A (my) brief history of Combinatorial Analysis on the λ-Calculus

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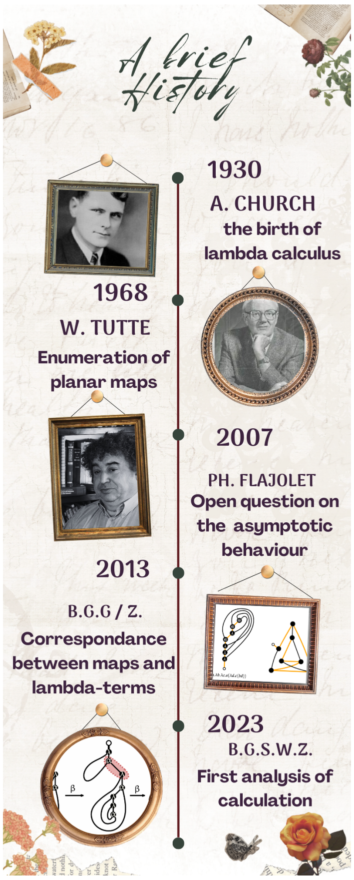
(Non-exhaustive) list of recent contributors on the topics

K. Asada, M. Bendkowski, O. Bodini, R. David, S. Dovgal, W. Fang, D. Gardy, A. Giorgetti, B. Gittenberger, Z. Golebiewski, K. Grygiel, H.K. Hwang, A. Jacquot, N. Kobayashi, J.Kozic, P. Lescanne, P. Tarau, G. Theyssier, C. Raffalli, A. Singh, R. Sin'Ya, T. Tsukada, M. Wallner, M. Zaionc, N. Zeilberger...

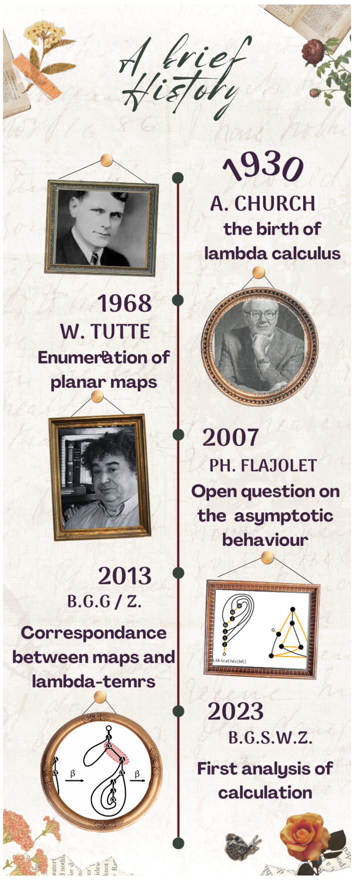
Special thanks to Pr.Dr. Alexandros Singh for many of the figures in this talk

My plan : A timeline



- What is a λ-terms (for a combinatorist)
- What is a map (for a combinatorist)
- The start of the modern history (for a combinatorist)
- Seems so far away and yet so close
- Towards the grail (for me)

A. Church and the λ-terms



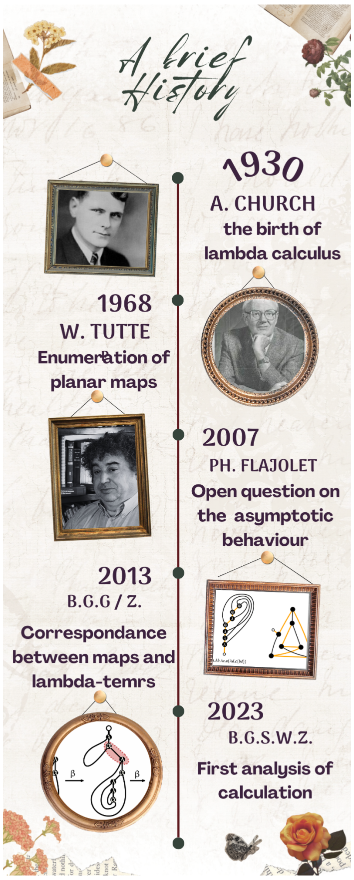
What is a λ-terms ?

$$T ::= a \mid (T * T) \mid \lambda a.T$$

a : variables

$(T * T)$: application $\qquad \lambda a.T$: abstraction

A. Church and the λ-terms



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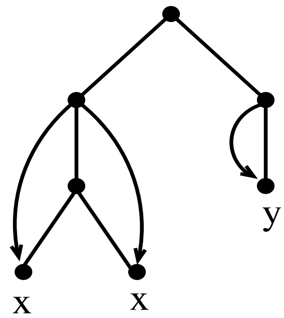
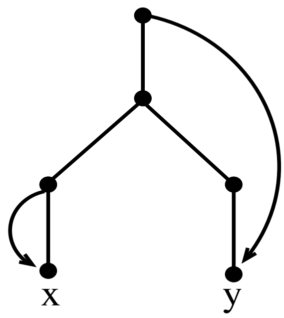
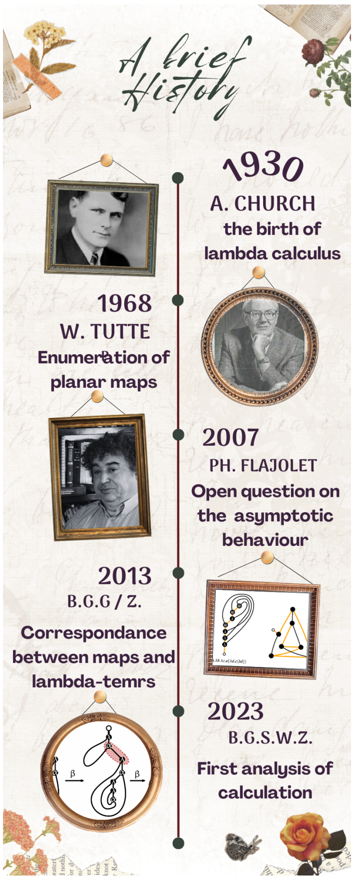
$(\lambda x.(x * x) * \lambda y * y)$



