Correlation decay and limit theorems for maximum weight matching on sparse random graphs

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(Based on an ongoing project with Arnab Sen)

Maximum weight matching

- Let G be a finite graph.
- A matching M of G is a subgraph of G such that all vertices have degree at most 1.
- Put i.i.d. $\mathrm{Exp}(1)$ weights (w_e) on the edges.
- For a matching M, define $W(M) = \sum_{e \in M} w_e$.
- Maximum weight matching M_G: the matching M such that $W(M)$ is maximized.
- Define $\mathsf{W}_G = W(\mathsf{M}_G)$.
- Goal: Understand M_G and W_G when G is "large".

Recurrence relation

- Aldous–Steele: To understand M_G and W_G, can look for a recurrence relation.
- What happens if we remove a vertex v from G?
- Two cases: either $v \in \mathsf{M}_G$ or $v \notin \mathsf{M}_G$.
- Hence

$$\mathsf{W}_G = \max\{\mathsf{W}_{G\setminus\{v\}}, \max_{u:u\sim v}(w_{(uv)} + \mathsf{W}_{G\setminus\{u,v\}})\}.$$

- Define the *bonus* at v in G by $\mathsf{B}(v,G) = \mathsf{W}_G - \mathsf{W}_{G\setminus\{v\}}$. Then

$$\begin{aligned}\mathsf{B}(v,G) &= \max_{u:u\sim v}\{w_{(uv)} - \mathsf{B}(u, G\setminus\{v\}), 0\}\\ &= \max_{u:u\sim v}(w_{(uv)} - \mathsf{B}(u, G\setminus\{v\}))_+.\end{aligned}$$

Previous results

Aldous–Steele '04:

- Showed that $\lim_{n\to\infty} \frac{\mathbb{E}\mathsf{W}_{G_n}}{n}$ exists, where G_n is uniform random tree with n vertices.
- Actually holds for any continuous distribution, and has an explicit formula if $w_e \sim \mathrm{Exp}(1)$.
- Rough idea: Remove an edge from a finite tree G_n, view these components as rooted trees $G_n^1(e)$ and $G_n^2(e)$ (with roots being the vertices of e).
- If $e \notin \mathsf{M}_{G_n}$, then $\mathsf{W}_{G_n} = \mathsf{W}_{G_n^1(e)} + \mathsf{W}_{G_n^2(e)}$.
- Otherwise, it equals $w_e + (\mathsf{W}_{G_n^1(e)} - \mathsf{B}(o_1, G_n^1(e))) + (\mathsf{W}_{G_n^2(e)} - \mathsf{B}(o_2, G_n^2(e)))$.
- Thus $e \in \mathsf{M}_{G_n}$ iff $w_e > \mathsf{B}(o_1, G_n^1(e)) + \mathsf{B}(o_2, G_n^2(e))$.

Previous results

- So

$$\mathsf{W}_{G_n} = \sum_{e \in E(G_n)} w_e \mathbf{1}_{\{w_e > \mathsf{B}(o_1, G_n^1(e)) + \mathsf{B}(o_2, G_n^2(e))\}}.$$

- Taking expectation,

$$\mathbb{E}\mathsf{W}_{G_n} = (n-1)\mathbb{E}w\mathbf{1}_{\{w > \mathsf{B}(\mathbf{o}_1, G_n^1(\mathbf{e})) + \mathsf{B}(\mathbf{o}_2, G_n^2(\mathbf{e}))\}},$$

where $\mathbf{e} = (\mathbf{o}_1, \mathbf{o}_2) \sim \mathrm{Unif}(E(G_n))$.

- Can show that $\mathsf{B}(\mathbf{o}_1, G_n^1(\mathbf{e}))$, $\mathsf{B}(\mathbf{o}_2, G_n^2(\mathbf{e}))$ converge in distribution, and their distributions satisfy some fixed point equations.

