Cantor order statistics: without applications.

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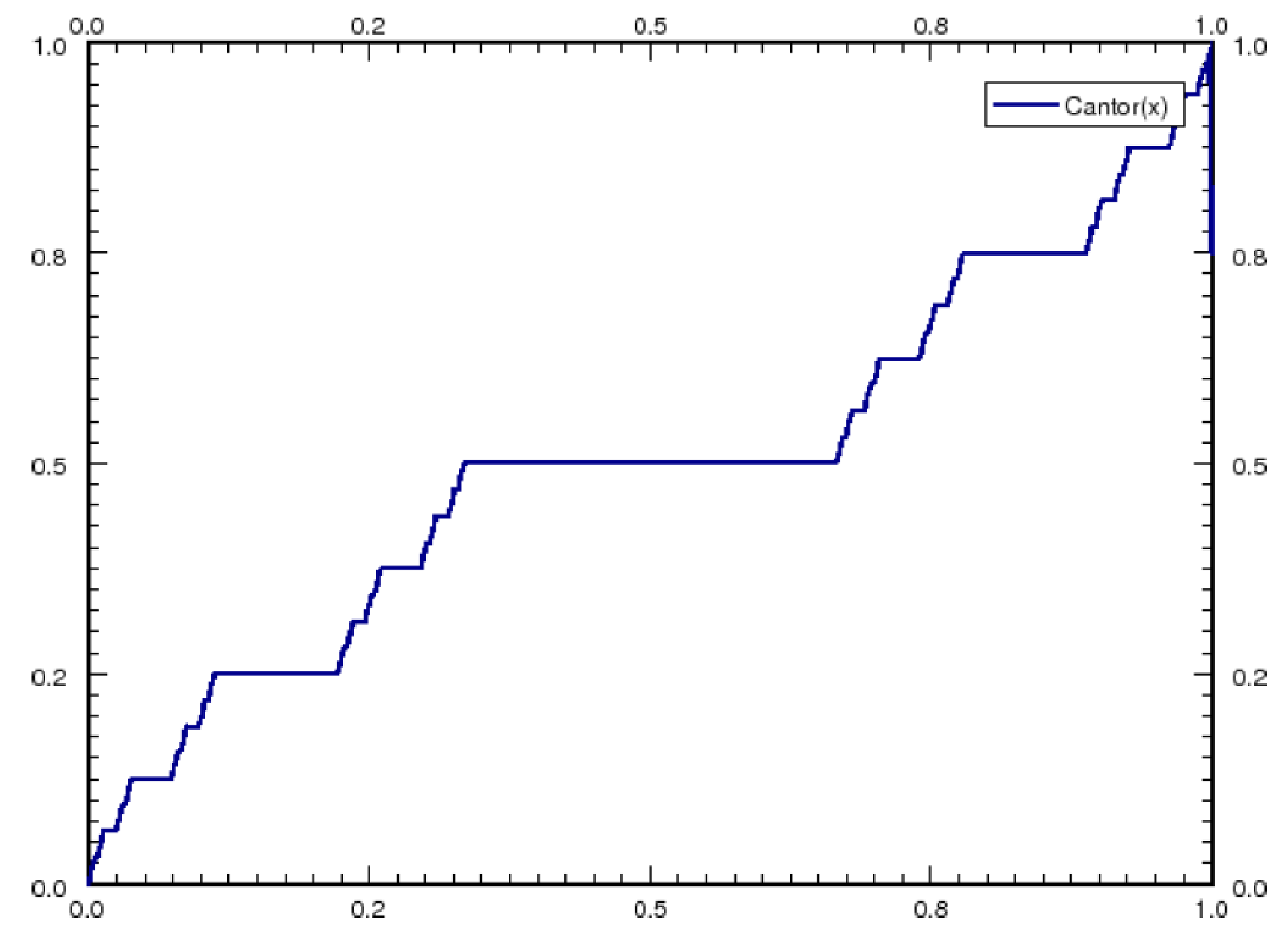
0.0
0.2
0.5
0.8
1.0
Cantor(x)

A classical Cantor r.v. admits the representation:

$$X = \sum_{j=1}^{\infty} \frac{2B_j}{3^j}$$

in terms of i.i.d. Bernoulli(1/2) r.v.'s.