$\lambda y.(\lambda x.x * \lambda x.y)$

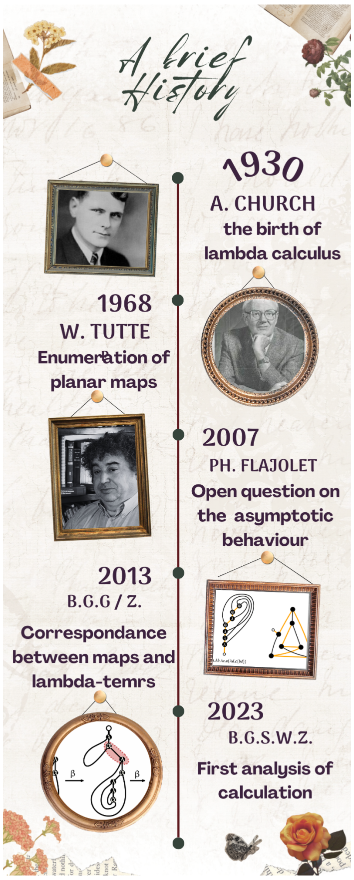


A. Church and the λ-terms



- λ-terms can be close : no free variable.

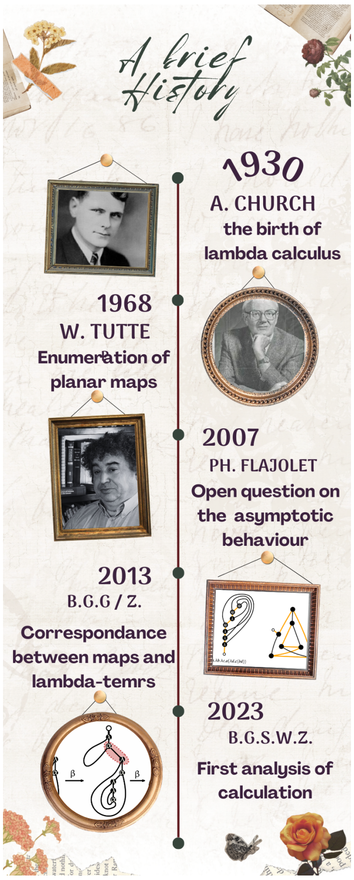
A. Church and the λ-terms



- λ-terms can be close : no free variable.
- Considered up to α-conversion : renaming of variables.

$$(\lambda x.(x * y)) = (\lambda z.(z * t)) \neq (\lambda y.(y * y))$$

A. Church and the λ-terms



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$$(\lambda x.(x * y)) = (\lambda z.(z * t)) \neq (\lambda y.(y * y))$$

- linear : each abstraction binds exactly one variable.

$$(\lambda x.x) * (\lambda y.y)$$

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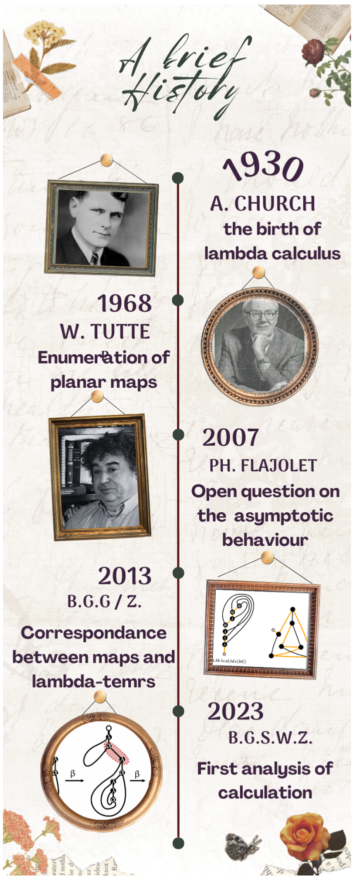

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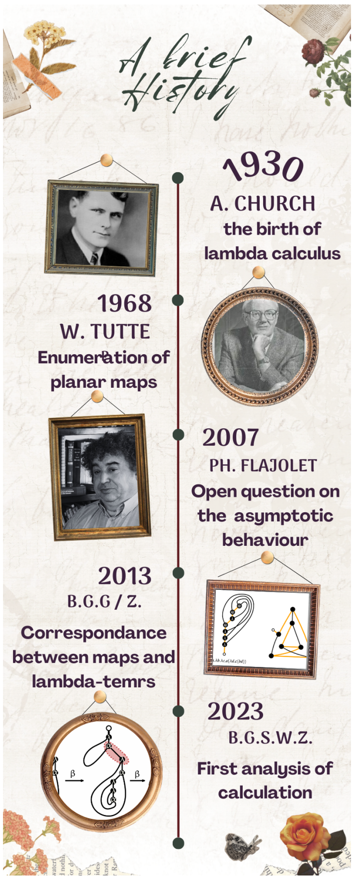


Where is the calculation ?

Calculation = β-reduction

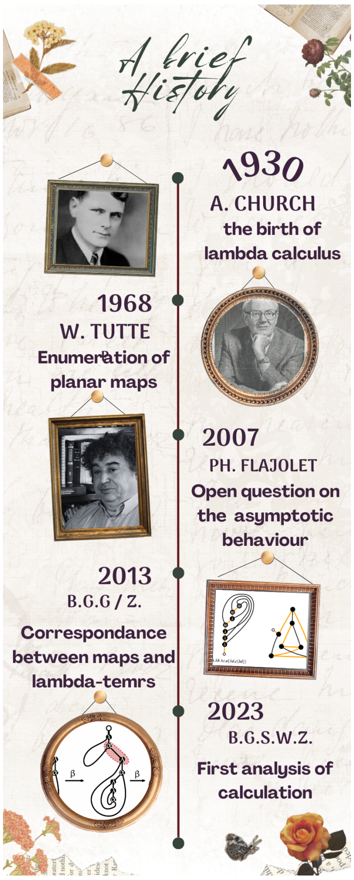
If you have somewhere in your λ-term a redex :

$$(\lambda x.T) * Q$$

Then we can apply a β-reduction that corresponds to replace all the instances of x in T by Q :

$$(\lambda x.T) * Q = T[x \leftarrow Q]$$

A. Church and the λ-terms



Where is the calculation ?

