, is defined as the joint conditional distribution of the states $X_{0:m} = (X_0, \ldots, X_m)$, given the observations $Z_{0:m}$. Consequently, η_m is the marginal of $\eta_{0:m}$ with respect to the last state X_m. Given a sequence $\{z_m\}_{m \in \mathbb{N}}$ of fixed observations, $\{\eta_{0:m}\}_{m\in\mathbb{N}}$ forms a Feynman–Kac model (see 1), with Markov kernels $\{M_m\}_{m\in\mathbb{N}}$ and potential functions $g_m := g(\cdot, z_m)$, for $m \in \mathbb{N}$, on X.

We now evaluate the proposed algorithm numerically for two HMMs: (i) a linear Gaussian state-space model (for which the filter and the joint-smoothing distribution flows are available in a closed form), and (ii) the stochastic volatility model proposed in [20]. The `PPG` algorithm used in this section is given in 3 (in S2).

Linear Gaussian state-space model (LGSSM). We first consider an LGSSM

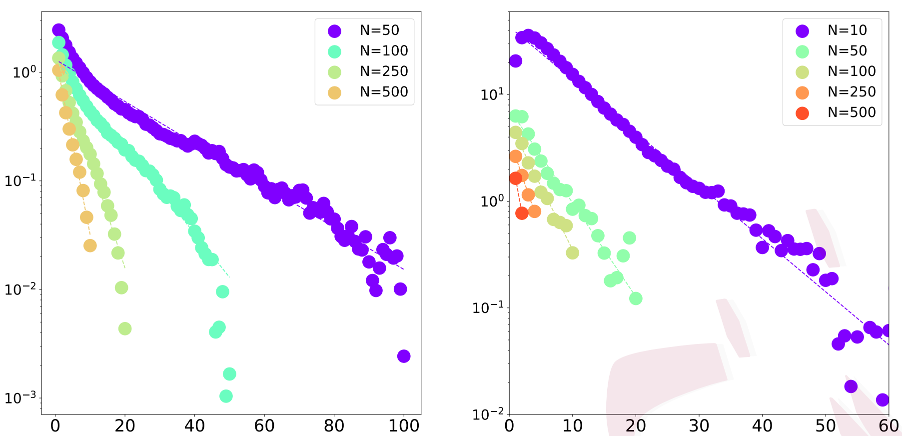
$$X_{m+1} = AX_m + Q\epsilon_{m+1}, \quad Z_m = BX_m + R\zeta_m, \quad m \in \mathbb{N}, \tag{5.1}$$

where $\{\epsilon_m\}_{m\in\mathbb{N}^*}$ and $\{\zeta_m\}_{m\in\mathbb{N}}$ are sequences of independent standard normally distributed random variables. The matrices A, Q, B, and R are assumed to be known 5×5 matrices (see section S1.1 for the precise values). In this framework, we aim to compute the expectation of the *one-lag state covariance* $h_n(x_{0:n}) := \sum_{m=0}^{n-1} x_m x_{m+1}^\intercal$ under the joint-smoothing distribution $\eta_{0:n}$ for observations generated by simulation under the given parameters with $n = 10^3$. In the LGSSM case, the *disturbance smoother* (see [7, Algorithm 5.2.15]) provides the exact values of $\eta_{0:n}h_n$, which allows us to assess numerically the bias of the `PARIS` and `PPG` estimators.

In this setting, we calculate the bias for batch sizes $N \in \{10, 25, 50, 100, 500\}$ and an increasing number k of iterations by averaging the `PPG` estimator over 10^4 independent runs. 1a shows the bias of the `PPG` estimates of the first diagonal entry of the one-lag covariance. For each batch size N, we estimate and display the regression function $k \mapsto \mathrm{e}^{ak+b}$ to illustrate the exponential decrease of the `PPG` bias, which is consistent with 2.

2a displays, for a given budget $C = 5 \times 10^3$, the bias of the estimates of $\eta_{0:n}h_n$ using the `PARIS` and the `PPG` for different batch sizes N and different numbers $k = C/N$ of iterations and burn-in periods $k_0 = \lfloor k/2 \rfloor$. The red line corresponds to zero (no bias), and the empirical

Statistica Sinica: Preprint
doi:10.5705/ss.202022.0215

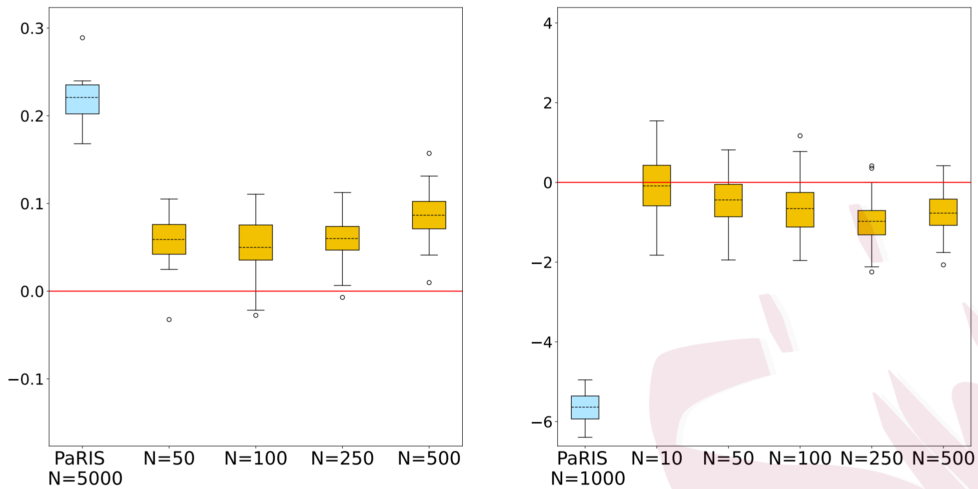


(a) LGSMM (b) StoVol

Figure 1: Output of the PPG roll-out estimator for the LGSSM (left panel) and the StoVol model (right panel). The curves describe the evolution of the bias with increasing k for different batch sizes N.

means are given by black-dashed lines. An extended comparison comprising different choices of k_0 and different budgets C is provided in S1. In order to estimate the bias for each algorithmic configuration, we average 10^3 independent replications of the corresponding estimator. Moreover, to assess the precision of the resulting bias estimator, we repeat this procedure 10^2 times, and present the bias estimates in a box plot. This enables us to form an idea of whether the PPG provides a statistically significant improvement in terms of bias. In this example, whatever the choice of the batch size is, the PPG bias is significantly reduced compared with the bias of the PARIS estimator. We further observe that a larger k leads to smaller bias.

