On Estimating the Parameters in Conditional Heteroskedasticity Model by Empirical Likelihood Estimation

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Outline

1. Introduction
2. Survey of Time Series Models and Related Estimation
3. Empirical Likelihood Estimation and its Application to Econometrical Models
4. Simulation Study And Data Analysis

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1. **Introduction**
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3. Empirical Likelihood Estimation and its Application to Econometrical Models
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- Since Owen (1988,1990), empirical likelihood estimation has become a popular tool for estimating the parameters in most well known statistical models without having to assume a known family of distributions for the data.
- The idea behind this method can be depicted in the following:
  - Let $\{x_i\}_{i=1}^n$ be i.i.d. $p \times 1$ random vectors from an unknown distribution F with mean vector μ.
  - The nonparametric EL function is defined by

$$L(F) = \prod_{i=1}^{n} dF(x_i) = \prod_{i=1}^{n} p_i, \tag{1}$$

    where $p_i = dF(x_i) = Pr(X = x_i)$.
  - (1) is maximized by the empirical distribution function $F_n(x) = n^{-1} \sum_{i=1}^{n} 1_{x_i < x}$.

- The empirical likelihood ratio (ELR, hereafter) is given by

$$R(F) = \frac{L(F)}{L(F_n)}, \tag{2}$$

and as a result $R(F) = \prod_{i=1}^{n} np_i$.

- Suppose that the parameter of interest is the mean μ of F. We may write the profile ELR function for the mean as

$$R(\mu) = max\{\prod_{i=1}^{n} np_i | \sum_{i=1}^{n} p_i x_i = \mu, p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}. \tag{3}$$

- Utilizing Lagrange multipliers, we can find that the maximizer of $\prod_{i=1}^{n} np_i$ subject to $\sum_{i=1}^{n} p_i x_i = \mu$, $p_i \geq 0$, and $\sum_{i=1}^{n} p_i = 1$ is given by

$$p_i(\mu) = n^{-1}\{1 + \lambda'(x_i - \mu)\}^{-1}, \qquad (4)$$

where $\lambda = \lambda(\mu)$ is defined according to the following equation

$$\sum_{i=1}^{n} \frac{(x_i - \mu)}{1 + \lambda'(x_i - \mu)} = 0. \qquad (5)$$

- Therefore the log ELR for the mean is

$$l(\mu) = -2\sum_{i=1}^{n} log\{np_i(\mu)\} = 2\sum_{i=1}^{n} log\{1 + \lambda'(x_i - \mu)\}. \qquad (6)$$

- Owen(1988,1990) proved that, under $\mu = \mu_0$, $l(\mu_0)$ converges in distribution to $\mathcal{X}^2_{(p)}$ as n$\rightarrow \infty$, where μ_0 is the true parameter.

- Inspired by the estimating equations used in Maximum likelihood estimation (MLE), Qin and Lawless(1994,1995) connected the theories of empirical likelihood and general estimating equations.
  - Assume that θ is associated with F, and the information about θ and F is available through r estimating equations $g_1, \ldots, g_r$ such that $E_F\{g_j(x,\theta)\} = 0$ for j=1,. . . ,r.
  - When the case is $p = r$, the log ELR statistic becomes

$$l(\theta) = 2\sum_{i=1}^{n} log\{1 + \lambda' g(x_i,\theta)\}, \tag{7}$$

where $g(x_i,\theta) = (g_1(x_i,\theta), \ldots, g_r(x_i,\theta))'$, and the Lagrange multiplier λ satisfies

$$\sum_{i=1}^{n} \frac{g(x_i,\theta)}{1 + \lambda' g(x_i,\theta)} = 0. \tag{8}$$

- Mykland(1995) generalized the definition of the empirical likelihood for i.i.d. data to statistical models with a martingale structure by using the concept of dual likelihood.
  - It is to find a ”score function” $m(\theta)$ which is a martingale for the true value of the parameter θ such that $m(\hat{\theta}) = 0$, where $\hat{\theta}$ is the estimated value of θ.
- Kitamura(1997) considered the empirical likelihood for generalized estimating equation(GEE) and extended the results of Qin and Lawless(1994) to blockwise EL for weakly dependent processes.
  - The method of blockwise empirical likelihood can be used to deal with the dependence for time series data.
  - The blockwise ELR has also been shown to be asymptotically chi-squared distributed.

- Monti(1997) applied the EL method to ARMA models in spectral domain by Whittle's approach.
- Chan and Chuang(2002) applied EL estimation to unstable and stable AR models with innovations that form a sequence of martingale differences.
  - They set the score function to be zero to be as the estimating equation by using the conditional least squares estimation.
  - They also obtain the limiting properties of the log ELR statistic for the stable and unstable AR processes.

Introduction

- In recent decades, some econometrical models have been exploited to modeling the volatility emerging in financial markets.
- Volatility, which stands for the conditional deviation of the underlying asset, is an important factor in option pricing.
- Since Engle (1982) proposed the celebrated ARCH models, a lot of studies and generalizations on this models have been developed, to name a few, the GARCH model of Bollerslev (1986), the EGARCH model of Nelson (1991), among others.

- Apropos of the estimation methods, for example:
  - Weiss(1986) studied the asymptotic properties of quasi-maximum likelihood estimation(QMLE) for the ARCH model.
  - Brorsen(1995) adopted MLE to estimate the parameters in a GARCH model where the residuals have a conditional stable distribution.
  - Cheng and Tsay(2005) used generalized method of moments(GMM) to estimate the parameters in EGARCH model.

