An Introduction to Mathematical Models of Neuronal Networks

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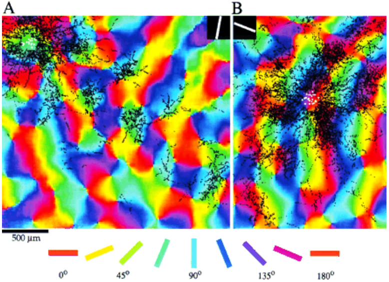
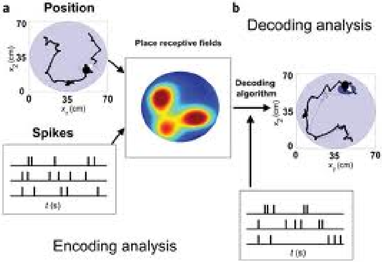
How is brain working?

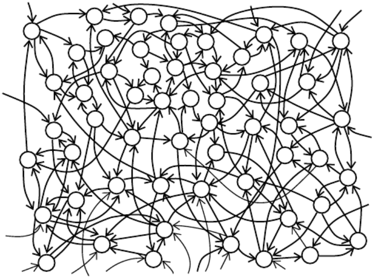
- Novel experiments are revealing the mystery of how brain is working.
- Imaging techniques: fMRI, EEG, Chemical imaging, etc.
- Multi-electrode recording of parallel spiking of population of neurons.





How is brain working?

- How do underlying neuronal networks perform computations?
- How can we reveal the role of network connectivity?

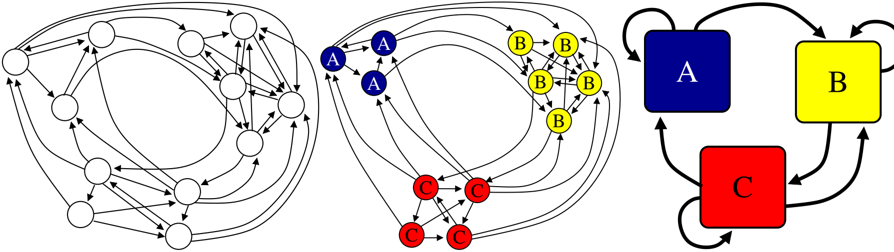


Theoretical approach

- One possible way of tracking principles or mechanisms is to propose mathematical models of neural activity at different space and time scales.
- From analyzing or simulating models to extract new ideas that can be tested in real experiments.

Coarse graining

- Basic assumption for mean-field models



Wilson-Cowan models

- Neural field model
- Ad hoc averaging approach
- Invalid predictions for synchrony

$$\tau_e \frac{dE}{dt} + E = \sigma_e S_e(W_{EE}E - W_{EI}I + P_E)$$

$$\tau_i \frac{dI}{dt} + I = \sigma_i S_i(W_{IE}E - W_{II}I + P_I)$$

$$S_{e/i}(x) = \frac{1}{1 + exp(\frac{(\theta_{e/i} - x)}{K_{e/i}}}$$

Population density models

- Spike-density approach
- Quick simulation of spiking networks
- Capturing the reset mechanism of neurons

$$\frac{\partial \rho^k}{\partial t} = -\frac{\partial J^k}{\partial v} + \delta(v - v_{reset}) J^k(v_{th}, t - \tau_{ref})$$

$$J^k(v,t) = J_l^k(v,t) + J_e^k(v,t) + J_i^k(v,t)$$

$$J_l^k(v,t) = -\frac{1}{\tau}(v - E_r)\rho^k(v,t)$$

$$J_e^k(v,t) = \nu_e^k(t) \int_{E_i}^{v} F_e\left(\frac{v - v'}{E_e - v'}\right)\rho^k(v',t)dv'$$

$$J_i^k(v,t) = \nu_i^k(t) \int_{v}^{v_{th}} F_i\left(\frac{v - v'}{E_i - v'}\right)\rho^k(v',t)dv'$$

$$r^k(t) = J^k(v_{th}, t)$$

$$\nu_{e/i}^k(t) = \nu_{e/i,o}^k(t) + \sum_{j \in P_{E/I}} W_{jk} \int_0^\infty \alpha_{jk}(t') r^j(t - t')dt'$$

Why kinetic theory models?

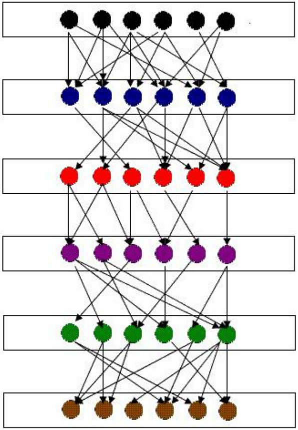
- Multielectrode recording reveals the importance of cross-correlation among population activity in the encoding and decoding aspects of neural systems.
- To improve mean-field models so that they can capture higher order dynamics of neural systems.

