Fringe subtrees and additive functionals

Recurring themes in the study of random trees

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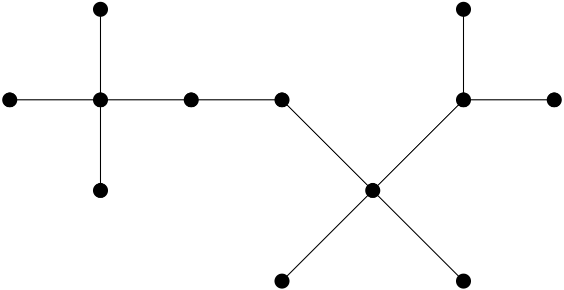
Department of Mathematics

Taipei, 28 June 2023

Why study trees?



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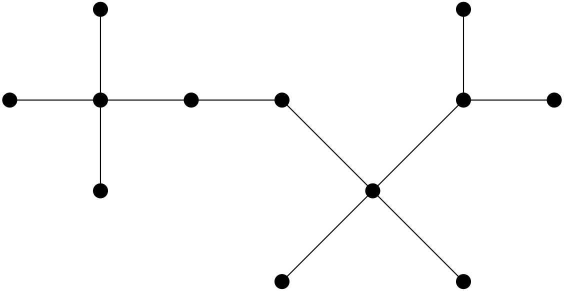

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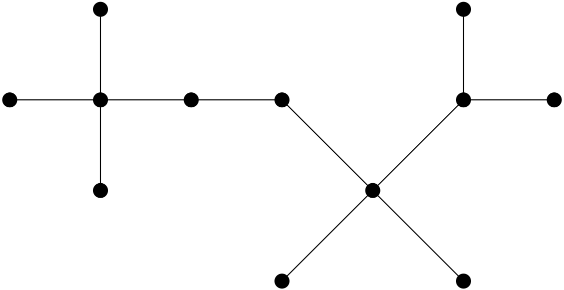

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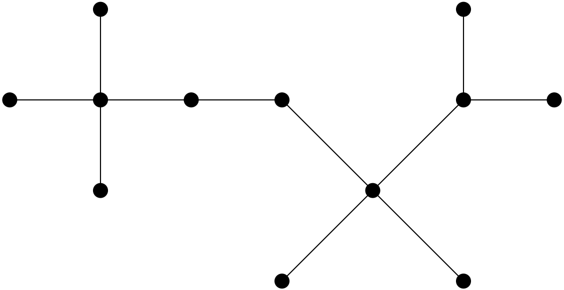

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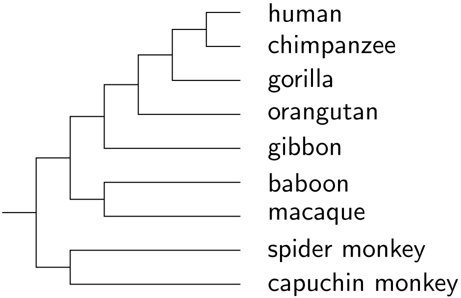

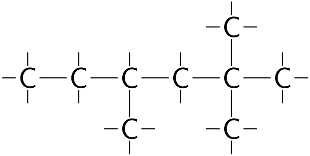

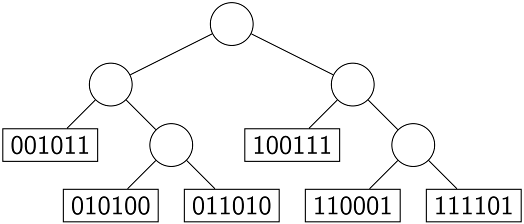

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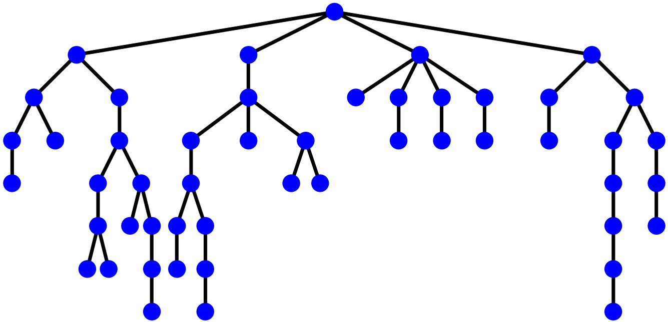

A random tree with 50 vertices.

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- Branching processes (e.g. Galton–Watson trees),
- Increasing tree models (e.g. recursive trees),
- Models based on random strings or permutations (e.g. tries, binary search trees).

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In particular, this is the case for *simply generated families of trees*.