We consider the following generalized Cantor r.v.

$$X = \sum_{j=1}^{\infty} (1-\phi)\phi^{j-1} B_j$$

In terms of i.i.d. Bernoulli (p) r.v.'s

Notation:

$$X \sim GC(\phi, p)$$

The case in which p=1/2 was investigated by Lad and Taylor(1992) and subsequently by Hosking (1994).

The classical Cantor case corresponds to p=1/2 and $\phi = 1/3$.

The general model

$$X = \sum_{j=1}^{\infty} (1 - \phi)\phi^{j-1} B_j$$

$0 < p < 1$ $\quad$ $0 < \phi < 1/2$

The constraint $0 < \phi < 1/2$ is needed to guarantee that we have a singular continuous r.v. , and thus a true analog of the classical Cantor r.v.

Since the generalized Cantor distribution has support in the interval $[0, 1]$, its right continuous inverse function (or quantile function) is also a valid distribution function.

A random variable Y with this quantile function as its distribution function will be said to have an inverse generalized Cantor distribution and we write: $Y \sim IGC(\phi, p)$

Note this usage is not customary, but it seems reasonable

Compare:

Inverse gamma distribution

Inverse Gaussian distribution

- Properties of the generalized Cantor distr.
- A skewed version (idea:Hosking 1994).
- The corresponding inverse distribution.
- Bivariate and multivariate extensions.

Properties of the generalized Cantor distribution.

$$X = \sum_{j=1}^{\infty} (1-\phi)\phi^{j-1} B_j$$

$$= (1-\phi) B_1 + \phi \widetilde{X}$$

where B_1 and $\widetilde{X}$ are independent

and $X =^d \widetilde{X}$.

Properties of the generalized Cantor distribution.

The corresponding moment generating function can be easily identified from either of these two representations of X. Thus:

$$M_X(t) = [1 - p + pe^{(1-\phi)t}]M_X(\phi t).$$

$$M_X(t) = \prod_{j=1}^{\infty}[1 - p + pe^{(1-\phi)\phi^{j-1}t}].$$

Properties of the generalized Cantor distribution.

Using the recursive representation we can compute moments (as done by Lad and Taylor when p=1/2):

$$E(X^k) = E[\{(1-\phi)B_1 + \phi\widetilde{X}\}^k]$$

$$= \sum_{j=0}^{k-1} \binom{k}{j} p(1-\phi)^{k-j}\phi^j E(X^j) + \phi^k E(X^k)$$

since $E(B_1^j) = p$ for $j > 0$.

Properties of the generalized Cantor distribution.

Thus:

$$E(X^k) = \frac{p\sum_{j=0}^{k-1}\binom{k}{j}(1-\phi)^{k-j}\phi^j E(X^j)}{1-\phi^k}, \quad k = 1, 2, \ldots.$$

In particular:

$$E(X) = \frac{p(1-\phi)}{1-\phi} = p,$$

$$\mathrm{var}(X) = \frac{p(1-p)(1-\phi)}{1+\phi}.$$

Properties of the generalized Cantor distribution.

Though both the mean and variance can be more easily obtained using the series representation

$$X = \sum_{j=1}^{\infty} (1 - \phi)\phi^{j-1} B_j$$

Properties of the generalized Cantor distribution.

Note: $E(X) > var(X)$

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In fact X is more undersdispersed than a Bernoulli (p) r.v.

Note that if $X \sim GC(\phi, p)$

then $1 - X \sim GC(\phi, 1 - p)$.

Properties of the generalized Cantor distribution.

Consistent asymptotically normal estimates of the parameters are available via the method of moments:

$$\hat{p} = M_1,$$

$$\hat{\phi} = \frac{M_1 - M_2}{M - 1 + M_2 - 2M_1^2},$$

in which $M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$.

Expected values of Order Statistics of the generalized Cantor distribution.

Here we follow Hosking (1994) who dealt with the case p=1/2.

We wish to evaluate

$$\mu_{i:n} = E(X_{i:n})$$

But it will suffice to evaluate

$$\mu_{1:n} = E(X_{1:n})$$

Expected values of Order Statistics of the generalized Cantor distribution.

$$X = \begin{cases} \phi\widetilde{X}, & \text{with probability } (1-p) \\ (1-\phi)+\phi\widetilde{X}, & \text{with probability } p. \end{cases}$$

So

$$X < \phi \text{ with probability } (1-p)$$

$$X > 1-\phi \text{ with probability } p.$$

Expected values of Order Statistics of the generalized Cantor distribution.

