Time-Simultaneous Prediction Bands: A New Look at the Uncertainty Involved in Forecasting Mortality

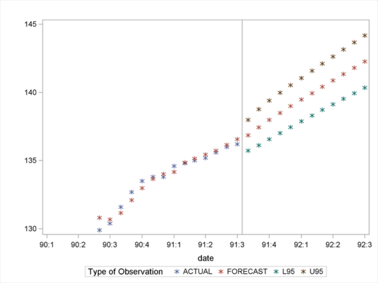
Wai-Sum Chan, PhD, CStat, FSA, CERA,
Professor of Finance
The Chinese University of Hong Kong

1. Introduction

- In recent years, there has been a new wave of work that is focused on the forecasting of uncertainty in mortality projections.
- Conventionally, isolated (point-wise) prediction intervals (IPI) are used to quantify the uncertainty in future mortality rates.
- A pointwise interval only reflects uncertainty in a variable at a single time point.
- In situations when the path or trajectory of future mortality rates is important, a band of pointwise intervals might lead to invalid inference.
- The primary objective of this paper is to demonstrate how simultaneous prediction bands (SPB) can be created for prevalent stochastic models.

What is IPI?

- Most time-series computer packages calculate and plot isolated (point-wise) prediction intervals (IPIs) for multiple forecasts.
- The following example shows an output from a standard forecasting package.
- 95% prediction intervals are included.
- What is the meaning of the confidence level 95% here?



What is SPB?

- SPB is a time-simultaneous prediction band with coverage probability $0 < 1 - \alpha \leq 1$ for a random trajectory $\mathbf{y}$ if

$$\Pr(\mathbf{y} \in \mathbf{SPB}) = \Pr\left(\bigcap_{s=1}^{S} (l_s \leq y_{T+s} \leq h_s)\right) = 1 - \alpha.$$

- SPB is useful when the whole path (or trajectory) of future mortality rates is important.
- The following graph shows a hypothetical 95% SPB.

7th IASC-ARS Σ joint 2011 Taipei Symposium

Why IPI $\neq$ SPB?

- Unless all trajectories develop very orderly (say, perfectly correlated), the probability that a trajectory lies completely inside all ISI would be less than $1 - \alpha$.
- It can be seen from an extreme example that if the forecasts are uncorrelated to each other, the probability of S consecutive predictions are within the ISI is $(1 - \alpha)^S$.
- For $S = 20$ and $\alpha = 5\%$, this probability becomes 35.85%!
- In general, predictions from a stochastic mortality model are neither perfectly correlated nor totally uncorrelated; therefore, we cannot use IPI to replace SPB.

2. Data

- Historic mortality data for US and Canadian (unisex) populations from age 60 to 99 and from year 1951 to 2004.
- The required data, death counts and exposures-to-risk, are obtained from the Human Mortality Database (2010).
- Note that the methods we propose do not require a specific choice of a sample age range and a sample period. They are chosen just purely for illustration purposes.

3. Mortality Models

- We provide a brief review of two mortality models which we use to illustrate the concept of time-simultaneous prediction bands (SPB).

- They are the Cairns-Blake-Dowd (CBD) model and the generalized CBD model with a cohort effect (GCBD).

- Note that the methods we propose do not require a specific choice of stochastic mortality models. They are chosen just purely for illustration purposes.

7th IASC-ARS $\sum$ joint 2011 Taipei Symposium

The Cairns-Blake-Dowd (CBD) Model

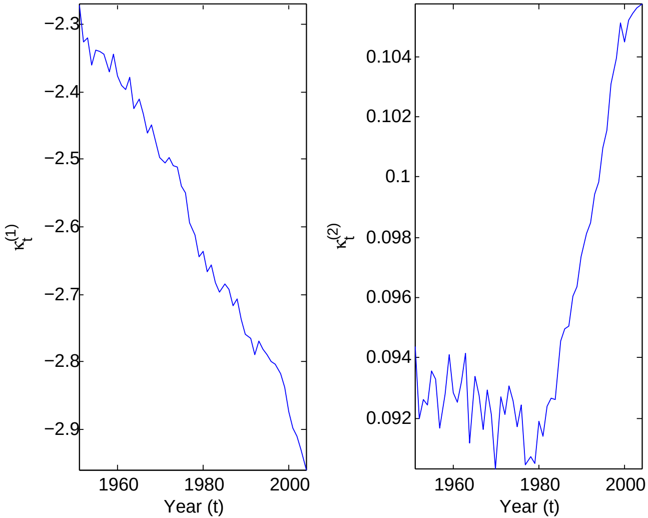
- Cairns et al. (2006) propose a two-factor stochastic mortality model