- In this thesis, we'll use the empirical likelihood method to estimate parameters in conditional heteroskedasticity models.
- We will describe how to choose the estimating equations in (7) for time series models.

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AR(P) Models:

- Let $\{Y_t\}_{t\geq 1}$ be an AR(p) process given by

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + Z_t, \tag{9}$$

  where $p \geq 1$ is a positive integer and $\{Z_t\}_{t\geq 1}$ is a sequence of white noises with mean zero and variance $\sigma^2 > 0$.
- Set $X_t = Y_t - \mu$, where $\mu = \frac{\phi_0}{1-\phi_1-\cdots-\phi_p}$ is the expectation of Y_t, $\forall t$. Then the equation (9) can be rewritten by

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t. \tag{10}$$

- Multiplying each side of (10) by $X_{t-j}, j = 0, \ldots, p$, and taking expectations, we obtain the Yule-walker equations,

$$\Gamma_p \phi = \gamma_p, \tag{11}$$

and

$$\sigma^2 = \gamma(0) - \phi' \gamma_p, \tag{12}$$

where $\phi = (\phi_1, \ldots, \phi_p)'$ and Γ_p is the covariance matrix $[\gamma(i-j)]_{i,j=1}^{p}$ and $\gamma_p = (\gamma(1), \gamma(2), \ldots, \gamma(p))'$.

- To estimate ϕ, we obtain a set of equations for the estimators $\hat{\phi}$ and $\hat{\sigma}^2$ of ϕ and σ^2, namely

$$\hat{\Gamma}_p \hat{\phi} = \hat{\gamma}_p, \tag{13}$$

and

$$\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\phi}' \hat{\gamma}_p. \tag{14}$$

Dividing each side of (13) by $\hat{\gamma}(0)$, we obtain

$$\hat{\phi} = \hat{R}_p^{-1}\hat{\rho}_p \tag{15}$$

and

$$\hat{\sigma}^2 = \hat{\gamma}(0)[1 - \hat{\rho}_p'\hat{R}_p^{-1}\hat{\rho}_p], \tag{16}$$

where $\hat{\rho}_p = (\hat{\rho}(1), \ldots, \hat{\rho}(p))' = \hat{\gamma}_p/\hat{\gamma}(0)$ and $\hat{R}_p = \hat{\gamma}_p/\hat{\gamma}(0)$.

ARMA Models:

- Let $\{X_t\}_{t\geq 1}$ be an *ARMA*(p,q) process defined by

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}, \quad (17)$$

  where $\{Z_t\} \sim WN(0, \sigma^2)$.
- If the distribution of $\{Z_t\}$ is known, we may obtain the maximum likelihood estimates of the parameter vectors $\phi = (\phi_1, \ldots, \phi_p)'$, $\theta = (\theta_1, \ldots, \theta_q)'$ and that of the white noise variance σ^2.

- The one-step predictors $\hat{X}_{i+1}$ and their mean squared errors are then given by,

$$\hat{X}_{i+1} = \begin{cases} \sum_{j=1}^{i} \theta_{ij}(X_{i+1-j} - \hat{X}_{i+1-j}), & 1 \leq i < m = max(p,q), \\ \phi_1 X_i + \cdots + \phi_p X_{i+1-p} + \sum_{j=1}^{q} \theta_{ij}(X_{i+1-j} - \hat{X}_{i+1-j}), & i \geq \end{cases}$$

and

$$E(X_{i+1} - \hat{X}_{i+1})^2 = \sigma^2 \gamma_i, \quad (19)$$

where $\theta_{n,n-k} = \nu_k^{-1}(\kappa(n+1,k+1) - \sum_{j=0}^{k-1} \theta_{k,k-j}\theta_{n,n-j}\nu_j)$, $k = 0, 1, \ldots, n-1$, $\nu_k = \| X_{n+1} - \hat{X}_{n+1} \|^2$ and $\kappa(i,j) = E(X_i X_j)$.

- The Gaussian likelihood of the vector of observations $X = (X_1, \ldots, X_n)'$ is given by

$$L(\phi, \theta, \sigma^2) = (2\pi\sigma^2)^{-n/2}(\gamma_0 \ldots \gamma_{n-1})^{-1/2} exp[-\frac{1}{2}\sigma^2 \sum_{j=1}^{n}(X_j - \hat{X}_j)^2/\gamma \qquad (20)$$

- Differentiating $lnL(\phi, \theta, \sigma^2)$ partially with respect to σ^2 and noting that $\hat{X}_j$ and γ_j are independent of σ^2, we deduce that the maximum likelihood estimators $\hat{\phi}, \hat{\theta}$ and $\hat{\sigma}^2$ satisfying

$$\hat{\sigma}^2 = n^{-1}S(\hat{\phi}, \hat{\theta}), \qquad (21)$$

where

$$S(\hat{\phi}, \hat{\theta}) = \sum_{j=1}^{n}(X_j - \hat{X}_j)^2/\gamma_{j-1}, \qquad (22)$$

and $\hat{\phi}, \hat{\theta}$ are the minimizer of

$$l(\phi, \theta) = ln(n^{-1}S(\phi, \theta)) + n^{-1}\sum_{j=1}^{n} ln\gamma_{j-1}. \quad (23)$$

- An alternative procedure of estimation is to minimize the weighted sum of squares

$$S(\phi, \theta) = \sum_{j=1}^{n}(X_j - \hat{X}_j)^2/r_{j-1}, \quad (24)$$

with respect to ϕ and θ. The estimators obtained in this way is referred to as the “generalized least squares” (GLS) estimators of ϕ and θ.