We then can condition on the number of X´s for which the corresponding B_1 ´s are equal to 0, to obtain:

$$E(X_{1:n}) = \sum_{j=1}^{n} \binom{n}{j} (1-p)^j p^{n-j} \phi E(X_{1:j}) +$$

$$+ p^n [1 - \phi + \phi E(X_{1:n}).$$

Expected values of Order Statistics of the generalized Cantor distribution.

$$E(X_{1:1}) = E(X) = p,$$

$$E(X_{1:2}) = \frac{p^2(1+\phi-2p\phi)}{1-\phi+2p(1-p)\phi}.$$

$$E(X_{2:2}) = 2E(X_{1:1}) - E(X_{1:2})$$

$$= p\frac{2-2\phi+3p\phi-2p^2\phi-p}{1-\phi+2p(1-p)\phi},$$

Gini index.of the generalized Cantor distribution

$$G(X) = \frac{E(X_{2:2}) - E(X_{1:2})}{E(X_{2:2}) + E(X_{1:2})}$$

$$= \frac{(1-p)(1-\phi)}{1-\phi+2p(1-p)\phi}.$$

Gini index.of the generalized Cantor distribution

If $p = 1/2$,

$$G(X) = \frac{1-\phi}{2-\phi}$$

Gini index.of the generalized Cantor distribution

If $p = 1/2,$

$$G(X) = \frac{1-\phi}{2-\phi}$$

If $p = 1/2,$ and $\phi = 1/3,$

$$G(X) = 2/5.$$

The skew generalized Cantor distribution.

Hosking introduced an asymmetric version of the generalized Cantor distr. In the case p=1/2. The extension to 0<p<1 is as follows:

$$X = \begin{cases} \alpha \widetilde{X} & \text{with probability } (1-p), \\ (1-\beta) + \beta \widetilde{X} & \text{with probability } p. \end{cases}$$

$$\alpha > 0, \beta > 0, \alpha + \beta < 1 \text{ and } 0 < p < 1$$

The skew generalized Cantor distribution.

Recursion for moments:

$$E(X^k) = \frac{p\sum_{j=0}^{k-1}\binom{k}{j}(1-\beta)^{k-j}\beta^j E(X^j)}{1-(1-p)\alpha^k - p\beta^k}$$

$$E(X) = \frac{p(1-\beta)}{1-(1-p)\alpha - p\beta},$$

$$\begin{aligned} var(X) &= p\frac{(\beta-1)^2(\alpha(p-1)+p\beta+1)}{((p-1)\alpha^2 - p\beta^2+1)(\alpha(p-1)-p\beta+1)} \\ &\quad - \left(\frac{p(1-\beta)}{1-(1-p)\alpha-p\beta}\right)^2. \end{aligned}$$

The skew generalized Cantor distribution.

Recursion for expected sample minima:

$$E(X_{1:n}) = \sum_{j=1}^{n} \binom{n}{j} (1-p)^j p^{n-j} \alpha E(X_{1:j}) +$$

$$+ p^n [1 - \beta + \beta E(X_{1:n})]$$

The skew generalized Cantor distribution.

Thus, for example

$$E(X_{1:2}) = \frac{2(1-p)p\alpha\frac{p(1-\beta)}{1-(1-p)\alpha-p\beta} + p^2(1-\beta)}{1-(1-p)^2\alpha - p^2\beta}$$

Using which, we obtain the Gini index:

$$G(X) = \frac{(1-p)(1-\alpha)}{1-(1-p)^2\alpha - p^2\beta}$$

Recall

For generalized Cantor distr.

$$G(X) = \frac{(1-p)(1-\phi)}{1-\phi+2p(1-p)\phi}.$$

While for skew generalized Cantor distr.

$$G(X) = \frac{(1-p)(1-\alpha)}{1-(1-p)^2\alpha - p^2\beta}$$

The inverse distribution.

If $X \sim SGC(\alpha, \beta, p)$ then we denote its distribution function by $F_{SGC}(x; \alpha, \beta, p)$

Since $0 \leq X \leq 1$, the corresponding right continuous quantile function $F_{SGC}^{-1}(x; \alpha, \beta, p)$ is also a valid distr. fn.