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}), \tag{1}$$

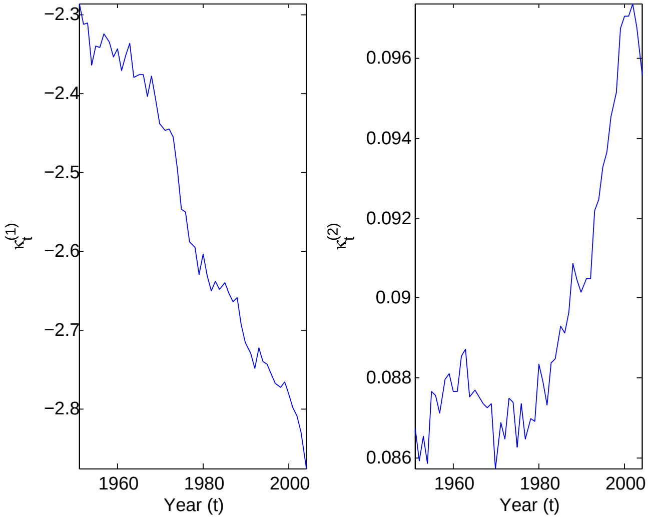
  where $q_{x,t}$ is the realized single-year death probability at age x and time t, $\bar{x}$ is the average age over the age range we consider, and $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ are period indexes.

- The maximum likelihood estimates (MLE) of $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$, $t = 1950, \ldots, 2004$, for Canada and US, are shown in the following graphs.

MLE of the CBD Model — Canada



MLE of the CBD Model — US



The Cairns-Blake-Dowd (CBD) Model

- After fitting equation (1) to historic death probabilities, the period indexes $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$ are modeled by a bivariate random walk with drift, that is,

$$\kappa_{t+1} = \kappa_t + \mu + CZ(t+1) \tag{2}$$

where $\kappa_t = (\kappa_t^{(1)}, \kappa_t^{(2)})'$, $\mu = (\mu_1, \mu_2)'$ is a constant 2×1 vector, C is a constant 2×2 upper triangular matrix, and $Z(t)$ is a 2-dimensional standard normal random vector.

The Cairns-Blake-Dowd (CBD) Model

- Trajectory of mortality predicted rates for a particular birth cohort (age x in year T) can be obtained by

$$\ln\left(\frac{\hat{q}_{x+s,T+s}}{1-\hat{q}_{x+s,T+s}}\right) = \kappa_T^{(1)}(s) + \kappa_T^{(2)}(s)(x+s-\bar{x}),$$

where $\kappa_T^{(1)}(s)$ and $\kappa_T^{(2)}(s)$ are the minimum sqaure error (MMSE) forecasts of $\kappa_{T+s}^{(1)}$ and $\kappa_{T+s}^{(2)}$, respectively.

7th IASC-ARS Σ joint 2011 Taipei Symposium

The GCBD Model

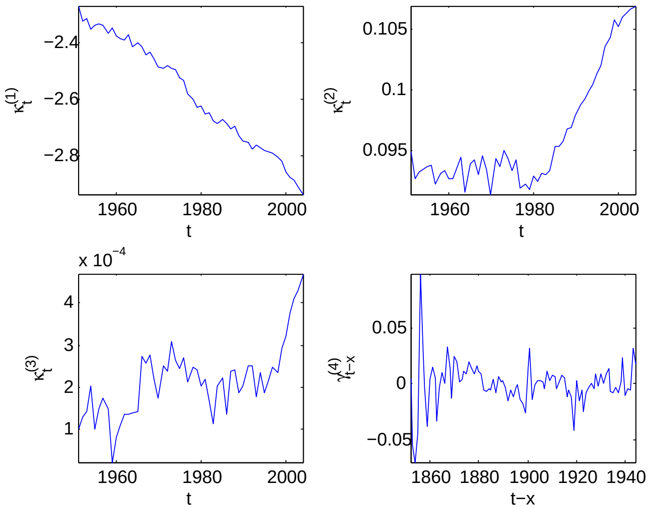
- To model cohort effects, we may consider the following generalization of the CBD model:

$$\ln\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = \kappa_t^{(1)} + \kappa_t^{(2)}(x-\bar{x}) + \kappa_t^{(3)}((x-\bar{x})^2 - \hat{\sigma}_x^2) + \gamma_{t-x}^{(4)}, \quad (3)$$

  where $\kappa_t^{(1)}$, $\kappa_t^{(2)}$, and $\kappa_t^{(3)}$ are period risk factors, $\gamma_{t-x}^{(4)}$ is a cohort risk factor, and $\hat{\sigma}_x^2$ is the mean of $(x-\bar{x})^2$ over the age range we consider.