Previous results

Gamarnik–Nowicki–Swirszcz '06:

- Show the same conclusion when G_n is sparese Erdős–Rényi random graph or random r-regular graph.
- For random r-regular graph, consider the distributional equation

$$\mathrm{Law}(B_i) = \mathrm{Law}\left(\max_{1\le j\le r}(w_j - B_j)_+\right), \tag{1}$$

  where w_j are i.i.d. with the same distribution as w_e, B_i are i.i.d. Also, all w_j and B_j are independent.
  If (1) has a unique solution, then the limit equals

$$\frac{1}{2}\mathbb{E}\left[\sum_{i=1}^{r} w_i \mathbf{1}_{\{w_i - B_i = \max_{1\le j\le r}(w_j - B_j) > 0\}}\right],$$

- If $w_e \sim \mathrm{Exp}(1)$, then (1) has a unique solution. Not known for general distribution.

Previous results

- Also showed a correlation decay result: if $e, e' \sim \mathrm{Unif}(E(G_n))$ are independent (where G_n is Erdős–Rényi or r-regular), and if (1) has a unique solution, then

$$\lim_{n\to\infty} \left[\mathbb{P}(e, e' \in \mathsf{M}_{G_n}) - \mathbb{P}(e \in \mathsf{M}_{G_n})\mathbb{P}(e' \in \mathsf{M}_{G_n})\right] = 0.$$

Previous results

Cao 21':

- For sparse Erdős–Rényi graph, if $w_e \sim \mathrm{Exp}(1)$, then

$$\frac{\mathsf{W}_{G_n} - \mathbb{E}\mathsf{W}_{G_n}}{\sqrt{\mathrm{Var}(\mathsf{W}_{G_n})}} \Rightarrow N(0,1).$$

- Based on Stein's method. Make use of a distributional equation similar to (1) to show a form of decorrelation.

Previous results

L.-Sen '22+:

- Consider a "regularized" version: for $\beta > 0$, define

$$F_n(\beta) = \log \left(\sum_{M \text{ matching in } G_n} \exp\left(\beta \sum_{e \in M} w_e \right) \right).$$

- When $\beta \to \infty$, then $F_n(\beta)/\beta \to \mathsf{W}_{G_n}$.
- For any (G_n) such that the maximal degree is uniformly bounded and for **any** edge weight distribution with $\mathbb{E}|w_e|^{2+\varepsilon} < \infty$, under some growth assumption on (G_n), $F_n(\beta)$ obeys a Gaussian CLT.
- No tree / random structure needed in this case.

Correlation decay

Notation: $\mathbb{B}_v^R(G) = \{u : \text{dist}(u,v) \leq R\}$.

Theorem (L.-Sen, '23+)

Let G be a finite graph with maximum degree bounded above by D. Let $v \in V(G)$. Suppose that for some $R \geq 1$, $\mathbb{B}_v^R(G)$ is a tree. Then there exists $c > 0$ (depending only on D) such that

$$\mathbb{E}(\mathsf{B}(v,G) - \mathsf{B}(v,\mathbb{B}_v^R(G)))^2 \leq e^{-cR}.$$

- Recall $\{v \in \mathsf{M}_G\} = \{\mathsf{B}(v,G) > 0\}$.
- If G is locally-tree like, then $\{v \in \mathsf{M}_G\}$ essentially depends only on a neighborhood of $v \to$ correlation decay.

Rough idea

- Recall:

$$\mathsf{B}(v, G) = \max_{u:u\sim v} (w_{(uv)} - \mathsf{B}(u, G \setminus \{v\}))_+.$$

- If G is a tree, $(w_{(uv)})_{u:u\sim v}, (\mathsf{B}(u, G \setminus \{v\}))_{u:u\sim v}$ are independent.
- In this case, by memeoryless property, $\mathsf{B}(v, G) = \max_{u:u\sim v} W_u \xi_u$ in distribution, where $W_u \sim \mathrm{Exp}(1)$, $\xi_u \sim \mathrm{Ber}(p_u)$, and W_u, ξ_u are independent.
- Same holds for the children $v1, \ldots, vk$ of v.
- Can show: $(p_{v1}, \ldots, p_{vk}) \mapsto p_v$ is essentially a contraction. So if two vertices are far away, the "information" of the ξ's cannot be passed to each other.
- $\implies \mathsf{B}(v, G)$ depends essentially on local neighborhood of v.