Distill connectivity patterns

- To develop kinetic theory models to capture cross-correlation, we need to distill connectivity patterns down to second order features.
- Unfortunately, it is still not clear what these key second order features should be.

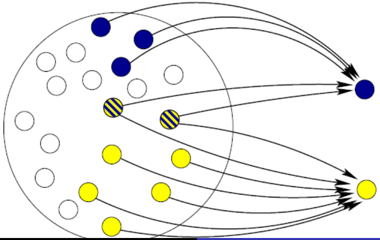
Distill connectivity patterns in Feedforward networks

For the first try, we have considered feedforward networks.



Distill connectivity from common input idea

- Two connectivity statistics
  - W^1: the average number of presynaptic neurons that project onto a postsynaptic neuron
  - W^2: the average number of presynaptic neurons that simultaneously project onto a pair of postsynaptic neurons
- Distill connectivity down to 2-parameter space $(W^1, \beta = W^2/W^1)$



Kinetic theory model of feed-forward networks

- Assume the input for any pair of neurons as correlated Poisson process (CPP).
- Impose conditional independence structure for CPP. In this way, we have a 2-parameter statistical model of CPP with first and second order population activity as parameters.
- Compute the first and second order activity by population density equations.
- Derivation of output-input coupling equation from above model of CPP and two distilled connectivity statistics.

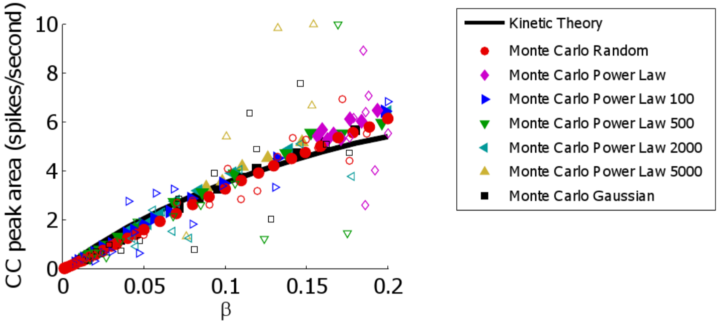
Kinetic theory model of feed-forward networks

Population density equations:

$$
\begin{aligned}
\frac{\partial \rho^k}{\partial t}(v_1, v_2, t) &= \frac{1}{\tau}\frac{\partial}{\partial v_1}\left[(v_1 - E_r)\rho^k(v_1, v_2, t)\right] + \frac{1}{\tau}\frac{\partial}{\partial v_2}\left[(v_2 - E_r)\rho^k(v_1, v_2, t)\right] \\
&+ \sum_m \nu_{m,0}(t)\left[\int_{v_{\mathrm{reset}}}^{v_1} f_{mA}(v_1 - \theta_1)\rho^k(\theta_1, v_2, t)d\theta_1 - \rho^k(v_1, v_2, t)\right] \\
&+ \sum_m \nu_{0,m}(t)\left[\int_{v_{\mathrm{reset}}}^{v_2} f_{mA}(v_2 - \theta_2)\rho^k(v_1, \theta_2, t)d\theta_2 - \rho^k(v_1, v_2, t)\right] \\
&+ \sum_{\substack{m,n \\ m\geq 1 \\ n\geq 1}} \nu_{m,n}(t)\left[\int_{v_{\mathrm{reset}}}^{v_1}\int_{v_{\mathrm{reset}}}^{v_2} f_{mA}(v_1 - \theta_1)f_{nA}(v_2 - \theta_2)\rho^k(\theta_1, \theta_2, t)d\theta_2 d\theta_1 \right. \\
&\qquad \left. - \rho^k(v_1, v_2, t)\right] \\
&+ \delta(v_1 - v_{\mathrm{reset}})J^k_{\mathrm{reset},1}(v_2, t) + \delta(v_2 - v_{\mathrm{reset}})J^k_{\mathrm{reset},2}(v_1, t) \\
&+ \delta(v_1 - v_{\mathrm{reset}})\delta(v_2 - v_{\mathrm{reset}})J^k_{\mathrm{reset},3}(t).
\end{aligned}
$$

Test connectivity hypothesis and validity of kinetic theory model

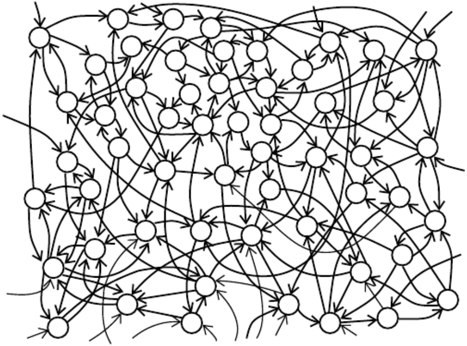
- Evidence from Monte Carlo simulation
- The performance of kinetic theory model



Recurrent networks of homogeneous population of neurons

Question: Is common input idea sufficient for recurrent networks?