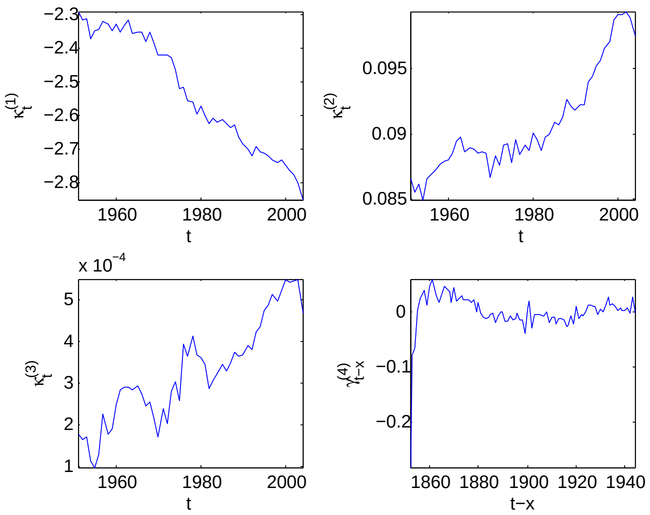
- The maximum likelihood estimates (MLE) of $\kappa_t^{(1)}$, $\kappa_t^{(2)}$, $\kappa_t^{(3)}$ and $\gamma_{t-x}^{(4)}$, $t = 1950, \ldots, 2004$, for Canada and US, are shown in the following graphs.

MLE of the GCBD Model — Canada



7th IASC-ARS
Σ joint 2011
Taipei Symposium

MLE of the GCBD Model — US



The GCBD Model

- Having fitted equation (3) to historic data, the period indexes are modeled by a trivariate random walk with drift:

$$\kappa_{t+1} = \kappa_t + \mu + CZ(t+1), \tag{4}$$

where $\kappa_t = (\kappa_t^{(1)}, \kappa_t^{(2)}, \kappa_t^{(3)})'$, $\mu = (\mu_1, \mu_2, \mu_3)'$ is a constant 3×1 vector, C is a constant 3×3 upper triangular matrix, and $Z(t)$ is a 3-dimensional standard normal random vector.

The GCBD Model

- Trajectory of mortality predicted rates for a particular birth cohort (age x in year T) can be obtained by

$$\ln\left(\frac{\hat{q}_{x+s,T+s}}{1-\hat{q}_{x+s,T+s}}\right) = \kappa_T^{(1)}(s) + \kappa_T^{(2)}(s)(x+s-\bar{x}) + \kappa_T^{(3)}(s)((x+s-\bar{x})^2 - \hat{\sigma}_x^2) + \gamma_{T-x}^{(4)}, \quad (5)$$

where $\kappa_T^{(i)}(s) = \kappa_T^{(i)} + s\mu_i$, $i = 1, 2, 3$, is the MMSE forecast of $\kappa_{T+s}^{(i)}$.

4. Simulation-Based SPB

- Consider the cohort of individuals who are aged x at the forecast origin T. Assuming that the forecast horizon is S, the trajectory of interest would be $\mathbf{m} = (m_{x+1,T+1}, \ldots, m_{x+S,T+S})$. From the stochastic components of the mortality models, we simulate a learning sample of size N (we use $N = 5000$ in this paper):

$$\mathbf{M} = \{\mathbf{m}^{(n)}\}_{n=1}^{N} = \{(m^{(n)}_{x+1,T+1}, \ldots, m^{(n)}_{x+S,T+S})\}_{n=1}^{N}$$

- Note that CBD models is based on $q_{x,t}$ rather than $m_{x,t}$. To obtain central death rates, we apply the constant force of mortality (CFM) assumption:

$$m_{x,t} = -\ln(1 - q_{x,t}),$$

Method 1: Adjusted Intervals

- Our goal is to construct a time-simultaneous prediction band such that $\lceil (1-\alpha)N \rceil$ of the trajectories in $\mathbf{M}$ are completely inside the band at every time point, while $\lfloor \alpha N \rfloor$ are outside the band at one or more points of time.