Calculation = β-reduction

Example :

$$S := \lambda t.(\lambda y.\lambda x.(x * x * y)) * (\lambda z.(z * t))$$

Here $T := x * x * y$ and $Q := \lambda z.(z * t)$

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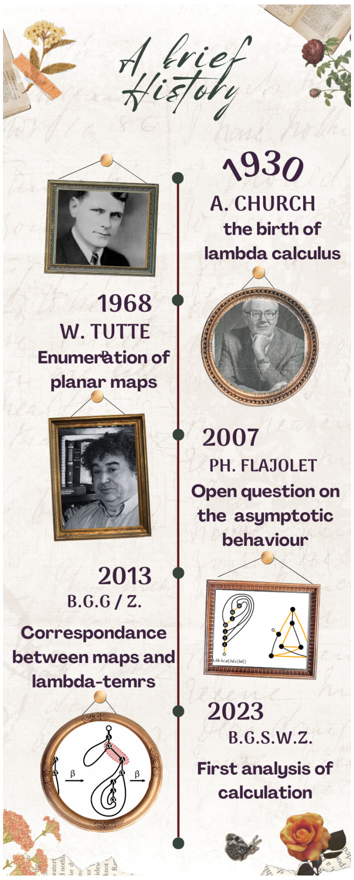


Where is the calculation ?

Calculation = β-reduction

Example :

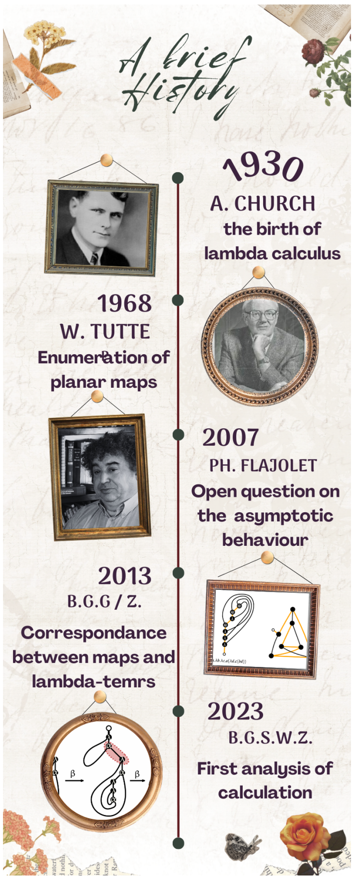
$$S := \lambda t.(\lambda y.\lambda x.(x * x * y)) * (\lambda z.(z * t))$$

Here $T := x * x * y$ and $Q := \lambda z.(z * t)$

So after β-reduction, we get

$$S := \lambda t.\lambda y.((\lambda z.(z * t)) * (\lambda z.(z * t)) * y)$$

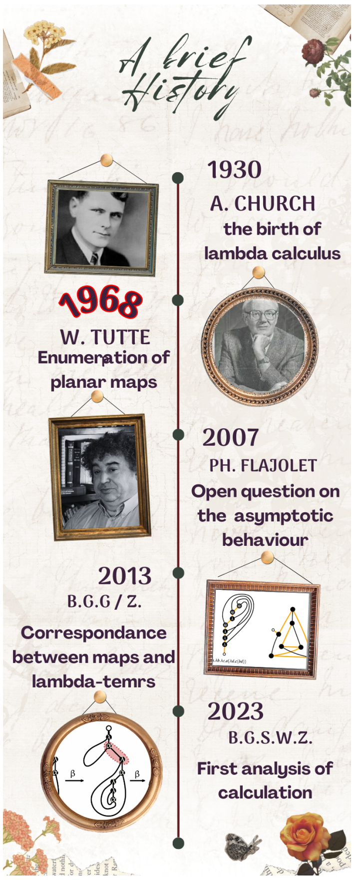
A. Church and the λ-terms



We have a model of calculation !

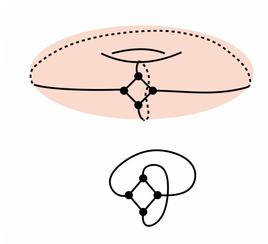
- Base of functional programming by introducing the notion of type.
- Lambda calculation is equivalent in computing power to Turing machines (Turing-Complete).
- Curry-Howard's correspondence between proofs and λ-terms

W. Tutte and the Maps



What is a map ?

A map is the representation of a graph without crossing edges on a surface.



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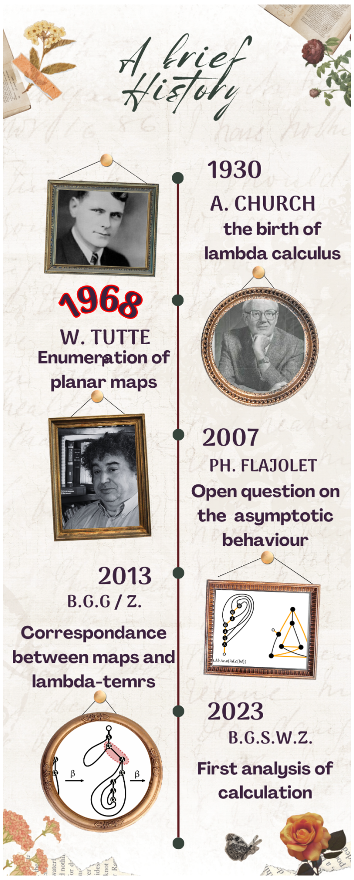