Statistica Sinica: Preprint
doi:10.5705/ss.202022.0215



(a) LGSSM (b) StoVol

Figure 2: `PARIS` and `PPG` bias dispersions for the LGSSM and StoVol model as a function of the mini-batch size N for fixed computational budgets $C = Nk$ of 5×10^3 (LGSSM) and 10^3 (StoVol model) and with $k_0 = \lfloor 2^{-1} k \rfloor$ burn-in steps.

Stochastic volatility (StoVol). As a second example, consider the stochastic volatility model

$$X_{m+1} = \phi X_m + \sigma_\epsilon \epsilon_{m+1}, \quad Z_m = \beta \exp(X_m/2)\zeta_m, \quad m \in \mathbb{N}, \tag{5.2}$$

where $\{\epsilon_m\}_{m \in \mathbb{N}^*}$ and $\{\zeta_m\}_{m \in \mathbb{N}}$ are as in the previous example, and the model parameters ϕ, β, and σ_ϵ are set to 0.975, 0.63, and 0.16, respectively. The reference value is calculated by running the `PARIS` with 5×10^4 particles. In this setting, we repeated the experiments of the previous example for the same additive functional and number $n = 10^3$ of observations, produced by simulation under the parameters above. The computational budget was set to $C = 10^3$. As in the LGSSM example, the bias decay with respect to the iteration index k is displayed in 1b, and the comparison with the PARIS is shown in 2b. The comments from the previous example apply to this StoVol model context as well. More in-depth numerical assessments of the proposed PPG estimator are found in S1.2. In particular, in S1.2, we compare our estimator with the Rhee–Glynn-type estimator with ancestor sampling proposed by [21], showing that the variance of the latter is significantly larger than that of the PPG for a given computational effort.

6. Proofs

6.1 Proof of 1

Using the identity

$$\eta_0 Q_0 \cdots Q_{n-1} \mathbb{1}_{\mathsf{X}_n} = \prod_{m=0}^{n-1} \eta_m Q_m \mathbb{1}_{\mathsf{X}_{m+1}}$$

and that each kernel Q_m has a transition density, write, for $h \in \mathsf{F}(\mathcal{X}_{0:n})$,

$$\begin{aligned}
\eta_{0:n} h &= \int \cdots \int h(x_{0:n}) \, \eta_0(\mathrm{d}x_0) \prod_{m=0}^{n-1} \left(\frac{\eta_m[q_m(\cdot, x_{m+1})] \, \lambda_{m+1}(\mathrm{d}x_{m+1})}{\eta_m Q_m \mathbb{1}_{\mathsf{X}_{m+1}}} \right) \left(\frac{q_m(x_m, x_{m+1})}{\eta_m[q_m(\cdot, x_{m+1})]} \right) \\
&= \int \cdots \int h(x_{0:n}) \, \eta_n(\mathrm{d}x_n) \prod_{m=0}^{n-1} \frac{\eta_m(\mathrm{d}x_m) \, q_m(x_m, x_{m+1})}{\eta_m[q_m(\cdot, x_{m+1})]} \qquad (6.1) \\
&= \left(\overleftarrow{Q}_{0,\eta_0} \otimes \cdots \otimes \overleftarrow{Q}_{n-1,\eta_{n-1}} \otimes \eta_n \right) h,
\end{aligned}$$

which establishes the proof.

6.2 Proof of 1

Lemma 1. *For all $n \in \mathbb{N}$, $\boldsymbol{x}_n \in \mathbf{X}_n$, and $h \in \mathsf{F}(\boldsymbol{\mathcal{X}}_{n+1} \otimes \mathcal{X}_{n+1})$,*

$$\begin{aligned}
&\iint h(\boldsymbol{x}_{n+1}, z_{n+1}) \, \boldsymbol{Q}_n(\boldsymbol{x}_n, \mathrm{d}\boldsymbol{x}_{n+1}) \, \mu(\boldsymbol{x}_{n+1})(\mathrm{d}z_{n+1}) \\
&\qquad = \iint h(\boldsymbol{x}_{n+1}, z_{n+1}) \, \mu(\boldsymbol{x}_n) Q_n(\mathrm{d}z_{n+1}) \, \boldsymbol{M}_n\langle z_{n+1}\rangle(\boldsymbol{x}_n, \mathrm{d}\boldsymbol{x}_{n+1}). \qquad (6.2)
\end{aligned}$$

Statistica Sinica: Preprint
doi:10.5705/ss.202022.0215

In addition, for all $h \in \mathsf{F}(\boldsymbol{\mathcal{X}}_0 \otimes \mathcal{X}_0)$,

$$\iint h(\boldsymbol{x}_0, z_0)\, \boldsymbol{\eta}_0(\mathrm{d}\boldsymbol{x}_0)\, \mu(\boldsymbol{x}_0)(\mathrm{d}z_0) = \iint h(\boldsymbol{x}_0, z_0)\, \boldsymbol{\eta}_0\langle z_0\rangle(\mathrm{d}\boldsymbol{x}_0)\, \eta_0(\mathrm{d}z_0). \tag{6.3}$$

