Bijections between planar maps and planar linear normal λ-terms with connectivity condition

Wenjie Fang
LIGM, Université Gustave Eiffel
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A crash course on λ-calculus

A λ-**term** is:

- Atom: variables $x, y, z, \ldots$;
- Application $t\ u$: apply a λ-term t to u as a function;
- Abstraction $\lambda x.t$: turn a λ-term t into a function of x.

λ-**calculus**, which is Turing complete, consists of:

- α-renaming: changing the name of a variable;
- β-**reduction**: $(\lambda x.t)\ u \to_\beta t[x \leftarrow u]$;
- (sometimes) η-conversion: $\lambda x.t\ x \Leftrightarrow_\eta t$.

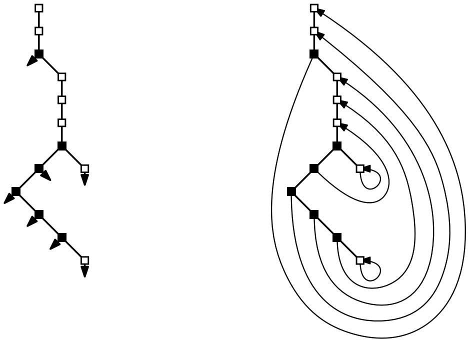
β-reduction admits unique normal form (no β-redex) if there is one.

λ-terms, according to a combinatorialist

λ-term: unary-binary tree (skeleton) + variable-abstraction map

Linear λ-term: the variable-abstraction map being bijective (so closed)

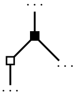
$$t = \lambda u.\lambda v.u(\lambda w.\lambda x.\lambda z.v(w(x(\lambda y.y)))z(\lambda k.k))$$



(RL-)planar term: counter-clockwise variable-abstraction map

Linear planar : unique choice, so just unary-binary tree!

Known enumeration of various families of λ-terms

- closed: no free variable
- unitless: no closed sub-term
- normal: no β-reduction, *i.e.*, avoiding 

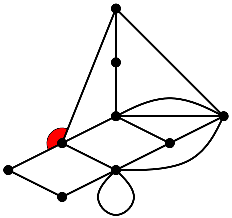
λ-terms	Maps	OEIS
linear	general cubic	A062980
planar	planar cubic	A002005
unitless	bridgeless cubic	A267827
unitless planar	bridgeless planar cubic	A000309
β-normal linear/$\sim$	general	A000698
β-normal planar	planar	A000168
β-normal unitless linear/$\sim$	bridgeless	A000699
β-normal unitless planar	bridgeless planar	A000260

Noam Zeilberger, *A theory of linear typings as flows on 3-valent graphs*, LICS 2018

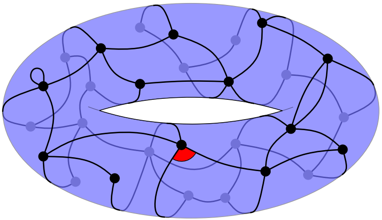
A lot of people and work: Bodini, Courtiel, Gardy, Giorgetti, Jacquot, Yeats, Zeilberger, ...

What is a map?

Combinatorial maps: (nice) drawings of graphs on a surface



sphere/planar ($g = 0$)



torus ($g = 1$)

We only consider rooted map, *i.e.*, with a marked corner.

Notions in graph theory: planar, loopless, bipartite, ...

Connectivity condition

Noam Zeilberger and Jason Reed (CLA 2019)

How about connectivity of the diagram on **planar linear normal terms**?

k-connected: breaking $k-1$ edges does not split the graph

- 1-connected: all (connected by their skeleton)
- 2-connected: unitless (bridge $\Leftrightarrow$ closed sub-term)
- 3-connected: ???

Motivated by type theory.

Conjecture (Zeilberger–Reed, 2019)

The number of 3-connected planar linear normal λ-terms with $n+2$ variables is

$$\frac{2^n}{(n+1)(n+2)}\binom{2n+1}{n},$$

which also counts bipartite planar maps with n edges (A000257).