- Let $PI_s = [l_s, h_s]$ be a pointwise prediction interval for $m_{x+s,T+s}$ with coverage probability $1-\alpha$. Then the limits l_s and h_s are set to the $\lfloor \alpha/2 \rfloor$th lowest and highest values in the sample, respectively.

- Kolsrud (2007) proposes a three-step iterative method, which he calls 'adjusted intervals,' to construct a time-simultaneous prediction band from a learning sample.

Method 1: Adjusted Intervals

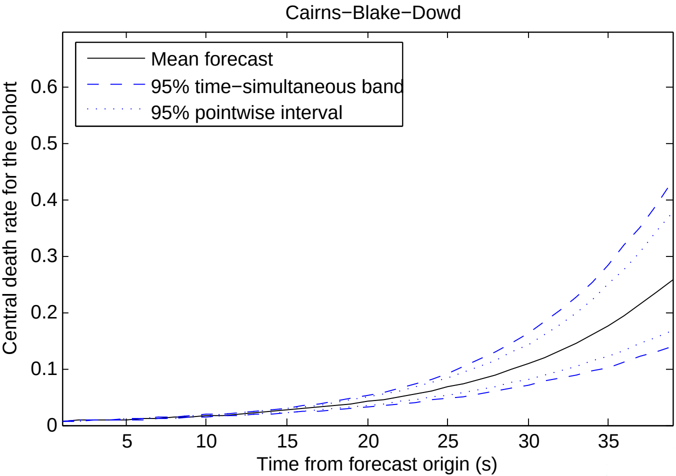
- The iterative algorithm:
  - Step 1: For each $s =, 1, \ldots, S$, widen the interval uniformly to include the nearest sample point above and the nearest sample point below.

$$PI_s^* = [l_s - \delta_s, h_s + \delta_s]$$

  - Step 2: Check the simultaneous coverage of all intervals in the learning sample **M**.
  - Step 3:If the simultaneous coverage is less than the prescribed level $1 - \alpha$, go to Step (1). Otherwise, terminate the algorithm.
- The resulting band of intervals would contain no less than $1 - \alpha$ of the trajectories in the learning sample.

7th IASC-ARS Σ joint 2011 Taipei Symposium

Illustration of Adjusted Intervals — Canada



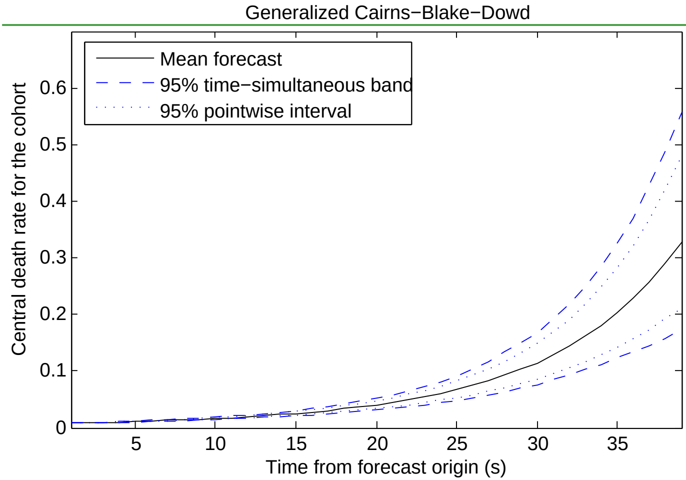
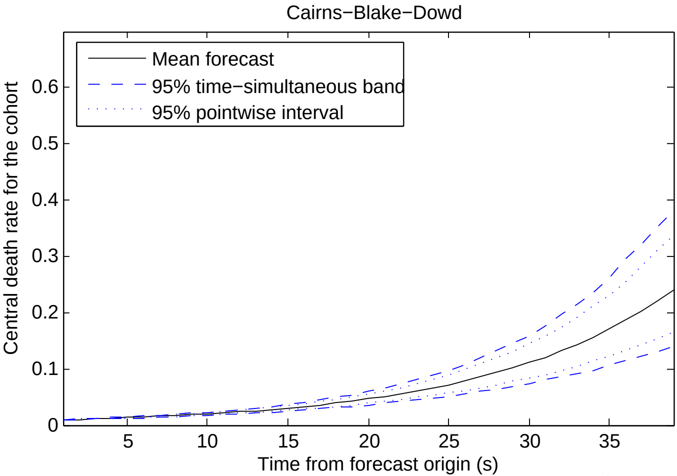
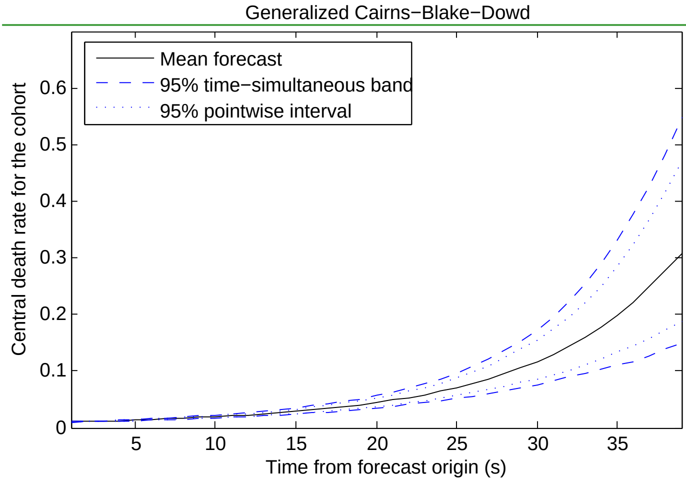