Proof. Because $\mu(\boldsymbol{x}_n)\, Q_n(\mathrm{d}z_{n+1}) = \boldsymbol{g}_n(\boldsymbol{x}_n)\, \Phi_n(\mu(\boldsymbol{x}_n))(\mathrm{d}z_{n+1})$, we may rewrite the right-hand side of (6.2) as

$$\begin{aligned}
&\iint h(\boldsymbol{x}_{n+1}, z_{n+1})\, \mu(\boldsymbol{x}_n) Q_n(\mathrm{d}z_{n+1})\, \boldsymbol{M}_n\langle z_{n+1}\rangle(\boldsymbol{x}_n, \mathrm{d}\boldsymbol{x}_{n+1}) \\
&\quad = \boldsymbol{g}_n(\boldsymbol{x}_n) \frac{1}{N} \sum_{i=0}^{N-1} \iint h(\boldsymbol{x}_{n+1}, z_{n+1})\, \Phi_n(\mu(\boldsymbol{x}_n))(\mathrm{d}z_{n+1}) \\
&\qquad \times \left(\Phi_n(\mu(\boldsymbol{x}_n))^{\otimes i} \otimes \delta_{z_{n+1}} \otimes \Phi_n(\mu(\boldsymbol{x}_n))^{\otimes (N-i-1)} \right) (\mathrm{d}\boldsymbol{x}_{n+1}) \\
&\quad = \boldsymbol{g}_n(\boldsymbol{x}_n) \frac{1}{N} \sum_{i=1}^{N} \int \cdots \int h((x_{n+1}^1, \ldots, x_{n+1}^{i-1}, z_{n+1}, x_{n+1}^{i+1}, \ldots, x_{n+1}^N), z_{n+1}) \\
&\qquad \times \Phi_n(\mu(\boldsymbol{x}_n))(\mathrm{d}z_{n+1}) \prod_{\ell \neq i} \Phi_n(\mu(\boldsymbol{x}_n))(\mathrm{d}x_{n+1}^\ell) \\
&\quad = \boldsymbol{g}_n(\boldsymbol{x}_n) \frac{1}{N} \sum_{i=1}^{N} \int h(\boldsymbol{x}_{n+1}, x_{n+1}^i)\, \boldsymbol{M}_n(\boldsymbol{x}_n, \mathrm{d}\boldsymbol{x}_{n+1}).
\end{aligned}$$

On the other hand, note that the left-hand side of (6.2) can be expressed as

$$\begin{aligned}
&\iint h(\boldsymbol{x}_{n+1}, z_{n+1})\, \boldsymbol{Q}_n(\boldsymbol{x}_n, \mathrm{d}\boldsymbol{x}_{n+1})\, \mu(\boldsymbol{x}_{n+1})(\mathrm{d}z_{n+1}) \\
&\qquad = \boldsymbol{g}_n(\boldsymbol{x}_n) \frac{1}{N} \sum_{i=1}^{N} \int h(\boldsymbol{x}_{n+1}, x_{n+1}^i)\, \boldsymbol{M}_n(\boldsymbol{x}_n, \mathrm{d}\boldsymbol{x}_{n+1}),
\end{aligned} \tag{6.4}$$

which establishes the identity. The identity (6.3) is established along similar lines. □

We establish 1 by induction. Thus, assume that the claim holds for n, and show that for all $h \in \mathsf{F}(\boldsymbol{\mathcal{X}}_{0:n+1} \otimes \mathcal{X}_{0:n+1})$,

$$\begin{aligned}
&\iint h(\boldsymbol{x}_{0:n+1}, z_{0:n+1})\, \boldsymbol{\gamma}_{0:n+1}(\mathrm{d}\boldsymbol{x}_{0:n+1})\, \mathbb{B}_{n+1}(\boldsymbol{x}_{0:n+1}, \mathrm{d}z_{0:n+1}) \\
&\qquad = \iint h(\boldsymbol{x}_{0:n+1}, z_{0:n+1})\, \gamma_{0:n+1}(\mathrm{d}z_{0:n+1})\, \mathbb{C}_{n+1}(z_{0:n+1}, \mathrm{d}\boldsymbol{x}_{0:n+1}).
\end{aligned} \tag{6.5}$$

To prove this, we process, using definition (2.4), the left-hand side of (6.5) according to

$$\begin{aligned}
&\iint h(\boldsymbol{x}_{0:n+1}, z_{0:n+1})\, \boldsymbol{\gamma}_{0:n+1}(\mathrm{d}\boldsymbol{x}_{0:n+1})\, \mathbb{B}_{n+1}(\boldsymbol{x}_{0:n+1}, \mathrm{d}z_{0:n+1}) \\
&= \iint \boldsymbol{\gamma}_{0:n}(\mathrm{d}\boldsymbol{x}_{0:n})\, \mathbb{B}_n(\boldsymbol{x}_{0:n}, \mathrm{d}z_{0:n}) \\
&\qquad\qquad \times \iint \bar{h}(\boldsymbol{x}_{0:n+1}, z_{0:n+1})\, \boldsymbol{Q}_n(\boldsymbol{x}_n, \mathrm{d}\boldsymbol{x}_{n+1})\, \mu(\boldsymbol{x}_{n+1})(\mathrm{d}z_{n+1}),
\end{aligned} \tag{6.6}$$

where we define the function

$$\bar{h}(\boldsymbol{x}_{0:n+1}, z_{0:n+1}) \coloneqq \frac{q_n(z_n, z_{n+1}) h(\boldsymbol{x}_{0:n+1}, z_{0:n+1})}{\mu(\boldsymbol{x}_n)[q_n(\cdot, z_{n+1})]}.$$