Simply generated families

On the set of all rooted ordered (plane) trees, we impose a weight function by first specifying a sequence $1 = w_0, w_1, w_2, \ldots$ and then setting

$$w(T) = \prod_{i \geq 0} w_i^{N_i(T)},$$

where $N_i(T)$ is the number of vertices of outdegree i in T. Then we pick a tree of given order n at random, with probabilities proportional to the weights. For instance,

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- $w_i = \frac{1}{i!}$ generates random rooted labelled trees.

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Simply generated trees and Galton–Watson trees are essentially equivalent. For example, a geometric distribution for branching will result in a random plane tree, a Poisson distribution in a random rooted labelled tree.

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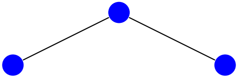

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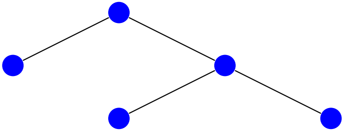

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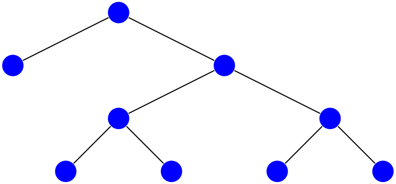

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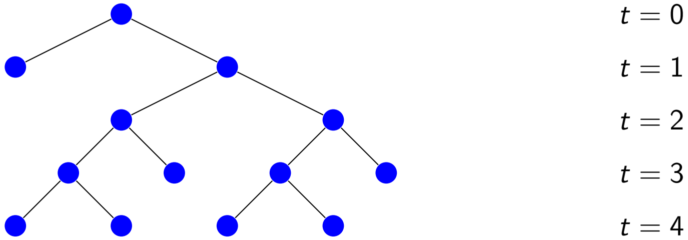

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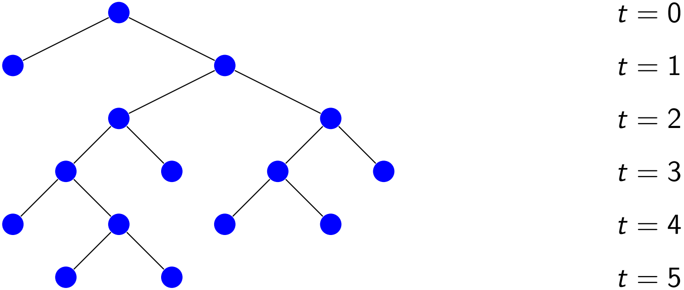

Simply generated and Galton–Watson trees

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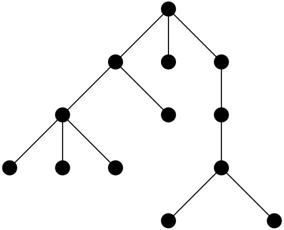

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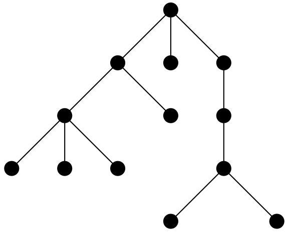

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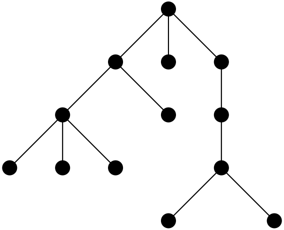

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as does *every* rooted ordered tree with 13 vertices.

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The model can be modified by not choosing a parent uniformly at random, but depending on the current outdegrees (to generate, for example, binary increasing trees or preferential attachment trees).

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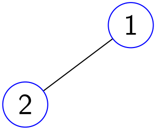

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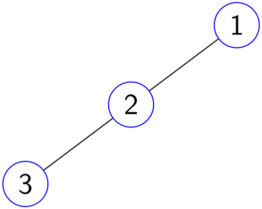

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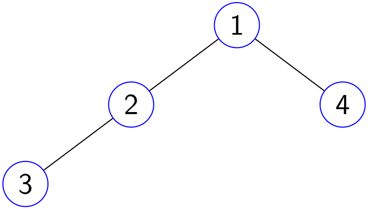

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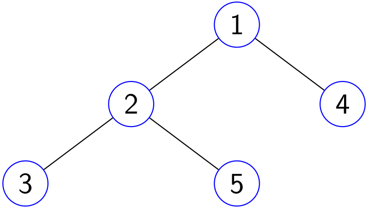

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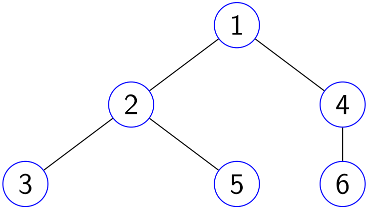