Generalized Cairns−Blake−Dowd
Mean forecast
95% time−simultaneous band
95% pointwise interval
Central death rate for the cohort
0.6
0.5
0.4
0.3
0.2
0.1
0
5
10
15
20
25
30
35
Time from forecast origin (s)

Illustration of Adjusted Intervals — US




Generalized Cairns−Blake−Dowd
Mean forecast
95% time−simultaneous band
95% pointwise interval
Central death rate for the cohort
0
0.1
0.2
0.3
0.4
0.5
0.6
5
10
15
20
25
30
35
Time from forecast origin (s)

Method 2: Chebyshev Bands

- The envelope of a (sub)sample is defined as the tightest band that contains all trajectories in the (sub)sample.
- As an example, the envelope of the learning sample $\mathbf{M}$ can be expressed as $([\min_n m^{(n)}_{x+s,T+s}, \max_n m^{(n)}_{x+s,T+s}])_{s=1}^{S}$.
- The idea behind Chebyshev bands is that we construct a time-simultaneous prediction band as the envelope of a subsample $\mathbf{M}^*$ that contains $\lceil (1-\alpha)N \rceil$ trajectories with the shortest distance to the mean trajectory

$$\bar{\mathbf{m}} = (\bar{m}_{x+1,T+1}, \ldots, \bar{m}_{x+S,T+S}).$$

7th IASC-ARS Σ joint 2011 Taipei Symposium

Method 2: Chebyshev Bands

- It is suggested that we measure the distant by the the weighted Chebyshev distance, which can be expressed as:

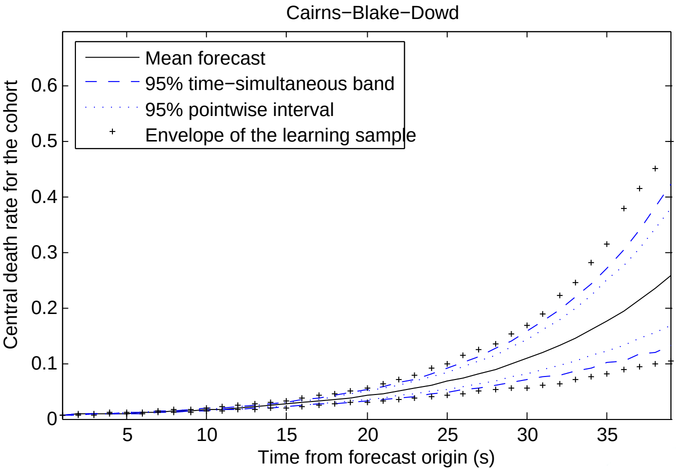
$$\mathcal{D} = \max_{s=1,\dots,S} \left(\frac{\left| m_{x+s,T+s} - \bar{m}_{x+s,T+s} \right|}{\sigma_s} \right),$$

where

$$\sigma_s = \sqrt{\frac{1}{N} \sum_{n=1}^{N} \left(m_{x+s,T+s} - \bar{m}_{x+s,T+s} \right)^2}$$

is the pointwise standard deviation s steps beyond the forecast origin.