Now, applying 1 to the inner integral and using

$$\mu(\boldsymbol{x}_n) Q_n(\mathrm{d}z_{n+1}) = \mu(\boldsymbol{x}_n)[q_n(\cdot, z_{n+1})]\, \lambda_{n+1}(\mathrm{d}z_{n+1})$$

yields, for every $\boldsymbol{x}_{0:n}$ and $z_{0:n}$,

$$\begin{aligned}
&\iint \bar{h}(\boldsymbol{x}_{0:n+1}, z_{0:n+1})\, \boldsymbol{Q}_n(\boldsymbol{x}_n, \mathrm{d}\boldsymbol{x}_{n+1})\, \mu(\boldsymbol{x}_{n+1})(\mathrm{d}z_{n+1}) \\
&= \iint \bar{h}(\boldsymbol{x}_{0:n+1}, z_{0:n+1})\, \mu(\boldsymbol{x}_n) Q_n(\mathrm{d}z_{n+1})\, \boldsymbol{M}_n\langle z_{n+1}\rangle(\boldsymbol{x}_n, \mathrm{d}\boldsymbol{x}_{n+1}) \\
&= \iint h(\boldsymbol{x}_{0:n+1}, z_{0:n+1})\, Q_n(z_n, \mathrm{d}z_{n+1})\, \boldsymbol{M}_n\langle z_{n+1}\rangle(\boldsymbol{x}_n, \mathrm{d}\boldsymbol{x}_{n+1}).
\end{aligned}$$

Inserting the previous identity into (6.6) and using the induction hypothesis yields

$$\begin{aligned}
&\iint h(\boldsymbol{x}_{0:n+1}, z_{0:n+1})\, \boldsymbol{\gamma}_{0:n+1}(\mathrm{d}\boldsymbol{x}_{0:n+1})\, \mathbb{B}_{n+1}(\boldsymbol{x}_{0:n+1}, \mathrm{d}z_{0:n+1}) \\
&= \iint \gamma_{0:n}(\mathrm{d}z_{0:n})\, \mathbb{C}_n(z_{0:n}, \mathrm{d}\boldsymbol{x}_{0:n}) \\
&\qquad\qquad \times \iint h(\boldsymbol{x}_{0:n+1}, z_{0:n+1})\, Q_n(z_n, \mathrm{d}z_{n+1})\, \boldsymbol{M}_n\langle z_{n+1}\rangle(\boldsymbol{x}_n, \mathrm{d}\boldsymbol{x}_{n+1}) \\
&= \iint h(\boldsymbol{x}_{0:n+1}, z_{0:n+1})\, \gamma_{0:n+1}(\mathrm{d}z_{0:n+1})\, \mathbb{C}_{n+1}(z_{0:n+1}, \mathrm{d}\boldsymbol{x}_{0:n+1}),
\end{aligned}$$

which establishes (6.5).

6.3 Proof of 3

First, define, for $m \in \mathbb{N}$,

$$\boldsymbol{P}_m : \mathbf{Y}_m \times \boldsymbol{\mathcal{Y}}_{m+1} \ni (\boldsymbol{y}_m, A) \mapsto \int \boldsymbol{M}_m(\boldsymbol{x}_{m|m}, \mathrm{d}\boldsymbol{x}_{m+1})\, \boldsymbol{S}_m(\boldsymbol{y}_m, \boldsymbol{x}_{m+1}, A). \tag{6.7}$$

For any given initial distribution $\boldsymbol{\psi}_0 \in \mathsf{M}_1(\boldsymbol{\mathcal{Y}}_0)$, let $\mathbb{P}^{\boldsymbol{P}}_{\boldsymbol{\psi}_0}$ be the distribution of the canonical Markov chain induced by the Markov kernels $\{\boldsymbol{P}_m\}_{m\in\mathbb{N}}$ and the initial distribution $\boldsymbol{\psi}_0$. With a slight abuse of notation we write, for $\boldsymbol{\eta}_0 \in \mathsf{M}_1(\boldsymbol{\mathcal{X}}_0)$, $\mathbb{P}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}$ instead of $\mathbb{P}^{\boldsymbol{P}}_{\boldsymbol{\psi}_0[\boldsymbol{\eta}_0]}$, where we define the extension $\boldsymbol{\psi}_0[\boldsymbol{\eta}_0](A) = \int \mathbb{1}_A(\mathbf{J}\boldsymbol{x}_0)\, \boldsymbol{\eta}_0(\mathrm{d}\boldsymbol{x}_0)$, for $A \in \boldsymbol{\mathcal{Y}}_0$. We preface the proof of 3 with some technical lemmas and a proposition.

Lemma 2. *For all $n \in \mathbb{N}$ and $(f_{n+1}, \tilde{f}_{n+1}) \in \mathsf{F}(\mathcal{X}_{n+1})^2$,*

$$\gamma_{n+1}(f_{n+1} B_{n+1} h_{n+1} + \tilde{f}_{n+1}) = \gamma_n\{Q_n f_{n+1} B_n h_n + Q_n(\tilde{h}_n f_{n+1} + \tilde{f}_{n+1})\}.$$

Proof. Pick arbitrary $\varphi \in \mathsf{F}(\mathcal{X}_{n:n+1})$ and, from definition (2.3) and that Q_n has a transition density, write

$$\begin{aligned}
&\iint \varphi(x_{n:n+1})\, \gamma_n(\mathrm{d}x_n)\, Q_n(x_n, \mathrm{d}x_{n+1}) \\
&\qquad = \iint \varphi(x_{n:n+1}) \gamma_n[q_n(\cdot, x_{n+1})]\, \lambda_{n+1}(\mathrm{d}x_{n+1})\, \frac{\gamma_n(\mathrm{d}x_n) q_n(x_n, x_{n+1})}{\gamma_n[q_n(\cdot, x_{n+1})]} \\
&\qquad = \iint \varphi(x_{n:n+1})\, \gamma_{n+1}(\mathrm{d}x_{n+1})\, \overleftarrow{Q}_{n,\eta_n}(x_{n+1}, \mathrm{d}x_n).
\end{aligned} \tag{6.8}$$

Now, by (2.10), it holds that

$$B_{n+1} h_{n+1}(x_{n+1}) = \int \overleftarrow{Q}_{n,\eta_n}(x_{n+1}, \mathrm{d}x_n) \left(\tilde{h}_n(x_{n:n+1}) + \int h_n(x_{0:n})\, B_n(x_n, \mathrm{d}x_{0:n-1}) \right);$$

therefore, by applying (6.8) with

$$\varphi(x_{n:n+1}) \coloneqq f_{n+1}(x_{n+1}) \left(\tilde{h}_n(x_{n:n+1}) + \int h_n(x_{0:n})\, B_n(x_n, \mathrm{d}x_{0:n-1}) \right),$$