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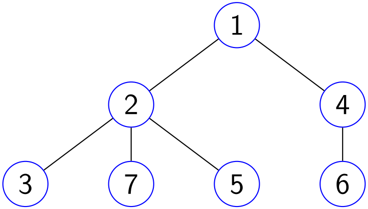

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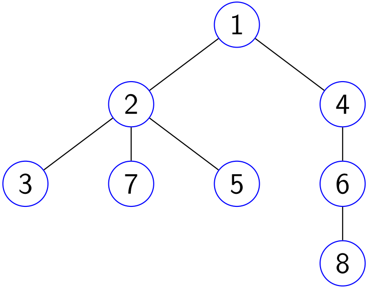

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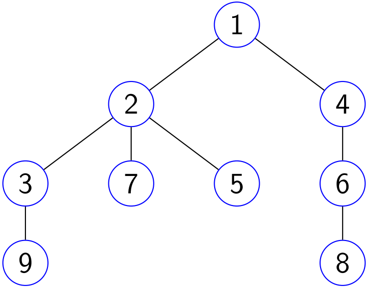

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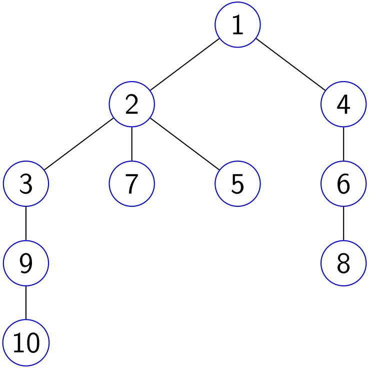

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- This procedure is repeated recursively.

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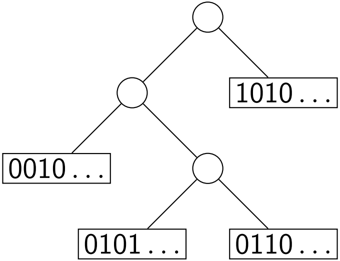

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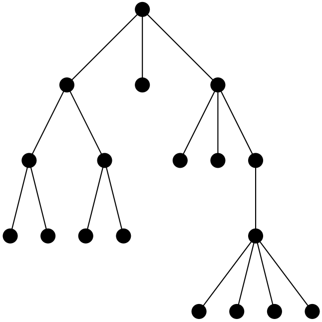
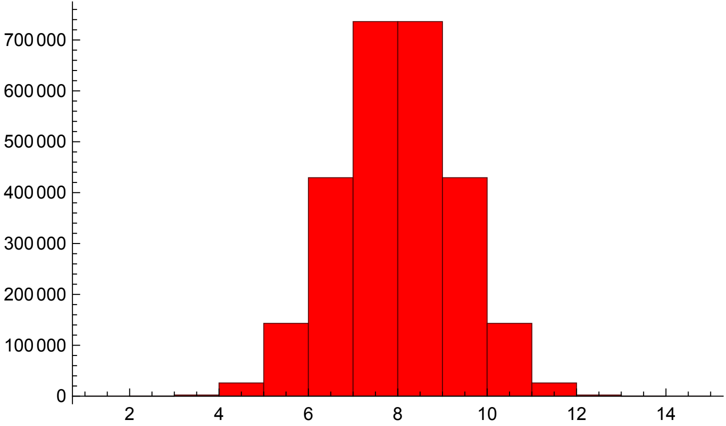
These questions become particularly relevant when n is large.

Some examples of parameters



The tree above has 11 leaves, 2 “cherries”, path length 44, 384 automorphisms and 3945 subtrees.

Distribution of parameters: some examples



Distribution of the number of leaves in plane trees with 15 vertices. Plane trees with n vertices and k leaves are counted by the Narayana numbers $N_{n,k} = \frac{1}{n-1}\binom{n-1}{k}\binom{n-1}{k-1}$.