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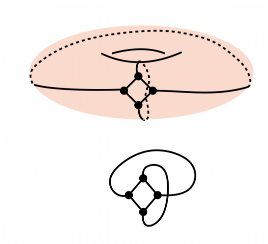
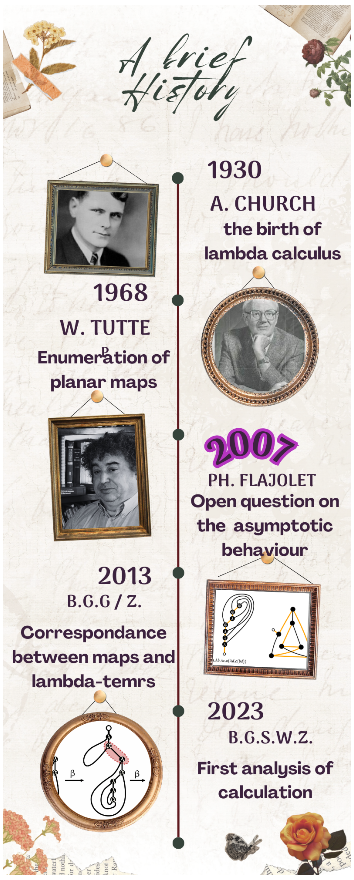


Here, we need maps on any surface (without genus restrictions).

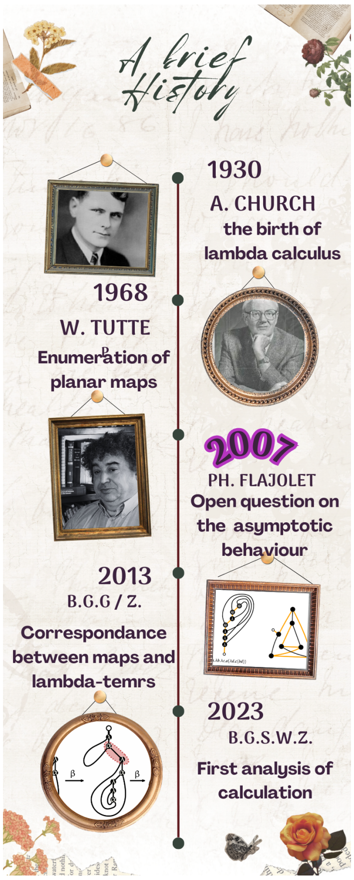
Ph. Flajolet and the Analytic Combinatoric



Can we have a combinatorial point of view on λ-calculus ?

- How count the number of λ-terms ?
- What is the asymptotic behaviour ?
- Asymptotic laws of parameters ?

Ph. Flajolet and the Analytic Combinatoric



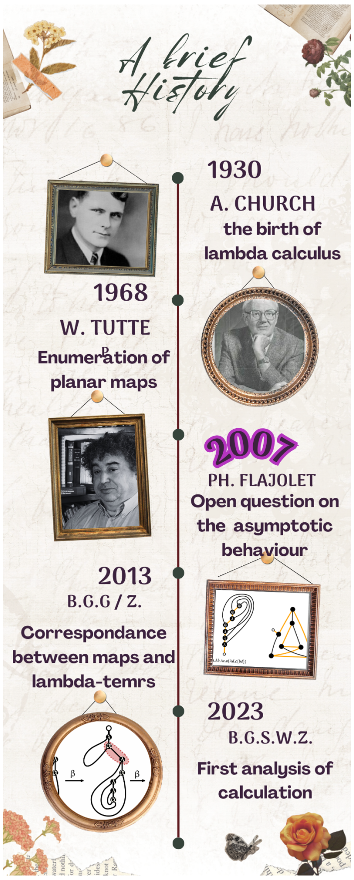
Can we have a combinatorial point of view on λ-calculus ?

Let's try the symbolic method !

$$\mathcal{L} = \mathcal{F} + \left(\mathcal{N} \times \mathcal{L}^2\right) + (\mathcal{U} \times \textit{subs}(\mathcal{F} \to \mathcal{F} + \mathcal{B}, \mathcal{L}))$$

- $\mathcal{L}$: the class of λ-terms with free variables
- $\mathcal{N}$ for the binary nodes (applications)
- $\mathcal{U}$ for the unary nodes (abstractions)
- $\mathcal{F}$ for the free leaves
- $\mathcal{B}$ for the binded leaves

Ph. Flajolet and the Analytic Combinatoric



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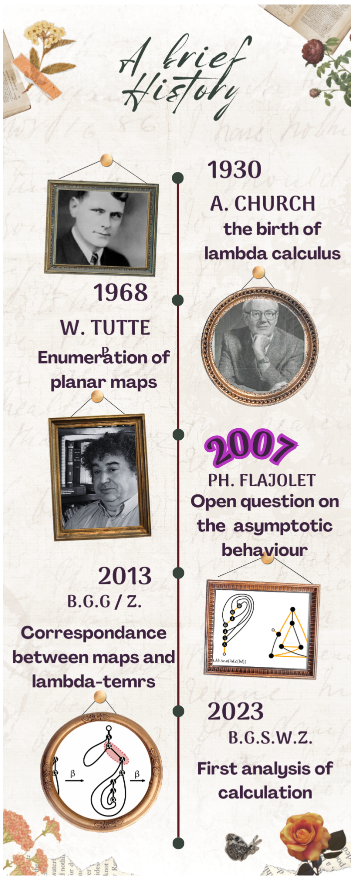
- Generating function

$$L(z, f) = fz + zL(z, f)^2 + zL(z, f + 1).$$

(size = number of nodes)

Ph. Flajolet and the Analytic Combinatoric



Can we have a combinatorial point of view on λ-calculus ?

$$L(z,f) = fz + zL(z,f)^2 + zL(z,f+1).$$

So, the series for the closed λ-terms begins as

$$\begin{aligned} L(z,0) &= [f^0]L(z,f) \\ &= z^2 + 2z^3 + 4z^4 + 13z^5 + 42z^6 + 139z^7 \\ &\quad +506z^8 + 1915z^9 + 7558z^{10} + \cdots \end{aligned}$$

Ph. Flajolet and the Analytic Combinatoric

$$L(z,0) = \frac{1}{2z}\left(1 - \sqrt{\Lambda(z)}\right)$$

with $\Lambda(z)$ equal to

$$1 - 2z + 2z\sqrt{1 - 2z - 4z^2 + 2z\sqrt{....\sqrt{1 - 2z - 4nz^2 + 2z\sqrt{...}}}}$$