Statistica Sinica: Preprint
doi:10.5705/ss.202022.0215

we obtain that

$$\begin{aligned}
\gamma_{n+1}(f_{n+1}B_{n+1}h_{n+1}) &= \iint \varphi(x_{n:n+1})\, \gamma_{n+1}(\mathrm{d}x_{n+1})\, \overleftarrow{Q}_{n,\eta_n}(x_{n+1}, \mathrm{d}x_n) \\
&= \iint \varphi(x_{n:n+1})\, \gamma_n(\mathrm{d}x_n)\, Q_n(x_n, \mathrm{d}x_{n+1}) \\
&= \gamma_n(Q_n f_{n+1} B_n h_n + Q_n \tilde{h}_n f_{n+1}).
\end{aligned}$$

Now, the proof is concluded by noting that because $\gamma_{n+1} = \gamma_n Q_n$, $\gamma_{n+1}\tilde{f}_{n+1} = \gamma_n Q_n \tilde{f}_{n+1}$. □

Lemma 3. *For every $n \in \mathbb{N}^*$, $h_n \in \mathsf{F}(\mathcal{Y}_n)$, and $\boldsymbol{\eta}_0 \in \mathsf{M}_1(\boldsymbol{\mathcal{X}}_0)$, it holds that*

$$\mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}[h_n(\boldsymbol{\upsilon}_n) \mid \boldsymbol{\xi}_{0|0}, \ldots, \boldsymbol{\xi}_{n|n}] = \mathbb{S}_n h_n(\boldsymbol{\xi}_{0|0}, \ldots, \boldsymbol{\xi}_{n|n}), \quad \mathbb{P}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}\text{-}a.s.$$

Proof. Pick arbitrary $v_n \in \mathsf{F}(\mathcal{X}_{0:n})$. We show that

$$\mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}[v_n(\boldsymbol{\xi}_{0|0}, \ldots, \boldsymbol{\xi}_{n|n}) h_n(\boldsymbol{\upsilon}_n)] = \mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}[v_n(\boldsymbol{\xi}_{0|0}, \ldots, \boldsymbol{\xi}_{n|n}) \mathbb{S}_n h_n(\boldsymbol{\xi}_{0|0}, \ldots, \boldsymbol{\xi}_{n|n})], \tag{6.9}$$

from which the claim follows. Using definition (6.7), the left-hand side of the previous identity may be rewritten as

$$\begin{aligned}
&\int \cdots \int \boldsymbol{\psi}_0[\boldsymbol{\eta}_0](\mathrm{d}\boldsymbol{y}_0) \prod_{m=0}^{n-1} \boldsymbol{P}_m(\boldsymbol{y}_m, \mathrm{d}\boldsymbol{y}_{m+1})\, h_n(\boldsymbol{y}_n) v_n(\boldsymbol{x}_{0|0}, \ldots, \boldsymbol{x}_{n|n}) \\
&\quad = \int \cdots \int \boldsymbol{\eta}_0(\mathrm{d}\boldsymbol{x}_{0|0}) \prod_{m=0}^{n-1} \boldsymbol{M}_m(\boldsymbol{x}_{m|m}, \mathrm{d}\boldsymbol{x}_{m+1})\, \boldsymbol{S}_0(\mathbf{J}\boldsymbol{x}_{0|0}, \boldsymbol{x}_1, \mathrm{d}\boldsymbol{y}_1) \\
&\qquad \times \prod_{m=0}^{n-1} \boldsymbol{S}_m(\boldsymbol{y}_m, \boldsymbol{x}_{m+1}, \mathrm{d}\boldsymbol{y}_{m+1})\, h_n(\boldsymbol{y}_n) v_n(\boldsymbol{x}_{0|0}, \ldots, \boldsymbol{x}_{n|n}) \\
&\quad = \int \cdots \int \boldsymbol{\eta}_0(\mathrm{d}\boldsymbol{x}_0) \prod_{m=0}^{n-1} \boldsymbol{M}_m(\boldsymbol{x}_m, \mathrm{d}\boldsymbol{x}_{m+1})\, \boldsymbol{S}_0(\mathbf{J}\boldsymbol{x}_0, \boldsymbol{x}_1, \mathrm{d}\boldsymbol{y}_1) \\
&\qquad \times \prod_{m=0}^{n-1} \boldsymbol{S}_m(\boldsymbol{y}_m, \boldsymbol{x}_{m+1}, \mathrm{d}\boldsymbol{y}_{m+1})\, h_n(\boldsymbol{y}_n) v_n(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n).
\end{aligned}$$

Thus, we conclude the proof by using the definition (3.2) of $\mathbb{S}_n$, together with Fubini's theorem.

□

Lemma 4. *For every $n \in \mathbb{N}^*$ and $h_n \in \mathsf{F}(\mathcal{Y}_n)$, it holds that*

$$\mathbb{E}_{\boldsymbol{\eta}_0}\left[\left(\prod_{m=0}^{n-1} \boldsymbol{g}_m(\boldsymbol{\xi}_{m|m})\right) h_n(\boldsymbol{\upsilon}_n)\right] = \int \boldsymbol{\gamma}_{0:n}\mathbb{S}_n(\mathrm{d}\boldsymbol{y}_n)\, h_n(\boldsymbol{y}_n).$$