ARMA Models

- The spectral domain approach is also popular in the estimations for ARMA models. In next section, we'll introduce the spectral-density approach proposed by Monti (1997) and illustrate how to applied EL estimation combined with Monti's method to estimating GARCH models.

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ARCH Models:

- An $ARCH(p)$ model is given by

$$a_t = \sigma_t \varepsilon_t \quad (25)$$
$$\sigma_t^2 = \beta_0 + \beta_1 a_{t-1}^2 + \cdots + \beta_p a_{t-p}^2 \quad (26)$$

where $\{\varepsilon_t\}_t \geq 0$ is a sequence of i.i.d.random variables with $E(\varepsilon_t) = 0$ and $Var(\varepsilon_t) = 1$.

- The ARCH models are designed to explain the following facts:
    - The mean corrected asset return is serially uncorrelated but not independent.
    - When properly transformed by a nonlinear functional, e.g. quadratic, absolute value, or indicator function, the dependence of the returns can be observed.

- One of the difficulties in studying volatility is that it can not be observed directly. However, volatility has some important characteristics:
  - There exist volatility clusters.
  - Volatility evolves over time in a continuous manner, i.e., it rarely jumps.
  - Volatility does not diverge.
- Therefore, it is reasonable to assume that volatility is stationary and can be fitted into an ARMA-like model.
- Set $\eta_t = a_t^2 - \sigma_t^2$, then $\{\eta_t\}$ form a sequence of martingale difference with respect to an increasing sequence of σ-fields $F_t = \sigma\{\varepsilon_s, s \leq t\}, t \in Z^+$.
- Substituting $\sigma_t^2 = a_t^2 - \eta_t$ into (26), (26) becomes

$$a_t^2 = \beta_0 + \beta_1 a_{t-1}^2 + \cdots + \beta_p a_{t-p}^2 + \eta_t, \tag{27}$$

It turns out that the sequance a_t^2 satisfies an $AR(p)$ model driven by the martingale differences $\{\eta_t\}$.

- The unknown parameter $\beta = (\beta_1, \cdots, \beta_p)'$ can be estimated by the conditional least squares(CLS) estimate $\hat{\beta}$ which maximizes the following statistics

$$-\frac{1}{2}\sum_{t=p+1}^{n}(a_t^2 - E(a_t^2 \mid F_{t-1}))^2 = -\frac{1}{2}\sum_{t=p+1}^{n}(a_t^2 - \beta' A_{t-1})^2, \tag{28}$$

where $A_t = (a_t^2, \cdots, a_{t-p+1}^2)'$.

- We denote the estimate obtained by CLS by $\hat{\beta}_{CLS}$ and we have

$$\hat{\beta}_{CLS} = (\sum_{t=p+1}^{n} A_{t-1}A_{t-1}')^{-1}\sum_{t=p+1}^{n} A_{t-1}a_t^2. \tag{29}$$

- Partial differentiating (28) with respect to β yields the score function

$$\sum_{t=p+1}^{n}(a_t^2 - \beta' A_{t-1})A_{t-1} = \sum_{t=p+1}^{n} m_t,$$

where $m_t = (a_t^2 - \beta' A_{t-1})A_{t-1}$.

- Let β_0 denote the true value for β. When $\beta = \beta_0$, $m_t = \eta_t A_{t-1}$ forms a sequence of martingale differences, and the score function then forms a martingale.
- It can be seen by Qin and Lawless (1994,1995) that the log ELR statistic of the autoregressive model is

$$l(\beta) = 2 \sum_{t=p+1}^{n} log(1 + \lambda' m_t), \qquad (30)$$

where λ satisfies

$$\sum_{t=p+1}^{n} \frac{m_t}{1 + \lambda' m_t} = 0. \qquad (31)$$

- For AR(p) models, Chuang and Chan (2002) obtained some asymptotic results for stable and unstable processes. Moreover, they applied EL to estimate the corresponding coefficients emerging in the general AR models disturbed by non-i.i.d. martingale-difference noises.
- Let

$$\phi(z) = 1 - \beta_1 z - \cdots - \beta_p z^p \tag{32}$$

denote the characteristic polynomial of the autoregressive model (27).Assume that the sequence $\{\eta_t\}_{t\geq 1}$ satisfies the moment condition $\sup_{t\geq p} E(|\eta_t|^{2+\alpha} \mid F_{t-1}) < \infty$ for some $\alpha > 0$. Similar to Chuang and Chan (2002), we may prove the following results.

Lemma 1.

Assume all roots of (32) lie outside the unit circle. Then

(i) $(\sum A_{t-1}A'_{t-1})^{\frac{1}{2}}(\hat{\beta}-\beta)$ converges in distribution to a multivariate normal distribution with mean 0 and covariance matrix $\sigma^2 I_p$, where I_p is the $p \times p$ identity matrix.

(ii) $l(\beta)$ given by (30) converges in distribution to $\mathcal{X}_p^2$.

Lemma 2.