Our contribution (1)

Katarzyna Grygiel and Guan-Ru Yu (CLA 2020): combinatorial characterization of 3-connected terms, partial bijective results

Theorem (F. 2023)

There is a direct bijection between 3-connected planar linear normal λ-terms with $n+2$ variables and bipartite planar maps with n edges.

What we do using bijections:

- Transfer of statistics (also about applications in λ-terms)
- Generating functions and probabilistic results also for free!

Proposition (F. 2023, from known results on maps by Liskovet)

Let $X_n = \#$ initial abstractions of a uniformly random 3-connected planar linear normal λ-term. When $n \to \infty$,

$$\mathbb{P}[X_n = k] \to \frac{k-1}{3}\binom{2k-2}{k-1}\left(\frac{3}{16}\right)^{k-1}.$$

Our contribution (2)

Theorem (F. 2023)

There is a direct bijection from planar linear normal λ-terms to planar maps, with its restriction to unitless terms giving loopless planar maps.

λ-terms	Maps	OEIS
β-normal linear/$\sim$	general	A000698
β-normal planar	planar	A000168
β-normal unitless linear/$\sim$	bridgeless	A000699
β-normal unitless planar	loopless planar	A000260

Known recursive bijection in (Zeilberger and Giorgetti, 2015) via LR-planar terms (clockwise, not stable by β-reduction...)

From λ-terms to unary-binary trees

Linear planar λ-terms $\Leftrightarrow$ unary-binary trees (with conditions)

Three statistics for a unary-binary tree S:

- $\mathrm{unary}(S)$: # unary nodes (abstractions)
- $\mathrm{leaf}(S)$: # leaves (variables)
- $\mathrm{excess}(S)$: $\mathrm{leaf}(S) - \mathrm{unary}(S)$ (free variables)

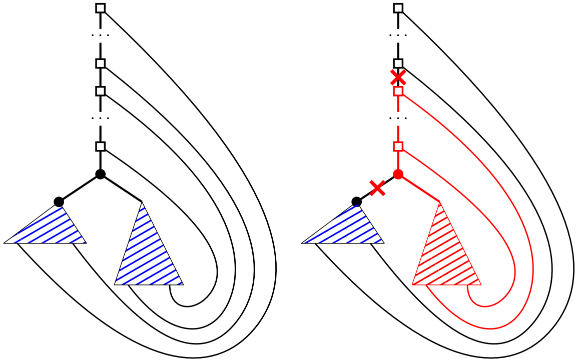
S_u: sub-tree of S induced by u

- Linear (or closed) $\Leftrightarrow$ $\mathrm{excess}(S) = 0$
- 1-connected (well-scoped) $\Leftrightarrow$ $\mathrm{excess}(S_u) \geq 0$ for all u
- 2-connected (or unitless) $\Leftrightarrow$ $\mathrm{excess}(S_u) > 0$ for all u non-root

Characterization of 3-connectedness (1)

Proposition (Grygiel and Yu, CLA 2020)

Let S be the skeleton of a 3-connected planar linear λ-term, then the left child of the first binary node is a leaf.



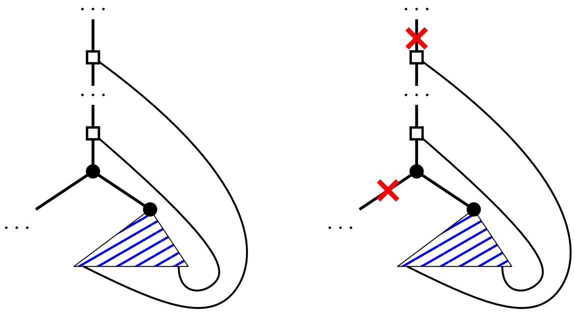
Reduced skeleton: the right sub-tree of the first binary node

Characterization of 3-connectedness (2)

Proposition (Proposed by Grygiel and Yu, CLA 2020)

S is the reduced skeleton of a 3-connected planar linear normal λ-term iff

- **(Normality)** *The left child of a binary node in S is never unary;*
- **(3-connectedness)** *For every binary node u with v its right child, $\mathrm{excess}(S_v) > \#$ consecutive unary nodes above u.*



Clearly necessary, but also sufficient!