Ph. Flajolet and the Analytic Combinatoric

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Ph. Flajolet and the Analytic Combinatoric

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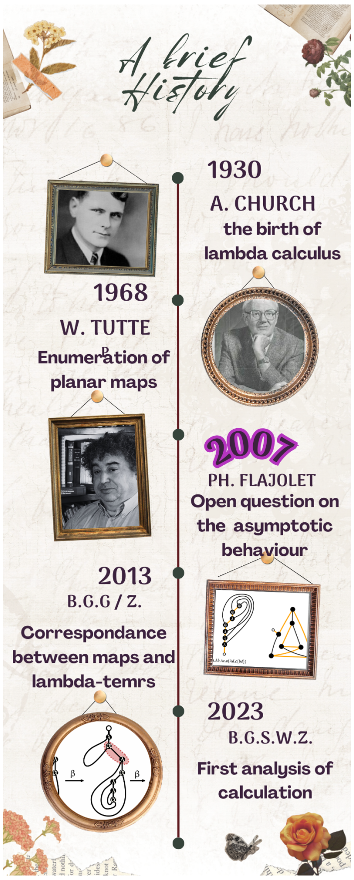
First idea : Restriction on the number of nested radicands...

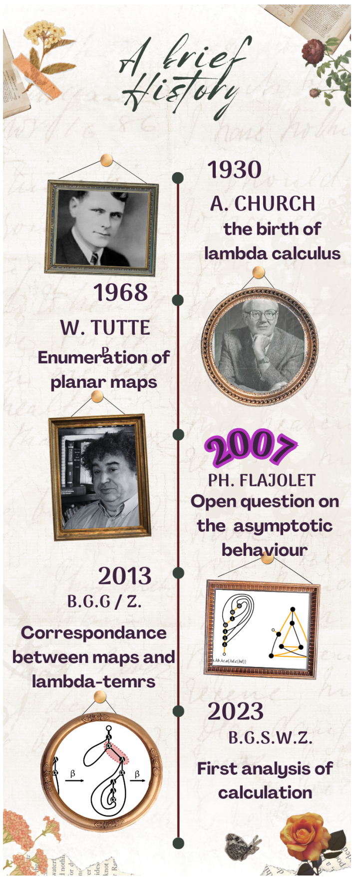
Ph. Flajolet and the Analytic Combinatoric



The bounded unary height λ-terms

Let fix k : maximum number of abstractions on a path from the root to a leave.
Denote $S^{(k)}$ the generating function of the k-bounded unary height λ-terms.

Ph. Flajolet and the Analytic Combinatoric



The bounded unary height λ-terms

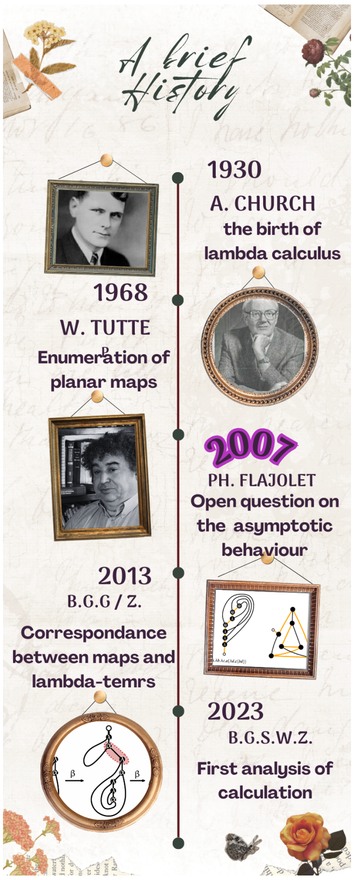
$S^{(k)} = P^{(0,k)}(z)$ is composed of k-nested radicals (square roots) :

$$P^{(k,k)}(z) = \frac{1-\sqrt{1-4kz^2}}{2z}$$

$$P^{(i,k)}(z) = \frac{1-\sqrt{1-4iz^2-4z^2P^{(i+1,k)}(z)}}{2z}$$

We'll be able to do the asymptotic study of its coefficients !

Ph. Flajolet and the Analytic Combinatoric



The bounded unary height λ-terms

$S^{(k)}$ is composed of k-nested radicals :
Where is "located" the dominant singularity as k increases ?

- $k = 1$: innermost radical
- $k = 2$: second internal radical
- $k = 3, 4, \ldots$: second internal radical
- $k = 9$: third internal radical !

Damned ! ! ! !

Ph. Flajolet and the Analytic Combinatoric

Values of k for which there are two dominant radicals (coalescence) ?

- Define $(u_k)_{k \geq 0}$ by $u_0 = 0$ and

$$u_k = u_{k-1}^2 + k \qquad \text{for} \qquad k > 0$$

- First values : $u_1 = 1$, $u_2 = 3$, $u_3 = 12$, $u_4 = 148$, $u_5 = 21909$, ...
- The sequel $(u_k)_{k \geq 0}$ is doubly exponential
- $\lim_{k \to \infty} u_k^{1/2^k} \simeq \chi = 1.36660956...$
- $u_k = \lfloor \chi^{2^k} \rfloor$

Ph. Flajolet and the Analytic Combinatoric

Define : $N_k = u_k^2 - u_k + k$.
First values : $N_1 = 1$, $N_2 = 8$, $N_3 = 135$, $N_4 = 21760$, $N_5 = 479982377$, ...

Theorem (BGG'2011)

Let i be such that $k \in [N_i, N_{i+1}[$.
By ordering radicals from inner to outer :

- *If $k \neq N_i$, the dominant radical of $S^{(k)}(z)$ is the i-th one ; the dominant singularity is algebraic in type $1/2$.*
- *If $k = N_i$, the two radicals of rank i and $(i+1)$ have the same singularity which is dominant ; this dominant singularity is algebraic in type $1/4$.*

Ph. Flajolet and the Analytic Combinatoric

Theorem (BGG'2011)

$[z^n]\mathcal{S}(N_k)_n \sim \frac{1}{\Gamma(3/4)} h_k n^{-5/4}(2u_k)^n$. *when n tends to* ∞

with $h_k = (-\frac{u_k}{2} w_{k-1,k})^{1/4} \prod_{i=k}^{n_k-1} \frac{1}{2u_{-i}}$.

and $w_{k-1,k}$ *is defined recursively by* $w_{0,k} = -4N_k/u_k$,

$w_{i,k} = -4(N_k - i)/u_k - 2 + 2u_{k-i}/u_k + w_{i-1,k}/(2u_{k-i})$.