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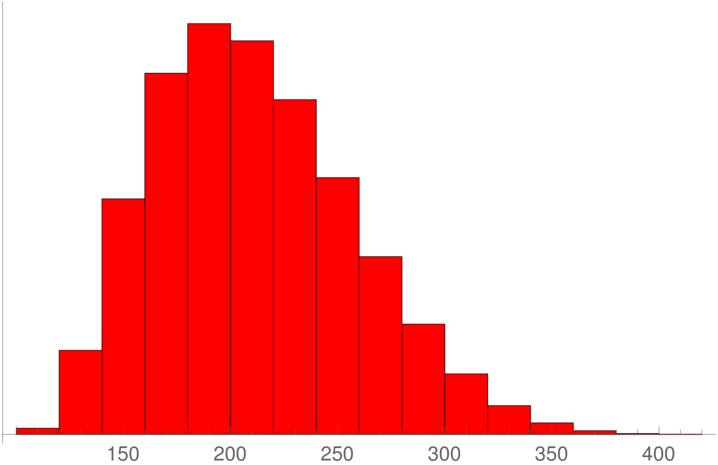

Distribution of the path length in (pruned) binary trees with 30 vertices.

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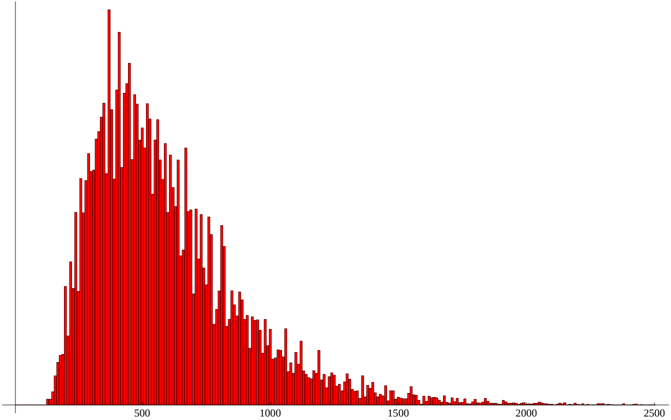

Distribution of the number of subtrees in labelled trees with 15 vertices.

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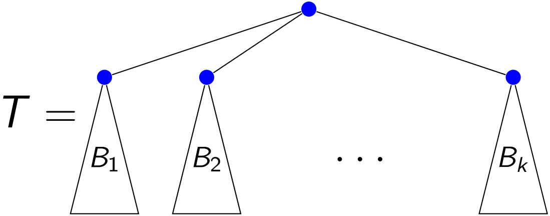

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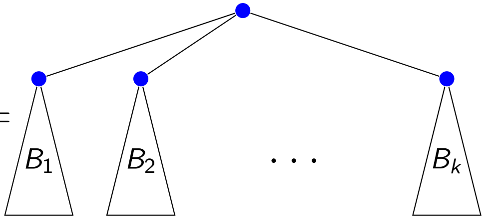

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Remark

The recursion remains true for the tree $T = \bullet$ of order 1 if we assume without loss of generality that $f(\bullet) = F(\bullet)$.

An equivalent definition

The fringe subtree T_v associated with a vertex v of a tree T is the subtree consisting of v and all its descendants.

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One can see by induction that the recursion

$$F(T) = F(B_1) + F(B_2) + \cdots + F(B_k) + f(T)$$

is equivalent to the formula

$$F(T) = \sum_v f(T_v).$$

Some examples

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$$f(T) = \begin{cases} 1 & \text{if the root of } T \text{ has outdegree } k, \\ 0 & \text{otherwise.} \end{cases}$$

Some more examples

- The path length, i.e., the sum of the distances from the root to all vertices, can be obtained from the toll function $f(T) = |T| - 1$:

$$P(T) = \sum_{i=1}^{k} (P(B_i) + |B_i|) = |T| - 1 + \sum_{i=1}^{k} P(B_i).$$

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- The log-product of the subtree sizes, also called the "shape functional", corresponds to $f(T) = \log |T|$. It is related to the number of linear extensions:

$$\mathrm{LE}(T) = \binom{|T| - 1}{|B_1|, |B_2|, \dots, |B_k|} \prod_{i=1}^{k} \mathrm{LE}(B_i),$$

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Even more examples

- The size of the automorphism group: if $c_1, c_2, \ldots, c_r$ are the multiplicities of the different isomorphism classes of branches, we have

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- The multiplicity of some eigenvalue λ:

$$N_\lambda(T) = \sum_{i=1}^{k} N_\lambda(B_i) + \epsilon_\lambda(T),$$

where $\epsilon_\lambda(T) \in \{-1, 0, 1\}$.

Back to AofA 2012 . . .

The number of subtrees (connected induced subgraphs) was first studied for simply generated trees by Meir and Moon in 1983. For labelled trees, the average number of subtrees is asymptotically equal to $(e/(e-1))^{3/2}e^{n/e}$.

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The asymptotic behaviour of the variance can be computed along the same lines to be of asymptotic order K^n for a constant $K \approx 2.15483 > e^{2/e}$, so the "standard" normalisation does not yield a limiting distribution.