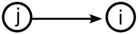
Answer: No!!



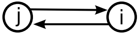
More second order connectivity motifs

Extend common input motif to more second order motifs with shared neurons.

$$\Pr(W_{ij} = 1) = E(W_{ij}) = p$$

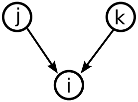


$$\alpha_{\text{recip}} = \frac{\text{cov}(W_{ij}, W_{ji})}{p^2}$$



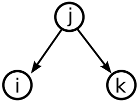
reciprocal connection

$$\alpha_{\text{conv}} = \frac{\text{cov}(W_{ij}, W_{ik})}{p^2}$$



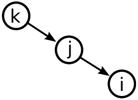
convergent connection

$$\alpha_{\text{div}} = \frac{\text{cov}(W_{ij}, W_{kj})}{p^2}$$



divergent connection

$$\alpha_{\text{chain}} = \frac{\text{cov}(W_{ij}, W_{jk})}{p^2}$$



chain connection

Selection of directed graph model with given network parameters

Network parameters: $N, p, \alpha_{recip}, \alpha_{conv}, \alpha_{div}, \alpha_{chain}$

Problem: There are many such probability distribution with given second order statistics.

Solutions:

1. The right way
   Choose the probability distribution with least structure, i.e., the maximum entropy solution. (Ising model.)

Selection of directed graph model with given network parameters

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Solutions:

1. The right way
   Choose the probability distribution with least structure, i.e., the maximum entropy solution. (Ising model.)
2. The wrong way
   Use a joint Gaussian distribution to define $\tilde{W}_{ij}$ and let $W_{ij} = 1$ if $\tilde{W}_{ij} > \theta$ for some threshold θ.

Selection of directed graph model with given network parameters

Network parameters: $N, p, \alpha_{recip}, \alpha_{conv}, \alpha_{div}, \alpha_{chain}$

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As matter of principle, choose the wrong way.

Influence of network structure on synchrony

Simulate network of excitatory phase response curve neurons.

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \frac{J}{pN} f(\theta_i) \sum_{j \neq i} W_{ij} \sum_k \delta(t - T_j^k) + \sigma \xi(t)$$

Measure steady state synchrony with order parameter.

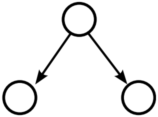
$$\text{synchrony} = \frac{1}{N} \left| \sum_{j=1}^{N} e^{i\theta_j} \right|$$

synchrony $= 0 \Rightarrow$ asynchrony
synchrony $= 1 \Rightarrow$ complete synchrony

$f =$ phase response curve, $T_j^k =$ time of kth spike of neuron j, $S =$ coupling strength, $\omega =$ intrinsic frequency, $\xi(t) =$ white noise

Does common input influence synchrony?

One idea: common input connections should encourage synchrony.



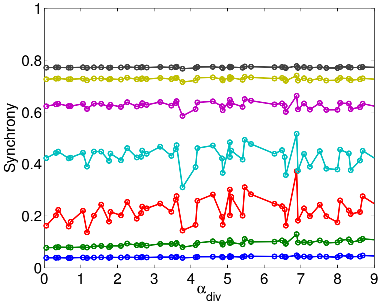
Intuition from feedforward networks:

common input
$\Rightarrow$ correlated input
 $\Rightarrow$ correlated output
  $\Rightarrow$ more correlated input downstream
   $\Rightarrow$ development of synchrony

Test for recurrent networks through simulations of second order networks model.

Simulation with divergence

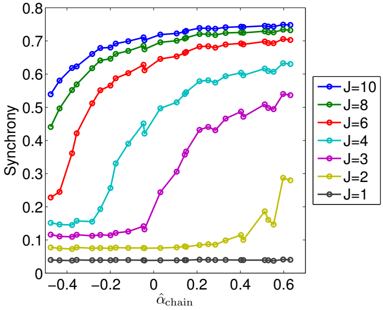
If fix α_{others}, synchrony doesn't appears to be a function of the frequency of divergences even increasing the coupling strength J.



Don't see evidence for strong effect of divergence.

Simulation with chains

If fix α_{others}, synchrony appears to be a function of the coupling strength J and the frequency of chains.