Numerically :

$N_1 := 1 : [z^n]\mathcal{S}(1)_n \sim 0.2426128012...\, n^{-5/4}2^n$

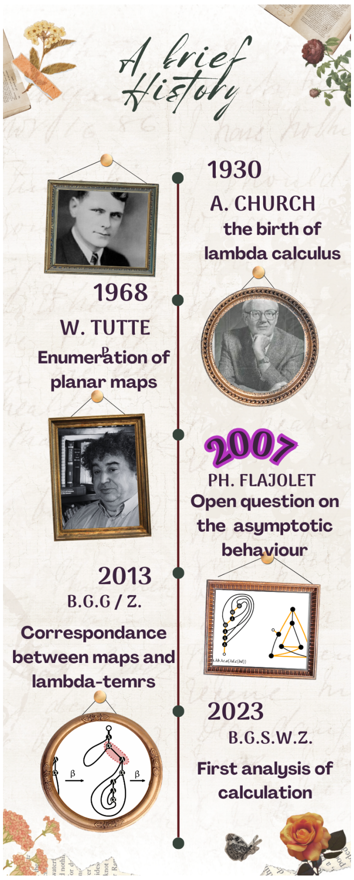
$N_2 := 8 : [z^n]\mathcal{S}(8)_n \sim 0.00009318885377...n^{-5/4}6^n$

$N_3 := 135 : [z^n]\mathcal{S}(135)_n \sim 7.116999389.... \times 10^{-158} n^{-5/4}24^n$

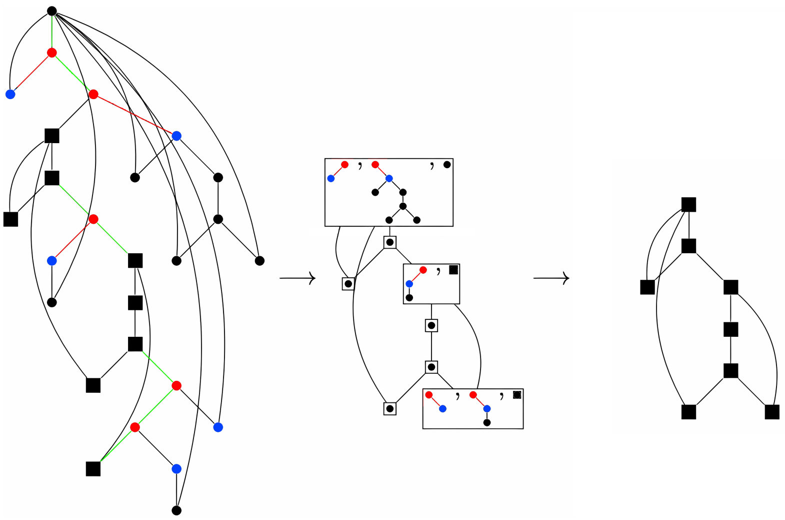
Ph. Flajolet and the Analytic Combinatoric

Function	Pos. of the dom. radical	Dom. singularity
$S^{(1)}$	{1,2}	0.5
$S^{(2)}$	2	0.3438
$S^{(3)}$	2	0.2760
...	...	...
$S^{(8)}$	{2,3}	0.1667
$S^{(9)}$	3	0.1571
...	...	...
$S^{(134)}$	3	0.0418
$S^{(135)}$	{3,4}	0.0417
$S^{(136)}$	4	0.0415
...	...	...

Ph. Flajolet and the Analytic Combinatoric

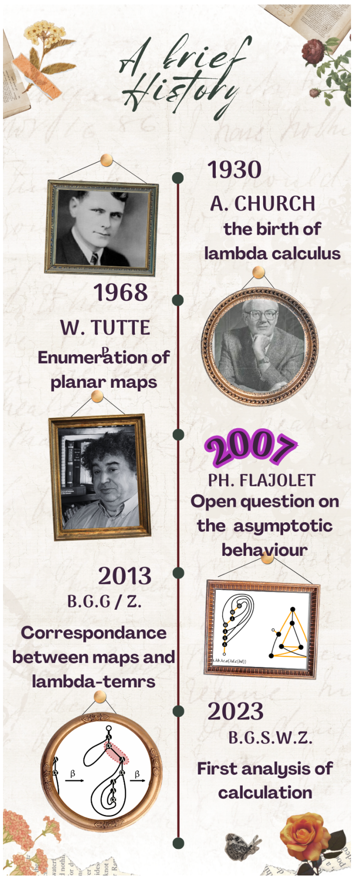


General λ-terms without catalytic variables [BGGJ'13]



$$L(z) = zM(z) + zL(z)^2 + zL\left(\frac{z}{1-2zM}\right)$$

Linear λ-terms and maps



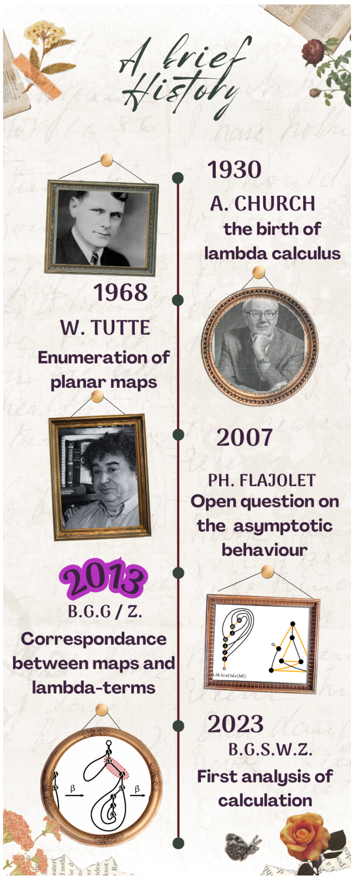
Asymptotic of the general λ-terms ?