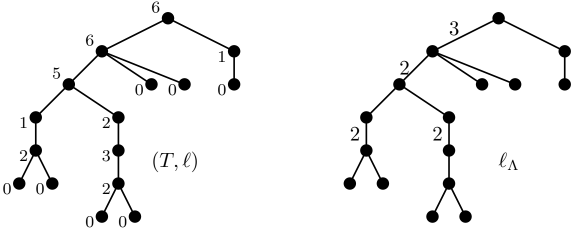
Degree trees

Degree tree: a plane tree T with a labeling ℓ on nodes with

- u is a leaf $\Rightarrow \ell(u) = 0$;
- u has children $v_1, \ldots v_k \Rightarrow s(u) - \ell(v_1) \leq \ell(u) \leq s(u)$, where $s(u) = k + \sum_{i=1}^{k} \ell(v_i)$.

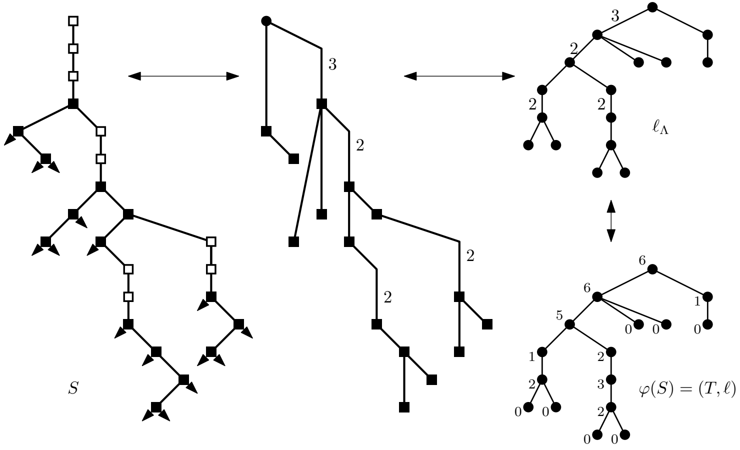
Contribution of each child : 1 (itself) $+ \ell(v_i)$ (its label)

Except for the first child: from 1 to its due contribution.



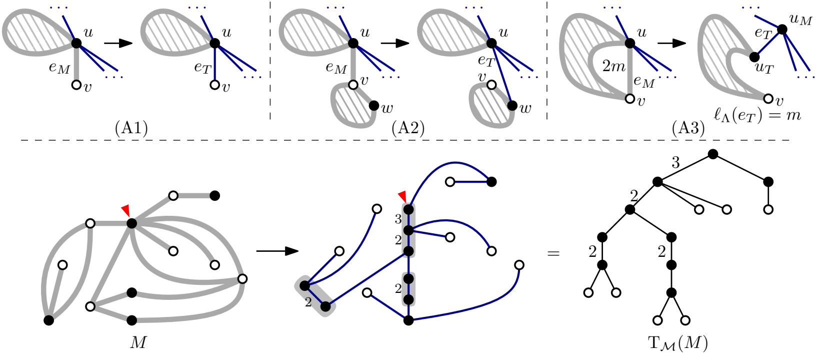
Edge labeling ℓ_Λ: the subtracted contribution (interchangeable with ℓ !)