Assume all roots of (32) lie either on or outside the unit circle. Let $\hat{\sigma}^2 = n^{-1}\sum(a_t^2 - \hat{\beta}' A_{t-1})$, then $\hat{\sigma}^2 \longrightarrow \sigma^2$ a.s.

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Lemma 3.

Let $0 < \alpha' < \alpha$.

(i) $\max_{p+1 \leq t \leq n} |\eta_t| = o(n^{\frac{1}{2+\alpha'}})$ a.s.

(ii) $n^{-1} \sum \eta_t^2 \longrightarrow \sigma^2$ a.s.

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Lemma 4.

(i) Let $M_{tn}, t = 1, \cdots, k_n$, be a $p \times 1$ martingale difference array adapted to a sequence of filtrations $\mathcal{G}_{tn}$ for each n. Let $U_{nn} = \sum_{t=1}^{k_n} M_{tn}M'_{tn}$ and $V_{nn} = \sum_{t=1}^{k_n} E(M_{tn}M'_{tn} \mid \mathcal{G}_{t-1,n})$. Suppose that $\sup_n P(\|V_{nn}\| > a) \to 0$ when $a \to \infty$ and for all $\delta > 0$, $\sum_{t=1}^{k_n} E(\|M_{tn}\|^2 1(\|M_{tn}\| > \delta) \mid \mathcal{G}_{t-1,n}) \to 0$ in probability. Then $V_{nn} - U_{nn} \to 0$ in probability.

(ii) Let X_t be random variables and $\mathcal{F}_t$ be a filtration. Suppose $\sup_t E(|X_t|^p \mid \mathcal{F}_{t-1}) < \infty$ a.s. for $p > 1$. If $\max_{1 \le t \le n} P(S_t \mid \mathcal{F}_{t-1}) \to 0$ a.s., then $\max_{1 \le t \le n} E(|X_t| 1\{S_t\} \mid \mathcal{F}_{t-1}) \to 0$ a.s.

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- Follow Chan and Wei (1988) and Chuang and Chan (2002) factorizing (32), we have $\phi(z) = (1-z)^a(1+z)^b \prod_{k=1}^{l}(1-2\cos\theta_k z + z^2)^{d_k}\psi(z)$, where $a+b+2d_1+\cdots+2d_l = p$. Let $(1-B)^j u_t(j) = \eta_t$, $j = 1,\cdots,a$, $(1+B)^j v_t(j) = \eta_t$, $j = 1,\cdots,b$, $(1-2\cos\theta_k B + B^2)^{d_k} y(d_k)_t(j) = \eta_t$, $j = 1,\cdots,d_k$, and $\psi(B)z_t = \eta_t$.
- Let $\mathbf{y}(d_k)_t = (n^{-1}y(d_k)_t(1), n^{-1}y(d_k)_{t-1}(1),\cdots,n^{-d_k}y(d_k)_t(d_k), n^{-d_k}y(d_k)_{t-1}(d_k))'$ for $k = 1,\cdots,l$. Define the matrices G_n and Q, as in Chan and Wei (1988), so that $(G_nQ)A_t = Y_t$, where $Y_t = (n^{-a}u_t(a),\cdots,n^{-1}u_t(1), n^{-b}v_t(b),\cdots, n^{-1}v_t(1), \mathbf{y}(d_1)'_t,\cdots,\mathbf{y}(d_l)'_t, n^{-\frac{1}{2}}z_t,\cdots,n^{-12}z_{t-q+1})'$. Multiplying A_t by the matrix G_nQ transforms the AR(p) model into its individual components and simplifies the analysis.

- Let $\tilde{\lambda} = (Q'G_n')^{-1}\lambda$, and rewrite (31) as

$$\sum_{t=p+1}^{n} \frac{n_t}{1 + \tilde{\lambda}' n_t} = 0, \qquad (33)$$

where $n_t = (G_n Q) m_t = Y_{t-1}\eta_t$.

Lemma 5.

(i) $\max_{p\le t\le n}\|Y_t\| = O(n^{-\frac{1}{2}}(loglogn)^{\frac{1}{2}})$ a.s.

(ii) $\max_{p+1\le t\le n}\|n_t\| = o(1)$ a.s.

(iii) $\sum Y_{t-1}Y'_{t-1}$ converges in distribution to a nonsingular limit and $\sum\|Y_{t-1}\|^2 = O_p(1)$.

(iv) $\sum n_t n'_t - \sum Y_{t-1}Y'_{t-1}\sigma^2$ converges in probability to 0.

(v) Let σ_n be the smallest eigenvalue of $\sum n_t n'_t$. Then σ_n converges in distribution to a strictly positive random variable.

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(v) Let σ_n be the smallest eigenvalue of $\sum n_t n'_t$. Then σ_n converges in distribution to a strictly positive random variable.

- Let $Q(\beta) = \sum_{t=p+1}^{n} m_t'(\sum_{t=p+1}^{n} m_t m_t')^{-1} \sum_{t=p+1}^{n} m_t$ and $S(\beta) = \hat{\sigma}^{-2}(\hat{\beta} - \beta)' \sum \mathbf{y}_{t-1}\mathbf{y}_{t-1}'(\hat{\beta} - \beta)$.
- Analogous to Chan and Chuang (2002), with Lemma 1 to Lemma 5, we have the following theorem.

Theorem 1.

Assume all roots of (32) lie either on or outside the unit circle, with at least one root lying on the unit circle. Then $Q(\beta)$, $S(\beta)$, and $l(\beta)$ all have the same limiting distribution.