Rough idea

- For general G such that v has a tree neighborhood, make use of monotonicity of bonuses.
- Can squeeze $\underline{\mathsf{B}}(v, \mathbb{B}_v^R(G)) \leq \mathsf{B}(v, G) \leq \overline{\mathsf{B}}(v, \mathbb{B}_v^R(G))$, where $\underline{\mathsf{B}}$, $\overline{\mathsf{B}}$ are bonuses by putting extreme boundary conditions on $\partial\mathbb{B}_v^R(G)$.
- Contraction $\implies \underline{\mathsf{B}}(v, \mathbb{B}_v^R(G)) \approx \overline{\mathsf{B}}(v, \mathbb{B}_v^R(G))$.

Application 1: law of large numbers

Theorem (L.-Sen, '23+)

Suppose that (G_n) *is a sequence of random graphs with* $|V(G_n)| = n$ *and maximal degree* D *that converges locally weakly to a rooted (random) tree* (T, o). *Then*

$$\lim_{n\to\infty} \frac{\mathbb{E}\mathsf{W}_{G_n}}{n} \to \frac{1}{2}\mathbb{E}\left[\sum_{v:v\sim o} w_{(ov)}\mathbf{1}_{\{(ov)\in\mathsf{M}_T\}}\right].$$

- One has $\frac{\mathbb{E}\mathsf{W}_{G_n}}{n} = \frac{1}{2}\mathbb{E}\left[\frac{\sum_{u,v:u\sim v} w_{(uv)}\mathbf{1}_{\{(uv)\in\mathsf{M}_{G_n}\}}}{n}\right] = \frac{1}{2}\mathbb{E}\left[\sum_{v:v\sim o_n} w_{(o_n v)}\mathbf{1}_{\{(o_n v)\in\mathsf{M}_{G_n}\}}\right]$, where $o_n \sim \mathrm{Unif}(V(G_n))$.
- Can make sense of M_T by correlation decay. (For fixed e, $\mathbf{1}_{\{e\in\mathsf{M}_{\mathbb{B}_o^R(T)}\}}$ will stabilize for large R.)

Application 2: quenched central limit theorem

Define

$$\rho_R(G) = \frac{\{e \in E(G) : \text{the } R\text{-neighborhood of } e \text{ is not a tree}\}}{|E(G)|}.$$

Using Stein's method, we can show a central limit theorem.

Theorem (L.-Sen, '23+)

Let G be a finite graph with maximum degree $\leq D$. There exist constants $C, c > 0$ depending only on D such that for all $1 \leq R \leq |V(G)|$,

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}\left(\frac{\mathsf{W}_G - \mathbb{E}\mathsf{W}_G}{\sqrt{\mathrm{Var}(\mathsf{W}_G)}} \leq t \right) - \Phi(t) \right| \leq C(\rho_R(G)^{1/2} + (D^R/|V(G)|)^{1/4} + e^{-cR}),$$

where Φ is the CDF of a standard Gaussian.

Application 3: annealed central limit theorem

Theorem (L.-Sen, '23+)

Let $d_1, d_2, \ldots$ be positive, integer-valued i.i.d. that are uniformly bounded. Let $\mathbb{G}_n$ be a uniformly chosen random simple graph with degree sequence $(d_1, \ldots, d_n)$. Then

$$\frac{\mathrm{W}_{\mathbb{G}_n} - \mathbb{E}\mathrm{W}_{\mathbb{G}_n}}{\sqrt{\mathrm{Var}(\mathrm{W}_{\mathbb{G}_n})}} \Rightarrow N(0,1).$$

- Previous CLT: only weights are random.
- This CLT: both weights and graphs are random. More involved!
- Difficulty: the distribution of $\mathbb{G}_n$ is complicated.

Idea of proof of the annealed CLT

- Decompose

$$\frac{W_{\mathbb{G}_n} - \mathbb{E}W_{\mathbb{G}_n}}{\sqrt{\mathrm{Var}(W_{\mathbb{G}_n})}} = \frac{W_{\mathbb{G}_n} - \mathbb{E}_w W_{\mathbb{G}_n}}{\sqrt{\mathrm{Var}_w(W_{\mathbb{G}_n})}} \cdot \sqrt{\frac{\mathbb{E}\mathrm{Var}_w(W_{\mathbb{G}_n})}{\mathrm{Var}(W_{\mathbb{G}_n})}} \cdot \sqrt{\frac{\mathrm{Var}_w(W_{\mathbb{G}_n})}{\mathbb{E}\mathrm{Var}_w(W_{\mathbb{G}_n})}} + \frac{\mathbb{E}_w W_{\mathbb{G}_n} - \mathbb{E}W_{\mathbb{G}_n}}{\sqrt{\mathrm{Var}(\mathbb{E}_w W_{\mathbb{G}_n})}} \cdot \sqrt{\frac{\mathrm{Var}(\mathbb{E}_w W_{\mathbb{G}_n})}{\mathrm{Var}(W_{\mathbb{G}_n})}}.$$