Illustration of Chebyshev Bands — Canada



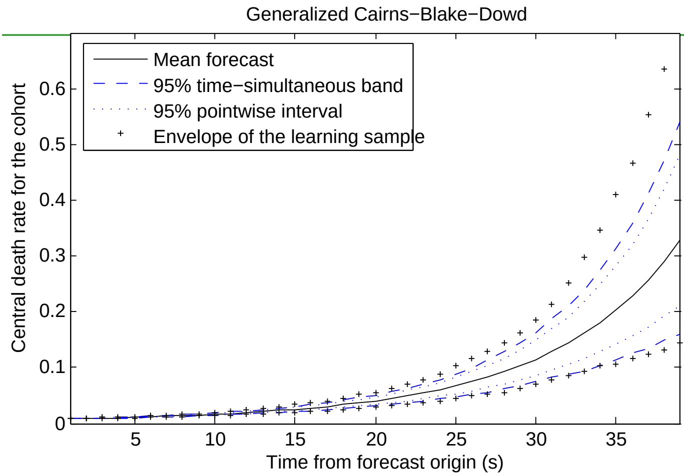
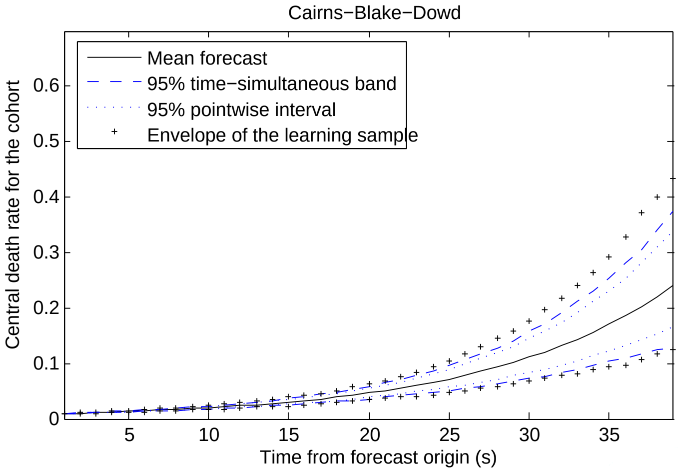
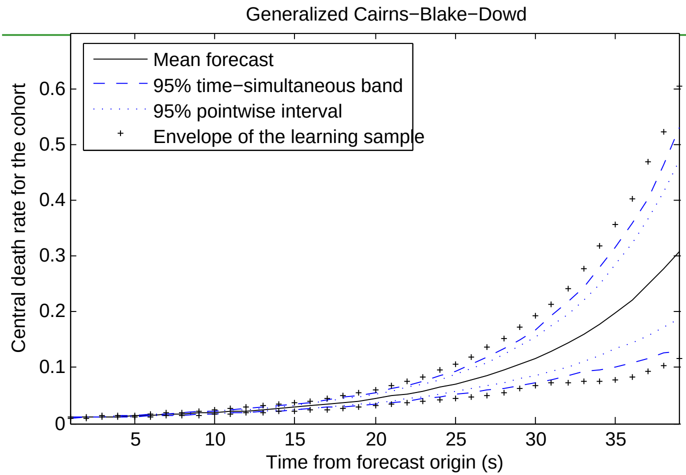

Generalized Cairns−Blake−Dowd
Mean forecast
95% time−simultaneous band
95% pointwise interval
Envelope of the learning sample
Central death rate for the cohort
0
0.1
0.2
0.3
0.4
0.5
0.6
5
10
15
20
25
30
35
Time from forecast origin (s)

Illustration of Chebyshev Bands — US




Generalized Cairns−Blake−Dowd
Mean forecast
95% time−simultaneous band
95% pointwise interval
Envelope of the learning sample
Central death rate for the cohort
0
0.1
0.2
0.3
0.4
0.5
0.6
5
10
15
20
25
30
35
Time from forecast origin (s)

5. Further Research

- Model Risk: Denuit (2009)
- Parameter Method: Li and Chan (2011)
- Non-linear Model: Azaïs et al. (2010)

Thank You!

7th IASC-ARS Σ joint 2011 Taipei Symposium