GARCH Models:

- Bollerslev(1986) generalized ARCH models to the Generalized ARCH models (GARCH).
  The $GARCH(p, q)$ model is given by

$$a_t = \sigma_t \varepsilon_t \quad (34)$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i a_{t-i}^2 + \sum_{j=1}^{q} \theta_j \sigma_{t-j}^2, \quad (35)$$

  where $\{\varepsilon_t\}$ is a sequence of i.i.d. random variables with mean zero and variance 1, $\alpha_0 > 0, \alpha_i \geq 0, \theta_j \geq 0$, and $\sum_{i=1}^{max(p,q)} (\alpha_i + \theta_j) < 1$.
- Set $\eta_t = a_t^2 - \sigma_t^2$. Then $\{\eta_t\}$ form a sequence of martingale differences with respect to a stochastic basis $F_t = \sigma\{\varepsilon_s, s \leq t\}, t \geq 1$.

- Substituting $\sigma_t^2 = a_t^2 - \eta_t$ into (35), we can rewrite the *GARCH* model as

$$a_t^2 = \alpha_0 + \sum_{i=1}^{max(p,q)} (\alpha_i + \theta_i) a_{t-i}^2 + \eta_t - \sum_{j=1}^{q} \theta_j \eta_{t-j}. \tag{36}$$

It is an *ARMA* form for the squared series a_t^2. Thus, a *GARCH* model can be regarded as an application of an *ARMA* model to the squared series a_t^2.

- We can rewrite (36) as

$$a_t^2 - \sum_{i=1}^{max(p,q)} (\alpha_i + \theta_i) a_{t-i}^2 = \alpha_0 + \eta_t - \sum_{j=1}^{q} \theta_j \eta_{t-j}. \tag{37}$$

Let $\phi_i = \alpha_i + \theta_i$, then (37) can be written in the form

$$\phi(B) a_t^2 = \alpha_0 + \theta(B) \eta_t, \tag{38}$$

where B is the backward operator($B\eta_t = \eta_{t-1}$),
$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$ and
$\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$ are admissible autoregressive and moving average operators.

- Monti (1997) proposed an EL method combined with periodogram to estimate the parameters appearing in ARMA models.
  Let $z_1, z_2, \ldots, z_T$ be T observations of the *ARMA* process and let $\bar{z}$ be their sample mean. Then the periodogram ordinate corresponding to frequency $\omega_j = 2\pi j/T$, for $j = 1, 2, \ldots, T-1$, is given by

$$I(\omega_j) = \frac{1}{2\pi T}[\{\sum_{t=1}^{T}(z_t - \bar{z})sin(\omega_j t)\}^2 + \{\sum_{t=1}^{T}(z_t - \bar{z})cos(\omega_j t)\}^2]. \tag{39}$$

- Since $I(\pi+\lambda) = I(\pi-\lambda)$, one can restrict his attention to the frequencies ω_j for $j = 1, 2, \ldots, n$, $n = [(T-1)/2]$. The spectral density function is given by

$$g(\omega, \beta) = \frac{\sigma^2}{2\pi} \frac{|\theta\{exp(-i\omega)\}|^2}{|\phi\{exp(-i\omega)\}|^2}, \tag{40}$$

where $\omega \in [-\pi, \pi]$ and $\beta = (\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q, \sigma^2)'$. Let $\beta = (\beta_{(1)}, \sigma^2)$, where $\beta_{(1)}$ is the parameter of interest. An approximating log-likelihood function is given by (Whittle,1953)

$$ln\{L(\beta)\} = -\sum_{j=1}^{n} ln\{g_j(\beta)\} - \sum_{j=1}^{n} \frac{I(\omega_j)}{g_j(\beta)}. \tag{41}$$

- ψ-functions is given by

$$\psi_j\{I(\omega_j),\beta\} = \{\frac{I(\omega_j)}{g_j(\beta)} - 1\}\frac{\partial ln\{g_j(\beta)\}}{\partial\beta}. \quad (42)$$

Thus, after maximization of (41) with respect to σ^2, the spectral log-likelihood function becomes

$$ln\{\hat{L}(\beta_{(1)})\} = -nln\{n^{-1}\sum_{j=1}^{n}\frac{I(\omega_j)}{g_j^1(\beta_{(1)})}\} - \sum_{j=1}^{n} ln\{g_j^1(\beta_{(1)})\}, \quad (43)$$

where

$$g_j^1(\beta_{(1)}) = \frac{1}{2\pi}\frac{|\theta\{exp(-i\omega_j)\}|^2}{|\phi\{exp(-i\omega_j)\}|^2}. \quad (44)$$

- The estimator of $\beta_{(1)}$ is the M-estimator corresponding to the ψ-function

$$\tilde{\psi}_j\{I(\omega_j), \beta_{(1)}\} = \frac{I(\omega_j)}{g_j^1(\beta_{(1)})}\left[\frac{\partial ln\{g_j^1(\beta_{(1)})\}}{\partial \beta_{(1)}} - n^{-1}\sum_{j=1}^{n} \frac{\partial ln\{g_j^1(\beta_{(1)})\}}{\partial \beta_{(1)}}\right]. \quad (45)$$