- Conditional on $\mathbb{G}_n$, $\frac{W_{\mathbb{G}_n} - \mathbb{E}_w W_{\mathbb{G}_n}}{\sqrt{\mathrm{Var}_w(W_{\mathbb{G}_n})}} \Rightarrow N(0,1)$ by the quenched CLT. (The convergence depends mainly on D but not $\mathbb{G}_n$.)
- $\left(\sqrt{\frac{\mathbb{E}\mathrm{Var}_w(W_{\mathbb{G}_n})}{\mathrm{Var}(W_{\mathbb{G}_n})}}\right)^2 + \left(\sqrt{\frac{\mathrm{Var}(\mathbb{E}_w W_{\mathbb{G}_n})}{\mathrm{Var}(W_{\mathbb{G}_n})}}\right)^2 = 1.$
- Can show (not easy): $\frac{\mathrm{Var}_w(W_{\mathbb{G}_n})}{\mathbb{E}\mathrm{Var}_w(W_{\mathbb{G}_n})} \to 1$ in probability.
- Remains to show: $\frac{\mathbb{E}_w W_{\mathbb{G}_n} - \mathbb{E}W_{\mathbb{G}_n}}{\sqrt{\mathrm{Var}(\mathbb{E}_w W_{\mathbb{G}_n})}} \Rightarrow N(0,1).$

Idea of proof of the annealed CLT

- Want: $\frac{\mathbb{E}_w \mathsf{W}_{\mathbb{G}_n} - \mathbb{E}\mathsf{W}_{\mathbb{G}_n}}{\sqrt{\mathrm{Var}(\mathbb{E}_w \mathsf{W}_{\mathbb{G}_n})}} \Rightarrow N(0,1)$.
- Configuration model (Bollobás): Put d_i many half-edges at vertex i, and connect the half-edges uniformly at random $\to$ a uniformly random *multigraph* $\mathbb{C}_n$ with degree sequence $(d_1, \ldots, d_n)$.
- Janson '20: Under some assumptions, one can transfer a CLT on $\mathbb{C}_n$ to one on $\mathbb{G}_n$.
- Remains to show $\frac{\mathbb{E}_w \mathsf{W}_{\mathbb{C}_n} - \mathbb{E}\mathsf{W}_{\mathbb{C}_n}}{\sqrt{\mathrm{Var}(\mathbb{E}_w \mathsf{W}_{\mathbb{C}_n})}} \Rightarrow N(0,1)$.

Idea of proof of the annealed CLT

- Want: $\frac{\mathbb{E}_w \mathsf{W}_{\mathbb{C}_n} - \mathbb{E}\mathsf{W}_{\mathbb{C}_n}}{\sqrt{\mathrm{Var}(\mathbb{E}_w \mathsf{W}_{\mathbb{C}_n})}} \Rightarrow N(0,1)$.
- Barbour–Röllin '19: If Y_n can be written as a sum that depends only on local neighborhoods of vertices in $\mathbb{C}_n$, then (under some technical assumptions) $\frac{Y_n - \mathbb{E}_{\mathbb{C}_n} Y_n}{\sqrt{\mathrm{Var}_{\mathbb{C}_n}(Y_n)}} \Rightarrow N(0,1)$.
- However $\mathbb{E}_w \mathsf{W}_{\mathbb{C}_n}$ does not just depend on local neighborhoods.
- Yet, thanks to correlation decay, it can be approximated by a random variable that depends only on local neighborhoods.

Open problems

- Universality? Can we extend the results to general distributions?
- Can we allow the maximal degree $\to \infty$ as $n \to \infty$?
- Maximum weight matching on graphs that are not locally tree-like?
- A better understanding on the structure of the maximum weight matching?

Thank you!