Proof. The claim of the lemma is a direct implication of 3; indeed, by applying the tower property and the latter, we obtain

$$\begin{aligned}
&\mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}\left[\left(\prod_{m=0}^{n-1} \boldsymbol{g}_m(\boldsymbol{\xi}_{m|m})\right) h_n(\boldsymbol{\upsilon}_n)\right] \\
&\quad = \mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}\left[\left(\prod_{m=0}^{n-1} \boldsymbol{g}_m(\boldsymbol{\xi}_{m|m})\right) \mathbb{S}_n h_n(\boldsymbol{\xi}_{0|0}, \ldots, \boldsymbol{\xi}_{n|n})\right] \\
&\quad = \int \cdots \int \boldsymbol{\eta}_0(\mathrm{d}\boldsymbol{x}_0) \prod_{m=0}^{n-1} \boldsymbol{g}_m(\boldsymbol{x}_m)\, \boldsymbol{M}_m(\boldsymbol{x}_m, \mathrm{d}\boldsymbol{x}_{m+1})\, \mathbb{S}_n h_n(\boldsymbol{x}_{0:n}) \\
&\quad = \int \boldsymbol{\gamma}_{0:n}\mathbb{S}_n(\mathrm{d}\boldsymbol{y}_n)\, h_n(\boldsymbol{y}_n).
\end{aligned}$$

□

Proposition 6. *For all $n \in \mathbb{N}^*$, $(N, M) \in (\mathbb{N}^*)^2$, and $(f_n, \tilde{f}_n) \in \mathsf{F}(\mathcal{X}_n)^2$,*

$$\int \boldsymbol{\gamma}_{0:n}\mathbb{S}_n(\mathrm{d}\boldsymbol{y}_n)\left(\frac{1}{N}\sum_{i=1}^{N}\{b_n^i f_n(x_{n|n}^i) + \tilde{f}_n(x_{n|n}^i)\}\right) = \gamma_n(f_n B_n h_n + \tilde{f}_n).$$

Proof. Applying 4 yields

$$\begin{aligned}
&\int \boldsymbol{\gamma}_{0:n}\mathbb{S}_n(\mathrm{d}\boldsymbol{y}_n)\left(\frac{1}{N}\sum_{i=1}^{N}\{b_n^i f_n(x_{n|n}^i) + \tilde{f}_n(x_{n|n}^i)\}\right) \\
&\qquad = \mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}\left[\left(\prod_{m=0}^{n-1} \boldsymbol{g}_m(\boldsymbol{\xi}_{m|m})\right) \frac{1}{N}\sum_{i=1}^{N}\{\beta_n^i f_n(\xi_{n|n}^i) + \tilde{f}_n(\xi_{n|n}^i)\}\right].
\end{aligned} \tag{6.10}$$

In the following, we repeatedly use the following filtrations. Let $\tilde{\mathcal{F}}_n := \sigma(\{\boldsymbol{\upsilon}_m\}_{m=0}^n)$ be the σ-field generated by the output of the `PARIS` (1) during the first n iterations. In addition, let $\mathcal{F}_n := \tilde{\mathcal{F}}_{n-1} \vee \sigma(\boldsymbol{\xi}_{n|n})$.

We proceed by induction. Thus, assume that the statement of the proposition holds for a given $n \in \mathbb{N}^*$, and consider, for arbitrarily chosen $(f_{n+1}, \tilde{f}_{n+1}) \in \mathsf{F}(\mathcal{X}_{n+1})^2$,

$$\begin{aligned}\mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}&\left[\left(\prod_{m=0}^{n} \boldsymbol{g}_m(\boldsymbol{\xi}_{m|m})\right) \frac{1}{N}\sum_{i=1}^{N}\{\beta^i_{n+1} f_{n+1}(\xi^i_{n+1|n+1}) + \tilde{f}_{n+1}(\xi^i_{n+1|n+1})\} \mid \tilde{\mathcal{F}}_n\right] \\ &= \left(\prod_{m=0}^{n} \boldsymbol{g}_m(\boldsymbol{\xi}_{m|m})\right) \mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}[\beta^1_{n+1} f_{n+1}(\xi^1_{n+1|n+1}) + \tilde{f}_{n+1}(\xi^1_{n+1|n+1}) \mid \tilde{\mathcal{F}}_n],\end{aligned}$$

where we use that the variables $\{\beta^i_{n+1} f_{n+1}(\xi^i_{n+1|n+1}) + \tilde{f}_{n+1}(\xi^i_{n+1|n+1})\}_{i=1}^N$ are conditionally independent and identically distributed (i.i.d.) given $\tilde{\mathcal{F}}_n$. Note that, by symmetry,

$$\begin{aligned}\mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}\left[\beta^1_{n+1} \mid \mathcal{F}_{n+1}\right] &= \int \boldsymbol{S}_n(\boldsymbol{\upsilon}_n, \boldsymbol{\xi}_{n+1|n+1}, \mathrm{d}\boldsymbol{y}_{n+1})\, b^1_{n+1} \\ &= \int\cdots\int\left(\prod_{j=1}^{M}\sum_{\ell=1}^{N}\frac{q_n(\xi^\ell_{n|n}, \xi^1_{n+1|n+1})}{\sum_{\ell'=1}^{N} q_n(\xi^{\ell'}_{n|n}, \xi^1_{n+1|n+1})}\delta_{(\xi^\ell_{n|n}, \beta^\ell_n)}(\mathrm{d}\tilde{x}^{1,j}_n, \mathrm{d}\tilde{b}^{1,j}_n)\right) \\ &\qquad\qquad \times \frac{1}{M}\sum_{j=1}^{M}\left(\tilde{b}^{1,j}_n + \tilde{h}_n(\tilde{x}^{1,j}_n, \xi^1_{n+1|n+1})\right) \\ &= \sum_{\ell=1}^{N}\frac{q_n(\xi^\ell_{n|n}, \xi^1_{n+1|n+1})}{\sum_{\ell'=1}^{N} q_n(\xi^{\ell'}_{n|n}, \xi^1_{n+1|n+1})}\left(\beta^\ell_n + \tilde{h}_n(\xi^\ell_{n|n}, \xi^1_{n+1|n+1})\right). \qquad (6.11)\end{aligned}$$