- Thus, it can be seen that the empirical likelihood ratio statistic of the *ARMA* model is

$$l(\beta_{(1)}) = 2\sum_{j=1}^{n} ln[1 + \hat{\xi}(\beta_{(1)})'\tilde{\psi}_j\{I(\omega_j), \beta_{(1)}\}], \quad (46)$$

where $\hat{\xi}(\beta_{(1)})$ satisfies

$$\sum_{j=1}^{n} \frac{\tilde{\psi}_j\{I(\omega_j), \beta\}}{1 + \hat{\xi}(\beta_{(1)})'\tilde{\psi}_j\{I(\omega_j), \beta_{(1)}\}} = 0, \quad (47)$$

- After estimating the parameter vector $\beta_{(1)} = (\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q)'$, we can use the unconditional mean of an *ARMA* model

$$E(a_t^2) = \frac{\alpha_0}{1 - \sum_{i=1}^{max(p,q)} (\alpha_i + \theta_i)} \tag{48}$$

  to obtain an estimate of $\hat{\alpha_0}$.
- Thanks to Qin and Lawless's (1994) idea, we may have another option on estimating the parameters in the GARCH models. The crucial step is to find the estimating equations.

- Consider the GARCH(1,1) model which can be written as below:

$$a_t^2 = \alpha_0 + (\alpha_1 + \theta_1)a_{t-1}^2 + \eta_t - \theta_1\eta_{t-1}. \tag{49}$$

First, we estimate the coefficients of the AR part in an ARMA model by constructing instrumental variables. (49) can be rewritten as

$$a_t^2 - (\alpha_1 + \theta_1)a_{t-1}^2 - \alpha_0 = u_t, \tag{50}$$

where $u_t = \eta_t - \theta_1\eta_{t-1}$.

- By utilizing the MA(1) structure of u_t, we have the following moment conditions

$$Ef_i(a_t^2, \alpha_0, \phi_1) = 0, \quad \text{for } i = 1, 2,$$

, where $\phi_1 = \alpha_1 + \theta_1$ and

$$f_1(a_t^2, \alpha_0, \phi_1) = (a_t^2 - \phi_1 a_{t-1}^2 - \alpha_0)a_{t-m},$$
$$f_2(a_t^2, \alpha_0, \phi_1) = (a_t^2 - \phi_1 a_{t-1}^2 - \alpha_0)a_{t-n},$$

for $m \neq n$ and $m, n \geq 2$. With the estimating equations, we can adopt EL estimation to obtain estimates $\hat{\phi}_1, \hat{\alpha}_0$ of ϕ_1 and α_0.

- After estimating the AR coefficients, we can estimate the MA coefficients in model (49). Consider

$$y_t = \eta_t - \theta_1 \eta_{t-1},$$

where $y_t = a_t^2 - \hat{\phi}_1 a_{t-1}^2 - \hat{\alpha}_0$ and $\eta_t \sim i.i.d.(0, \sigma^2)$. We may also obtain two moment conditions

$$Ef_i(y_t, \theta_1, \sigma^2) = 0, \quad for\, i = 1, 2,$$

where

$$f_1(y_t, \theta_1, \sigma^2) = y_t y_{t-1} + \sigma^2 \theta_1,$$
$$f_2(y_t, \theta_1, \sigma^2) = y_t^2 - \sigma^2(1 + \theta_1^2).$$

The above equations provide themselves as being the estimating equations for estimating θ_1 and σ^2.

- Unfortunately, according to our simulation results, such an approach is not good compared with EL estimation.

EGARCH Models:

Nelson (1991) proposed the EGARCH models.
Consider the EGARCH(1,1) model :

$$a_t = \sigma_t \varepsilon_t \tag{51}$$

$$ln(\sigma_t^2) = \alpha_0 + \beta_0 ln(\sigma_{t-1}^2) + g(\varepsilon_{t-1}), \tag{52}$$

where $\{\varepsilon_t\}$ is a sequence of i.i.d. random variables with $E(\varepsilon_t) = 0$ and $E(\varepsilon_t^2) = 1$, $g(\varepsilon_t) = \omega_0 \varepsilon_t + \phi_0[|\varepsilon_t| + E(|\varepsilon_t|)]$, and $\alpha_0, \beta_1, \omega$, and ϕ are real numbers.

The EGARCH Models

Let $\theta = [\alpha, \beta, \omega, \phi]^T$ and the true value be $\theta_0 = [\alpha_0, \beta_0, \omega_0, \phi_0]^T$. The four moment conditions for the EGARCH(1,1) model are $E[f_{t,1}(\theta_0)] = 0$, $E[f_{t,2}(\theta_0)] = 0$, $E[f_{t,3}(\theta_0)] = 0$, $E[f_{t,4}(\theta_0)] = 0$, where

$$f_{t,1}(\theta) = ln a_t^2 - \beta ln a_{t-1}^2 - \alpha - E(ln \varepsilon_t^2) + \beta E(ln \varepsilon_t^2), \quad (53)$$

$$f_{t,2}(\theta) = (ln a_t^2 - \beta ln a_{t-1}^2 - \alpha)^2 - (1 + \beta^2) E(ln \varepsilon_t^2)^2 - E(g(\varepsilon_t))^2 + 2\beta E(g(\varepsilon_t) ln \varepsilon_t^2) + 2\beta [E(ln \varepsilon_t^2)]^2, \quad (54)$$