$$L(z) = zM(z) + zL(z)^2 + zL(\frac{z}{1-2zM})$$

$$[z^n]L(z) \bowtie \left(\frac{4n}{e\ln(n)}\right)^{n/2}$$

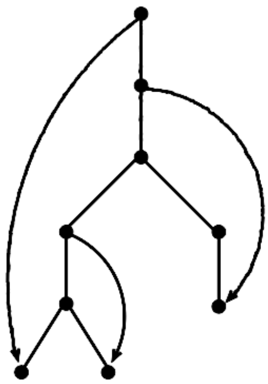
More precise asymptotic is still open

Linear λ-terms and Maps



The sub-class of linear λ-terms

Each abstraction binds exactly one variable.
$n+1$ variables $\rightarrow$ $n+1$ abstractions $\rightarrow$ n applications $\rightarrow$ size := $3n+2$



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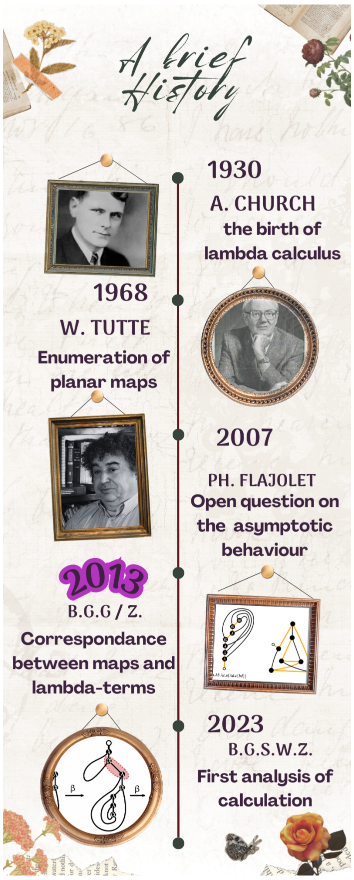
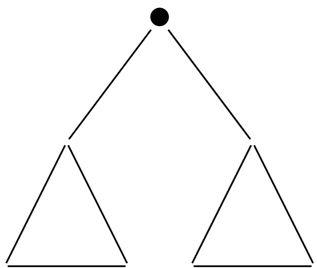
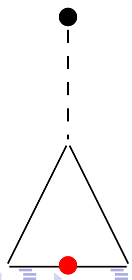


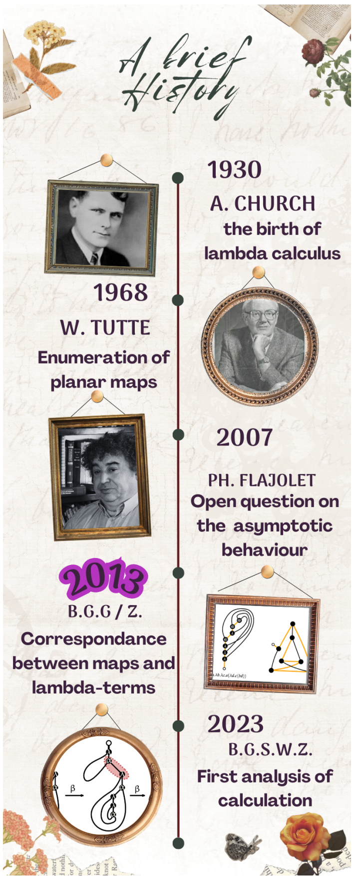
A catalytic specification for linear λ-terms

$$T(z,u) = uz + z\,T^2(z,u) + z\,\frac{\partial T}{\partial u}(z,u)$$





Linear λ-terms and Maps



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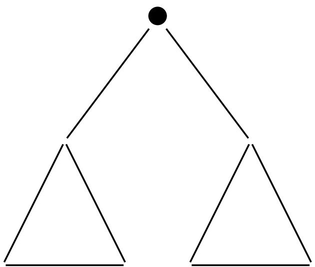
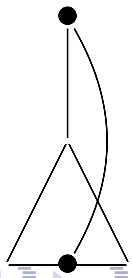
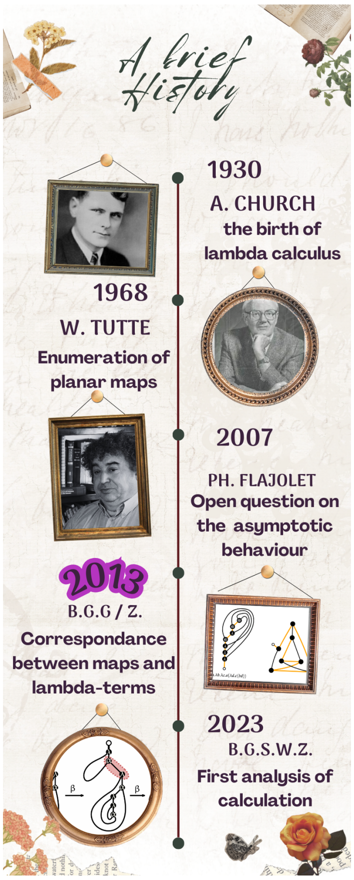