First bijection (1/2): 3-connected terms ⇔ degree trees



- Contribution of u to its parent = excess of its right sub-tree
- Unary nodes on right child ⇔ Subtraction on left child

First bijection (2/2): degree trees ⇔ bipartite planar maps



Existing direct bijection (F., 2021), using an exploration

Also related to Chapoton's new intervals in the Tamari lattice

Some statistics correspondences:

- Unary chains of length k ⇔ edge label k ⇔ faces of degree $2k$
- Initial unary chain ⇔ root label ⇔ degree of root face

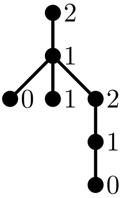
Connected terms and trees

Recall the conditions:

- 1-connected $\Leftrightarrow \text{excess}(S_u) \geq 0$ for all u
- 2-connected $\Leftrightarrow \text{excess}(S_u) > 0$ for all u non-root

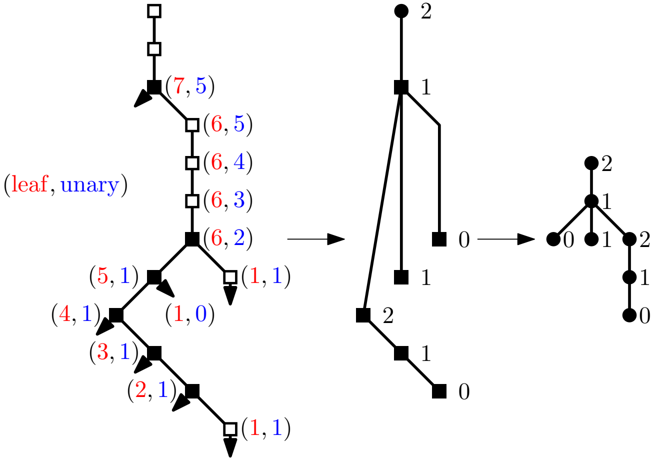
v-trees: a plane tree T with a labeling ℓ on nodes with

- Leaf $u \Rightarrow \ell(u) \in \{0, 1\}$;
- Non-root u with children $v_1, \ldots, v_k \Rightarrow 0 \leq \ell(u) \leq 1 + \sum_{i=1}^{k} \ell(v_i)$



A v-tree is positive if there is no label 0.

Second bijection (1/2)



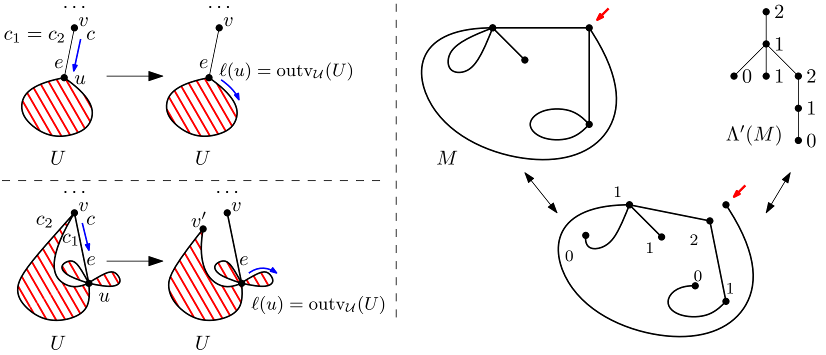
$$\lambda u.\lambda v.u(\lambda w.\lambda x.\lambda z.v(w(x(\lambda y.y)))z(\lambda k.k))$$

Excess of the **right child**! $0 \Leftrightarrow$ closed sub-term

Second bijection (2/2)

Direct bijection = "de-recusifying" a "new" recursive decomposition

$\mathrm{outv}_{\mathcal{U}}$: #vertices on outer face -1 (catalytic statistics)



Loop ⇔ component with 1 outer node ⇔ label 0 in tree

Restriction to loopless planar maps ⇔ unitless terms

Recapitulation

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Nearly the same bijection

Higher connectivity? Other enumeration consequences?

And types (1, 2, 9, 52, 344, 2482, 19028, 152570, 1266340, ...) ?

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Thank you for listening!