The EGARCH Models

$$
\begin{aligned}
f_{t,3}(\theta) &= (lna_t^2 - \beta lna_{t-1}^2 - \alpha)^3 - (1-\beta^3)E(ln\varepsilon_t^2)^3 \\
&\quad -3\beta(\beta-1)E(ln\varepsilon_t^2)^2E(ln\varepsilon_t^2) + 6\beta E(ln\varepsilon_t^2)E(g(\varepsilon_t)ln\varepsilon_t^2) \\
&\quad -\beta^2E(g(\varepsilon_t)(ln\varepsilon_t^2)^2) + E(g(\varepsilon_t))^3 - 3E(g(\varepsilon_t))^2E(ln\varepsilon_t^2) \\
&\quad +3\beta E((g(\varepsilon_t))^2ln\varepsilon_t^2), \qquad (55) \\
f_{t,4}(\theta) &= (lna_t^2 - \beta lna_{t-1}^2 - \alpha)^4 - (1+\beta^4)E(ln\varepsilon_t^2)^4 \\
&\quad +4\beta(1+\beta^2)E(ln\varepsilon_t^2)^3E(ln\varepsilon_t^2) - 4E(ln\varepsilon_t^2)^3E(g(\varepsilon_t)) \\
&\quad -6E(ln\varepsilon_t^2)^2E(g(\varepsilon_t))^2 - E(g(\varepsilon_t))^4 - 4E(ln\varepsilon_t^2)E(g(\varepsilon_t))^3 \\
&\quad +12\beta E(ln\varepsilon_t^2)^2E(g(\varepsilon_t ln\varepsilon_t^2)) - 12\beta E(ln\varepsilon_t^2)E(g(\varepsilon_t)ln\varepsilon_t^2) \\
&\quad +4\beta E((g(\varepsilon_t))^3ln\varepsilon_t^2) - 12\beta^2E(ln\varepsilon_t^2)E(g(\varepsilon_t)(ln\varepsilon_t^2)^2) \\
&\quad -6\beta^2E(g(\varepsilon_t ln\varepsilon_t^2))^2 - 6\beta^2[E(ln\varepsilon_t^2)^2]^2 \\
&\quad +4\beta^3E(g(\varepsilon_t)(ln\varepsilon_t^2)^3). \qquad (56)
\end{aligned}
$$

(53)∼(56) can then be treated as estimating equations and we can apply EL method to the underlying EGARCH(1,1) model.

TGARCH Models:

Zakoian(1994) introduced the TGARCH models.
A TGARCH(p,q) is defined by

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} (\alpha_i + \gamma_i N_{t-i}) a_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, \qquad (57)$$

where N_{t-i} is an indicator for negative a_{t-i}, that is,

$$N_{t-i} = \begin{cases} 1 & \text{if } a_{t-i} < 0, \\ 0 & \text{if } a_{t-i} \geq 0, \end{cases}$$

and α_i, γ_i, and β_i are nonnegative parameters.

We consider a TGARCH(1,1) model:

$$\begin{aligned} a_t &= \sigma_t \varepsilon_t, \\ \sigma_t^2 &= \alpha_0 + (\alpha_1 + \gamma_1 N_{t-1}) a_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \end{aligned}$$

If we assume that $\{\varepsilon_t\}_{t=1,\ldots,T}$, is t distributed with degree ν, then the likelihood function can be written as

$$L(\alpha_0, \alpha_1, \gamma_1, \beta_1) = \prod_{t=1}^{T} \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \frac{1}{\sqrt{\nu\pi}} (1 + \frac{\varepsilon_t^2}{\nu})^{-(\nu+1)/2}. \tag{58}$$

Taking the logarithm on both sides of (58) and differentiating $log\, L(\alpha_0, \alpha_1, \gamma_1, \beta_1)$ with $\alpha_0, \alpha_1, \gamma_1$, and β_1 respectively, we obtain four estimating equations. Thus, we can estimate the parameters, by EL estimation, in the TGARCH model.

Outline

1. Introduction
2. Survey of Time Series Models and Related Estimation
3. Empirical Likelihood Estimation and its Application to Econometrical Models
4. **Simulation Study And Data Analysis**

- 300 replicates of random samples each with size N=1000 are generated for each Example 1$\sim$4.
- In Example5, we'll model the volatility of WTI crude oil prices by a GARCH model.

Example 1.

Model–ARCH(2)

$$a_t = \sigma_t \varepsilon_t,$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \alpha_2 a_{t-2}^2,$$

where $\alpha_0 = 0.2$, $\alpha_1 = 0.28$, and $\alpha_2 = 0.12$.

Table 1. Estimated parameters of an ARCH(2) model by empirical likelihood estimation

Parameter value	Estimated value	95% confidence interval
α_0 (0.02)	0.0199	(0.0197, 0.0201)
α_1 (0.28)	0.2774	(0.2714, 0.2834)
α_2 (0.12)	0.1206	(0.1156, 0.1257)

Table 2. Estimated parameters of an ARCH(2) model by least squares estimation

Parameter(value)	Estimated value	95%confidence interval
α_0 (0.02)	0.0199	(0.0198, 0.0200)
α_1 (0.28)	0.2798	(0.2784, 0.2813)
α_2 (0.12)	0.1201	(0.1187, 0.1216)

Example 2.

Model–GARCH(1,1)

$$a_t = \sigma_t \varepsilon_t,$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \theta_1 \sigma_{t-1}^2,$$

where $\alpha_0 = 0.05$, $\alpha_1 = 0.37$, and $\theta_1 = 0.18$.