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- A priori estimates show that $f(T)$ is exponentially small on average if T is a large tree.
- This suffices to show that $\log(1 + s_1(T))$ satisfies a central limit theorem, along with convergence of all moments.
- For the total number of subtrees $s(T)$, we have $\log s(T) = \log(1 + s_1(T)) + O(\log |T|)$.

The number of subtrees

Theorem (SW, AofA 2012)

The logarithm $\log s(\mathcal{T}_n)$ *of the number of subtrees of a random labelled tree* $\mathcal{T}_n$ *of order* n *is asymptotically normally distributed, with mean and variance asymptotically equal to* μn *and* $\sigma^2 n$ *respectively, where the numerical values of* μ *and* σ^2 *are* $\mu \approx 0.35$ *and* $\sigma^2 \approx 0.04$*, respectively.*

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Remark

Computing the constants is surprisingly tricky: they are given by infinite sums that converge poorly.

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This condition also applies to other natural examples, but turns out to be a lot stronger than necessary.

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There exist constants μ and σ^2 such that mean and variance of $F(\mathcal{T}_n)$ for a random tree $\mathcal{T}_n$ with n vertices are $\mu_n \sim \mu n$ and $\sigma_n^2 \sim \sigma^2 n$.

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Moreover, the renormalised random variable

$$X_n = \frac{F(\mathcal{T}_n) - \mu n}{\sqrt{\sigma^2 n}}$$

converges weakly to a standard normal distribution.

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- The toll function f only depends on the size of the tree.
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- The toll function f is "local" (only depends on a small neighbourhood of the root), at least approximately.

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- the multiplicity of eigenvalues (N).

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Non-Gaussian limits

When the toll function is not sufficiently small, then non-Gaussian limit distributions can be observed. Perhaps the most prominent examples are the *path length* (sum of distances from the root) and the related *Wiener index* (sum of all distances).

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For simply generated trees/conditioned Galton–Watson trees and Pólya trees (rooted unordered trees), limits that can be expressed in terms of a Brownian excursion are observed.

Path length and Wiener index

Theorem (Takács 1993, Janson 2003, SW 2012)

For conditioned Galton–Watson trees (whose offspring distribution has finite variance) and for uniformly random Pólya trees, there exists a constant $\mu > 0$ *such that the path length* $D(\mathcal{T}_n)$ *and the Wiener index* $W(\mathcal{T}_n)$ *of a random tree* $\mathcal{T}_n$ *with* n *vertices have means* $\mu_n^D \sim \mu n^{3/2}$ *and* $\mu_n^W \sim \frac{\mu}{2} n^{5/2}$ *respectively.*

Path length and Wiener index

Theorem (cont.)

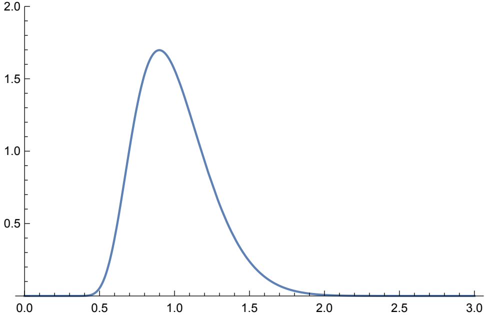
Moreover, the random variables

$$X_n = \frac{D(\mathcal{T}_n)}{\mu n^{3/2}} \qquad \text{and} \qquad Y_n = \frac{W(\mathcal{T}_n)}{\mu n^{5/2}}$$

converge weakly to random variables given in terms of a normalised Brownian excursion $e(t)$ on $[0,1]$:

$$\sqrt{\frac{8}{\pi}}\int_0^1 e(t)\,dt \quad \text{and} \quad \sqrt{\frac{8}{\pi}}\iint_{0<s<t<1}\Big(e(s)+e(t)-2\min_{s\le u\le t} e(u)\Big)\,ds\,dt.$$

Path length and Wiener index



The Airy distribution: limiting distribution of the path length.

Power toll functions

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The more general case $f(T) = |T|^\alpha$ has been studied in detail as well, as discussed in Jim Fill's talk (Fill + Kapur 2004, Fill + Flajolet + Kapur 2005, Delmas + Dhersin + Sciauveau 2018, Abraham + Delmas + Nassif 2022, Fill + Janson 2022, Fill + Janson + SW 2023+).

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A phase transition can be observed at $\mathrm{Re}\,\alpha = 0$.

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- Parameters that follow different types of recursion (e.g.: max instead of $\sum$).