The inverse distribution.

If Y has $F_{SGC}^{-1}(x;\alpha,\beta,p)$ as its distr.fn. then we write $Y \sim SGC^{-1}(\alpha,\beta,p)$ and say that Y has an inverse-skew-generalized-Cantor distribution.

The inverse distribution.

If $Y \sim SGC^{-1}(\alpha, \beta, p)$ then Y is discrete with a countable number of possible values. Like the corresponding r.v. X, it has a fractal structure.

The inverse distribution.

Thus Y can be described as follows:

$$Y = \begin{cases} (1-p)\widetilde{Y} \text{ with probability } \alpha \\ (1-p) \text{ with probability } (1-\alpha-\beta) \\ (1-p)+p\widetilde{Y} \text{ with probability } \beta \end{cases}$$

$$\alpha > 0, \beta > 0, \alpha + \beta < 1 \text{ and } 0 < p < 1$$

in which $\widetilde{Y} =^d Y.$

The inverse distribution.

Using this representation we find:

$$E(Y) = \frac{(1-\alpha)(1-p)}{1-\alpha+\alpha p-\beta p},$$

and eventually

$$var(Y) \;=\; \frac{(1-p)^2(1-\alpha)}{1-\alpha+\alpha p-\beta p}\,\frac{1-\alpha+\alpha p+\beta p}{1-\alpha(1-p)^2-\beta p^2} - \left(\frac{(1-\alpha)(1-p)}{1-\alpha+\alpha p-\beta p}\right)^2.$$

The inverse distribution.

For the inverse classical Cantor distribution
Corresponding to the choice

$$\alpha = \beta = 1/3 \text{ and } p = 1/2,$$

we have:

$$E(Y) = 1/2. \qquad var(Y) = 1/20.$$

On Moments of Inverse Distributions

Suppose that X satisfies $0 \le X \le 1$ and has d.f. F. Let Y have the corresponding inverse distribution, i.e., its d.f. is F^{-1}. How are the moments of X related to those of Y ?

On Moments of Inverse Distributions

Integrating by parts, one finds

$$E(Y^k) = \int_0^1 y^k dF^{-1}(y) = \int_0^1 [F(x)]^k dx$$

$$= x[F(x)]k|_0^1 - \int_0^1 xd[F(x)]^k$$

$$= 1 - E(X_{k:k}) = 1 - E[(1 - X)_{1:k}]$$

On Moments of Inverse Distributions

With k=1, we get

$$E(X) + E(Y) = 1$$

On Moments of Inverse Distributions

In the case of SGC variables, then

$$
\begin{aligned}
E(Y^k) &= \int_0^1 y^k dF^{-1}(y;\alpha,\beta,p) = \int_0^1 [F(x;\alpha,\beta,p)]^k dx \\
&= x[F(x;\alpha,\beta,p)]^k|_0^1 - \int_0^1 xd[F(x;\alpha,\beta,p)]^k \\
&= 1 - E(X_{k:k}).
\end{aligned}
$$

$$= E((1-X)_{1:k}) = E(Z_{1:k})$$

where the Z_i's are i.i.d. $SGC(\beta,\alpha,1-p)$ r.v.$'$s

A bivariate SGC random variable

$$
\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{cases}
\begin{pmatrix} \alpha_1 \widetilde{X} \\ \alpha_2 \widetilde{Y} \end{pmatrix} & \text{with probability } p_{00} \\[2ex]
\begin{pmatrix} (1-\beta_1) + \beta_1 \widetilde{X} \\ \alpha_2 \widetilde{Y} \end{pmatrix} & \text{with probability } p_{10} \\[2ex]
\begin{pmatrix} \alpha_1 \widetilde{X} \\ (1-\beta_2) + \beta_2 \widetilde{Y} \end{pmatrix} & \text{with probability } p_{01} \\[2ex]
\begin{pmatrix} (1-\beta_1) + \beta_1 \widetilde{X} \\ (1-\beta_1) + \beta_2 \widetilde{Y} \end{pmatrix} & \text{with probability } p_{11}
\end{cases} \quad : \quad (
$$

Multivariate SGC random variables

Multivariate SGC random variables

We'd need a bigger screen !!

Thank you for your attention.