Table 3. Estimated parameters of a GARCH(1,1) model by EL estimation without knowing the distribution of the innovations

Parameter(value)	Estimated value	95% confidence interval
α_0 (0.05)	0.0505	(0.0496, 0.0513)
α_1 (0.38)	0.3735	(0.3679, 0.3791)
θ_1 (0.18)	0.1784	(0.1740, 0.1827)

Table 4. Estimated parameters of a GARCH(1,1) model by MLE provided that the innovations are i.i.d. normally distributed

Parameter(value)	Estimated value	95% confidence interval
α_0 (0.05)	0.0497	(0.0494, 0.0500)
α_1 (0.38)	0.3703	(0.3691, 0.3715)
θ_1 (0.18)	0.1809	(0.1797, 0.1821)

Example 3.

Model–EGARCH(1,1)

$$a_t = \sigma_t \varepsilon_t,$$

$$ln(\sigma_t^2) = \alpha_0 + \beta_0 ln(\sigma_{t-1}^2) + \omega_0 \varepsilon_{t-1} + \phi_0[|\varepsilon_{t-1}| + E(|\varepsilon_{t-1}|)],$$

where $\alpha_0 = 0.05$, $\beta_0 = 0.63$, $\omega_0 = 0.07$, and $\phi_0 = 0.15$.

Table 5. Estimated parameters of a EGARCH(1,1) model by EL estimation without knowing the distribution of the innovation

Parameter(value)	Estimated value	95% confidence interval
α_0 (0.05)	0.0486	(0.0465, 0.0507)
β_0 (0.63)	0.6280	(0.6226, 0.6334)
γ_0 (0.07)	0.0705	(0.0685, 0.0726)
ϕ_0 (0.15)	0.1540	(0.1496, 0.1584)

Table 6. Estimated parameters of a EGARCH(1,1) model by MLE provided that the innovations are i.i.d. normally distributed

Parameter(value)	Estimated value	95% confidence interval
α_0 (0.05)	0.0499	(0.0496, 0.0501)
β_0 (0.63)	0.6298	(0.6296, 0.6301)
γ_0 (0.07)	0.0700	(0.0697, 0.0702)
ϕ_0 (0.15)	0.1500	(0.1497, 0.1502)

Example 4.

Model–TGARCH(1,1)

$$a_t = \sigma_t \varepsilon_t,$$

$$\sigma_t^2 = \alpha_0 + (\alpha_1 + \gamma_1 N_{t-1}) a_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

where $\alpha_0 = 0.05$, $\alpha_1 = 0.33$, $\gamma_1 = 0.15$, and $\beta_1 = 0.2$, and we assume $\{\varepsilon_t\}$ is a student-t distribution with degree $\nu = 5$.

Table 7. Estimated parameters of a TGARCH(1,1) model by empirical likelihood estimation

Parameter(value)	Estimated value	95% confidence interval
α_0 (0.05)	0.0515	(0.0489, 0.0541)
α_1 (0.33)	0.3339	(0.3237, 0.3442)
γ_1 (0.15)	0.1488	(0.1364, 0.1613)
β_1 (0.20)	0.1995	(0.1850, 0.2140)

Table 8. Estimated parameters of a TGARCH(1,1) model by MLE

Parameter(value)	Estimated value	95% confidence interval
α_0 (0.05)	0.0499	(0.0497, 0.0502)
α_1 (0.33)	0.3297	(0.3289, 0.3304)
γ_1 (0.15)	0.1499	(0.1491, 0.1507)
β_1 (0.20)	0.1998	(0.1991, 0.2006)

- From the above outcomes, although the confidence intervals obtained by the empirical likelihood method are wider than those by MLE or OLS, the estimates by EL are also close to true parameters.
- The best advantage using EL estimation is that it only exploit some moment conditions which are easier to obtain than the distributional information required for MLE.
- Moreover, OLS is intractable in many situations, e.g. in GARCH, EGARCH or TGARCH models.
  Therefore, EL approach provides itself as an option to estimate parameters in more complicated models.

Example 5.

- We use the data from the West Texas Intermediate Crude Oil (WTI) Prices.
- The WTI data was collected from 2007 to 2008.
- Let P_t be the price of WTI at time index t.The simple return is defined by $R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$.
- An GARCH(1,1) model is used to model the volatility of the return sequence derived by the WTI crude oil prices.
- We estimate the parameters in GARCH(1,1) model by using MLE (provided that the innovations are i.i.d. normally distributed) and EL estimation (without knowing the distribution of the innovations), respectively .

Table 9. Estimated parameters of a GARCH(1,1) model for the WTI data(2007/1/3 ∼ 2008/12/31) by maximum likelihood estimation

Parameter	value	Standard Error	T Statistic
α_0	7.3384e-006	5.5186e-006	1.3298
α_1	0.90922	0.023712	38.3452
θ_1	0.090776	0.02142	4.2379

Log Likelihood Value: 1161.37

Table 10. Estimated parameters of a GARCH(1,1) model for the WTI data(2007/1/3 ∼ 2008/12/31) by empirical likelihood estimation

Parameter	Value
α_0	4.7817e-007
α_1	0.8993
θ_1	0.0877

- The estimates by these two distinct methods are quite close.
- The outcomes show that the fitted model might have a unit root.

Thank you for your listening!