Thus, using the tower property,

$$\begin{aligned}&\mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}\left[\beta^1_{n+1} f_{n+1}(\xi^1_{n+1|n+1}) \mid \tilde{\mathcal{F}}_n\right] \\ &\quad = \int \Phi_n(\mu(\boldsymbol{\xi}_{n|n}))(\mathrm{d}x_{n+1})\, f_{n+1}(x_{n+1}) \sum_{\ell=1}^{N}\frac{q_n(\xi^\ell_{n|n}, x_{n+1})}{\sum_{\ell'=1}^{N} q_n(\xi^{\ell'}_{n|n}, x_{n+1})}\left(\beta^\ell_n + \tilde{h}_n(\xi^\ell_{n|n}, x_{n+1})\right),\end{aligned}$$

and, consequently, using definition (2.1),

$$\begin{aligned}&\left(\prod_{m=0}^{n} \boldsymbol{g}_m(\boldsymbol{\xi}_{m|m})\right) \mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}\left[\beta^1_{n+1} f_{n+1}(\xi^1_{n+1|n+1}) \mid \tilde{\mathcal{F}}_n\right] \\ &= \left(\prod_{m=0}^{n-1} \boldsymbol{g}_m(\boldsymbol{\xi}_{m|m})\right) \int \frac{1}{N}\sum_{i=1}^{N} q_n(\xi^i_{n|n}, x_{n+1}) \\ &\quad \times f_{n+1}(x_{n+1}) \sum_{\ell=1}^{N}\frac{q_n(\xi^\ell_{n|n}, x_{n+1})}{\sum_{\ell'=1}^{N} q_n(\xi^{\ell'}_{n|n}, x_{n+1})}\left(\beta^\ell_n + \tilde{h}_n(\xi^\ell_{n|n}, x_{n+1})\right) \lambda_{n+1}(\mathrm{d}x_{n+1}) \\ &= \left(\prod_{m=0}^{n-1} \boldsymbol{g}_m(\boldsymbol{\xi}_{m|m})\right) \frac{1}{N}\sum_{\ell=1}^{N}\left(\beta^\ell_n Q_n f_{n+1}(\xi^\ell_{n|n}) + Q_n(\tilde{h}_n f_{n+1})(\xi^\ell_{n|n})\right).\end{aligned}$$

Thus, applying the induction hypothesis,

$$
\begin{aligned}
&\mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}\left[\left(\prod_{m=0}^{n}\boldsymbol{g}_m(\boldsymbol{\xi}_{m|m})\right)\frac{1}{N}\sum_{i=1}^{N}\beta^i_{n+1}f_{n+1}(\xi^i_{n+1|n+1})\right] \\
&= \mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}\left[\left(\prod_{m=0}^{n-1}\boldsymbol{g}_m(\boldsymbol{\xi}_{m|m})\right)\frac{1}{N}\sum_{\ell=1}^{N}\left(\beta^\ell_n Q_n f_{n+1}(\xi^\ell_{n|n}) + Q_n(\tilde{h}_n f_{n+1})(\xi^\ell_{n|n})\right)\right] \\
&= \gamma_n\left(Q_n f_{n+1} B_n h_n + Q_n(\tilde{h}_n f_{n+1})\right).
\end{aligned} \tag{6.12}
$$

In the same manner, it can be shown that

$$
\mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}\left[\left(\prod_{m=0}^{n}\boldsymbol{g}_m(\boldsymbol{\xi}_{m|m})\right)\frac{1}{N}\sum_{i=1}^{N}\tilde{f}_{n+1}(\xi^i_{n+1|n+1})\right] = \gamma_n Q_n \tilde{f}_{n+1}. \tag{6.13}
$$

Now, by (6.12–6.13) and 2,

$$
\begin{aligned}
&\mathbb{E}^{\boldsymbol{P}}_{\boldsymbol{\eta}_0}\left[\left(\prod_{m=0}^{n}\boldsymbol{g}_m(\boldsymbol{\xi}_{m|m})\right)\frac{1}{N}\sum_{i=1}^{N}\{\beta^i_{n+1}f_{n+1}(\xi^i_{n+1|n+1}) + \tilde{f}_{n+1}(\xi^i_{n+1|n+1})\}\right] \\
&= \gamma_n\left(Q_n f_{n+1} B_n h_n + Q_n(\tilde{h}_n f_{n+1} + Q_n \tilde{f}_{n+1})\right) \\
&= \gamma_{n+1}(f_{n+1} B_{n+1} h_{n+1} + \tilde{f}_{n+1}),
\end{aligned}
$$

which shows that the claim of the proposition holds at time $n+1$.

It remains to check the base case $n = 0$, which holds trivially, because $\boldsymbol{\beta}_0 = \mathbf{0}$ and $B_0 h_0 = 0$ by convention, and the initial particles $\boldsymbol{\xi}_{0|0}$ are drawn from $\boldsymbol{\eta}_0$. This completes the proof. □

Proof of 3. The identity $\int \boldsymbol{\eta}_{0:n}(\mathrm{d}\boldsymbol{x}_{0:n})\, \mathbb{S}_n(\boldsymbol{x}_{0:n}, \mathrm{d}\boldsymbol{b}_n)\, \mu(\boldsymbol{b}_n)(\mathrm{id}) = \eta_{0:n} h_n$ follows immediately by letting $f_n \equiv 1$ and $\tilde{f}_n \equiv 0$ in 6, and using that $\boldsymbol{\gamma}_{0:n}(\mathbf{X}_{0:n}) = \gamma_{0:n}(\mathsf{X}_{0:n})$. Moreover, applying 1 yields

$$
\begin{aligned}
\int \eta_{0:n}\mathbb{C}_n\mathbb{S}_n(\mathrm{d}\boldsymbol{b}_n)\,\mu(\boldsymbol{b}_n)(\mathrm{id}) &= \iint \eta_{0:n}(\mathrm{d}z_{0:n})\,\mathbb{C}_n(z_{0:n}, \mathrm{d}\boldsymbol{x}_{0:n}) \int \mathbb{S}_n(\boldsymbol{x}_{0:n}, \mathrm{d}\boldsymbol{b}_n)\,\mu(\boldsymbol{b}_n)(\mathrm{id}) \\
&= \iint \boldsymbol{\eta}_{0:n}(\mathrm{d}\boldsymbol{x}_{0:n})\,\mathbb{B}_n(\boldsymbol{x}_{0:n}, \mathrm{d}z_{0:n}) \int \mathbb{S}_n(\boldsymbol{x}_{0:n}, \mathrm{d}\boldsymbol{b}_n)\,\mu(\boldsymbol{b}_n)(\mathrm{id}) \\
&= \int \boldsymbol{\eta}_{0:n}\mathbb{S}_n(\mathrm{d}\boldsymbol{b}_n)\,\mu(\boldsymbol{b}_n)(\mathrm{id}).
\end{aligned}
$$

Finally, the first identity holds because K_n leaves $\eta_{0:n}$ invariant. □

Supplementary Material

The supplementary material contains proofs for the technical propositions, lemmas and theorems as well as additional numerical investigations of different aspects of the `PPG` algorithm.