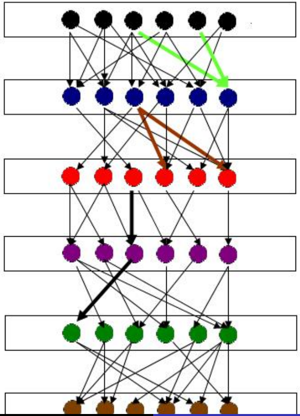
Chains, not common input, highly influence synchrony.

Issues

- Mechanism? Why the effects of divergence on feed-forward and recurrent networks are so different?
- Kinetic theory model for recurrent? How to develop models to capture the effect of chain on synchrony?

SOFFNETs

In the same framework, we can construct second order structures in Feed-forward networks. (No reciprocal)



SOFFNETs: Formalism

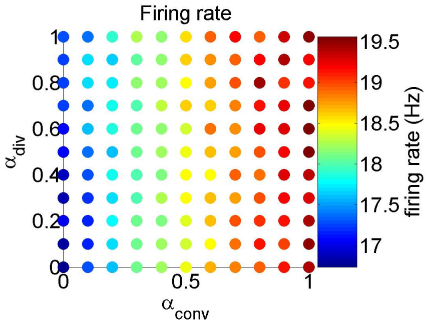
$$\Pr(W_{jk}^{s,s-1} = 1) = p$$
$$\Pr(W_{jk}^{s,s-1} = 1, W_{ji}^{s,s-1} = 1) = p^2(\alpha_{\text{conv}} + 1)$$
$$\Pr(W_{jk}^{s,s-1} = 1, W_{ik}^{s,s-1} = 1) = p^2(\alpha_{\text{div}} + 1)$$
$$\Pr(W_{jk}^{s,s-1} = 1, W_{ki}^{s-1,s-2} = 1) = p^2(\alpha_{\text{chain}} + 1)$$

Comparing this formalism to old connectivity statistics, we have

$$W^1 = pN$$
$$W^2 = p^2(\alpha_{div} + 1)N$$
$$\beta = p(1 + \alpha_{div})$$

Effect of convergence and divergence on FF networks

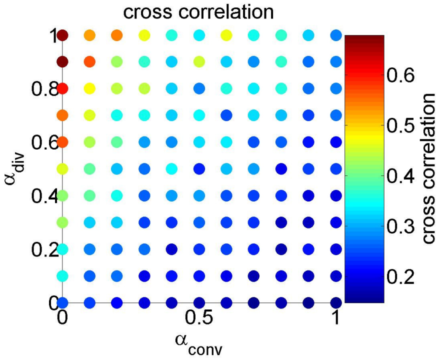
Effect on firing rate



Both convergences and divergences can influence the transmission of firing rate.

Effect of convergence and divergence on FF networks

Effect on cross-correlation



Both convergences and divergences can influence the transmission of cross-correlation.

Kinetic Theory Model?

In the case of uncorrelated in- and out-degree (i.e. $\alpha_{chain} = 0$), the idea is the maximum entropy distribution $P(N_1, N_2, N_3)$

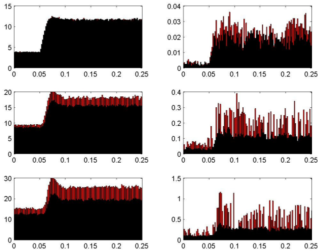
$$S(P) = -\sum_{N_1+N_2+N_3 \leq N} P(N_1, N_2, N_3) lnP(N_1, N_2, N_3)$$

maximized under the following constraint,

$$\begin{aligned}
EN_1 &= N[p - p^2(1 + \alpha_{div})] \\
EN_2 &= Np^2(1 + \alpha_{div}) \\
EN_3 &= N[p - p^2(1 + \alpha_{div})] \\
E(N_1 + N_2)^2 &= Np(1 - p) + N(N - 1)p^2\alpha_{conv} + N^2p^2 \\
E(N_2 + N_3)^2 &= Np(1 - p) + N(N - 1)p^2\alpha_{conv} + N^2p^2
\end{aligned}$$

Simulations with chain

$$(\alpha_{conv}, \alpha_{div}, \alpha_{chain}) = (1.0, 1.0, 0.0)(black)$$
$$(\alpha_{conv}, \alpha_{div}, \alpha_{chain}) = (1.0, 1.0, 0.05)(red)$$



Chains highly influence the transmission of firing rate and cross-correlation

Issues

- Mechanism of the effect from chains? It shows that chain plays an important role in information processing.
- Kinetic theory model or other models to capture the effect of chains?

Conclusion

- Kinetic theory model for feedforward networks with simple structure
- Second order networks
- The important role of chain motif
- A challenge for kinetic theory model in SONETs

Ongoing works

- Simple models to investigate the mechanism of chains
- Application of SONETs to brain functions

Thanks

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Thanks for your attention

Any comment or suggestion is welcome!!