Acknowledgments

The work of J. Olsson was supported by the Swedish Research Council, Grant 2018-05230. The work of E. Moulines was supported by ANR CHIA 002 – SCAI. The work of G. Cardoso was supported by the Investment of the Future Grant, ANR-10-IAHU-04, from the government of France through the Agence National de la Recherche.

References

[1] C. Andrieu and A. Doucet. Particle filtering for partially observed Gaussian state space models. *J. Roy. Statist. Soc. B*, 64(4):827–836, 2002.

[2] C. Andrieu and A. Doucet. Online Expectation–Maximization type algorithms for parameter estimation in general state space models. In *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, volume 6, pages 69–72, 2003.

[3] C. Andrieu, A. Doucet, and R. Holenstein. Particle Markov chain Monte Carlo methods (with discussion). *J. Roy. Statist. Soc. B*, 72:269–342, 2010.

[4] O. Cappé. Recursive computation of smoothed functionals of hidden Markovian processes using a particle approximation. *Monte Carlo Methods Appl.*, 7(1–2):81–92, 2001.

[5] O. Cappé. Online EM algorithm for hidden Markov models. *J. Comput. Graph. Statist.*, 20(3):728–749, 2011.

[6] O. Cappé, S. J. Godsill, and E. Moulines. An overview of existing methods and recent advances in sequential Monte Carlo. *IEEE Proceedings*, 95(5):899–924, 2007.

[7] O. Cappé, E. Moulines, and T. Rydén. *Inference in Hidden Markov Models*. Springer, 2005.

[8] N. Chopin and O. Papaspiliopoulos. *An Introduction to Sequential Monte Carlo*. Springer, 2020.

[9] N. Chopin and S. S. Singh. On particle Gibbs sampling. *Bernoulli*, 21(3):1855–1883, 2015.

[10] P. Del Moral. *Feynman-Kac Formulae. Genealogical and Interacting Particle Systems with Applications*. Springer, 2004.

[11] P. Del Moral. *Mean Field Simulation for Monte Carlo Integration*. CRC Press, 2013.

[12] P. Del Moral, A. Doucet, and S. S. Singh. A backward interpretation of Feynman–Kac formulae. *ESAIM: Mathematical Modelling and Numerical Analysis*, 44:947–975, 2010.

[13] P. Del Moral and A. Jasra. A sharp first order analysis of Feynman–Kac particle models, part II: Particle Gibbs samplers. *Stoch. Proc. Appl.*, 128(1):354–371, 2018.

[14] P. Del Moral, R. Kohn, and F. Patras. On particle Gibbs samplers. *Ann. Inst. H. Poincaré Probab. Statist.*, 52(4):1687–1733, 2016.

[15] R. Douc, A. Garivier, E. Moulines, and J. Olsson. Sequential Monte Carlo smoothing for general state space hidden Markov models. *Ann. Appl. Probab.*, 21(6):1201–2145, 2011.

[16] R. Douc and E. Moulines. Limit theorems for weighted samples with applications to sequential Monte Carlo methods. *Ann. Statist.*, 36(5):2344–2376, 2008.

[17] R. Douc, E. Moulines, P. Priouret, and P. Soulier. *Markov Chains.* Springer, 2018.

[18] P. Gloaguen, S. Le Corff, and J. Olsson. A pseudo-marginal sequential Monte Carlo online smoothing algorithm. *Bernoulli*, 28(4):2606–2633, 2022.

[19] S. J. Godsill, A. Doucet, and M. West. Monte Carlo smoothing for non-linear time series. *J. Am. Statist. Assoc.*, 50:438–449, 2004.

[20] J. Hull and A. White. The pricing of options on assets with stochastic volatilities. *J. Finance*, 42:281–300, 1987.

[21] P. E. Jacob, F. Lindsten, and T. B. Schön. Smoothing with couplings of conditional particle filters. *J. Am. Statist. Assoc.*, 115(530):721–729, 2020.

[22] A. Lee, S. S. Singh, and M. Vihola. Coupled conditional backward sampling particle filter. 48(5):3066–3089, 2020.

[23] F. Lindsten, M. I. Jordan, and T. B. Schön. Particle Gibbs with ancestor sampling. *J. Mach. Learn. Res.*, 15(1):2145–2184, 2014.

[24] J. Olsson and J. Westerborn. Efficient particle-based online smoothing in general hidden Markov models: The PaRIS algorithm. *Bernoulli*, 23(3):1951–1996, 2017.

[25] M. K. Pitt and N. Shephard. Filtering via simulation: Auxiliary particle filters. *J. Am. Statist. Assoc.*, 94(446):590–599, 1999.

[26] G. Poyiadjis, A. Doucet, and S. S. Singh. Particle methods for optimal filter derivative: application to parameter estimation. In *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, pages v/925–v/928, 18-23 March 2005.

[27] G. Poyiadjis, A. Doucet, and S. S. Singh. Particle approximations of the score and observed information matrix in state space models with application to parameter estimation. *Biometrika*, 98(1):65–80, 2011.

[28] S. Särkkä. *Bayesian Filtering and Smoothing.* Cambridge University Press, 2013.

[29] N. Whiteley. Discussion on particle Markov chain Monte Carlo methods. *J. Roy. Statist. Soc. B*, 72(3):306–307, 2010.