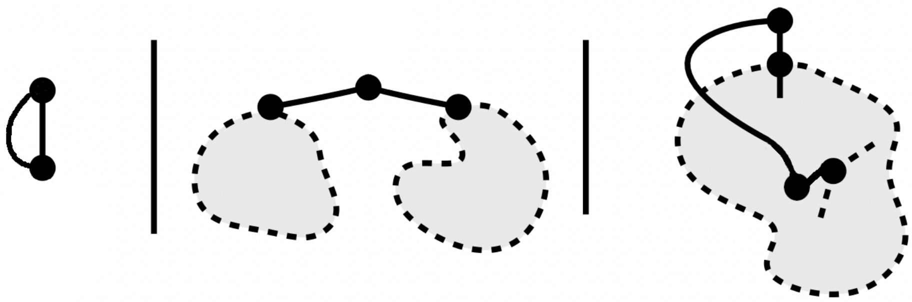
Linear λ-terms and Maps



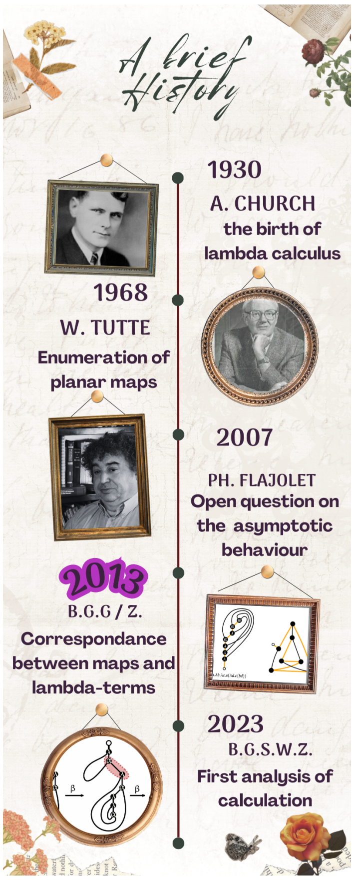
A specification (without catalytic variable) for linear λ-terms

Each abstraction binds exactly one variable.

$$T(z) = z^2 + zT(z)^2 + 2z^4 \frac{\partial T}{\partial z}(z)$$



Linear λ-terms and Maps



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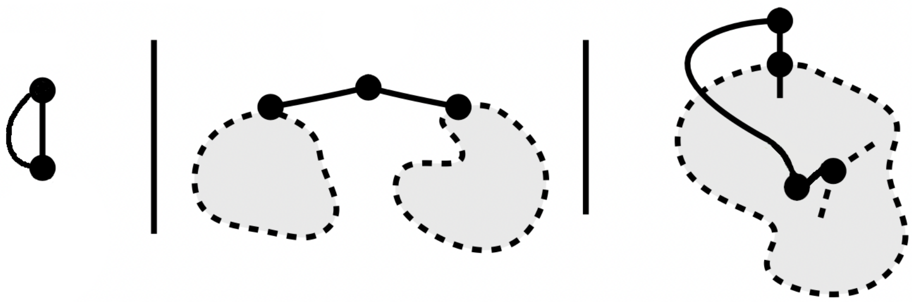
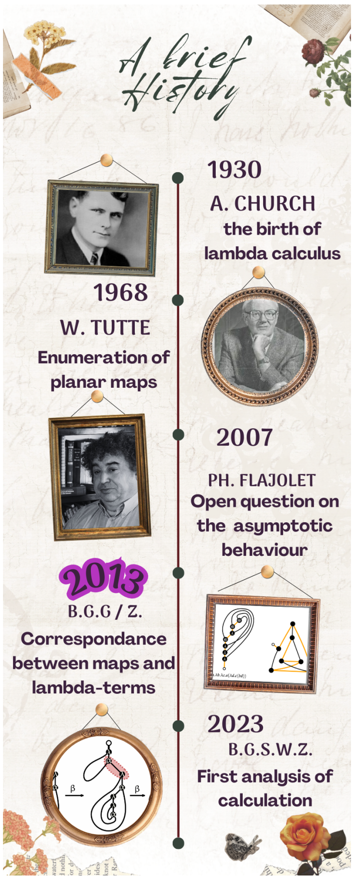
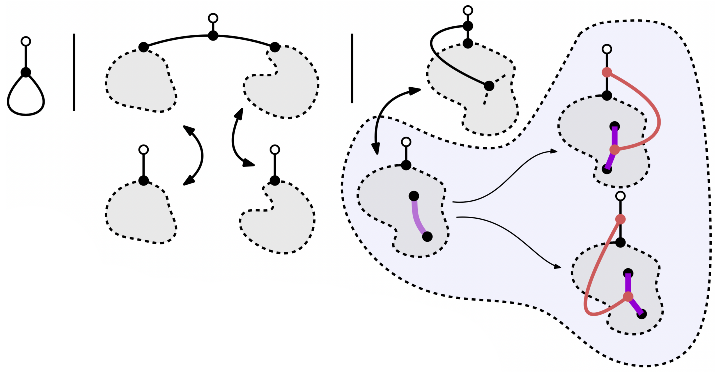


Linear λ-terms and Maps

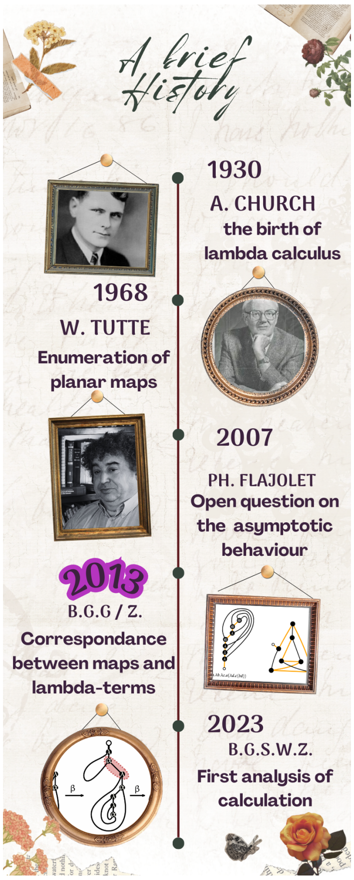


A specification for connected rooted trivalent maps

$$T(z) = z^2 + zT(z)^2 + 2z^4 \frac{\partial T}{\partial z}(z)$$



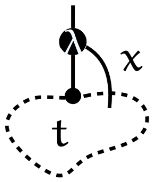
Linear λ-terms and Maps



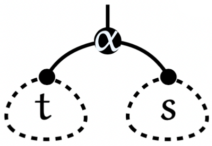
Linear λ-terms $\cong$ rooted trivalent maps

From closed terms to maps

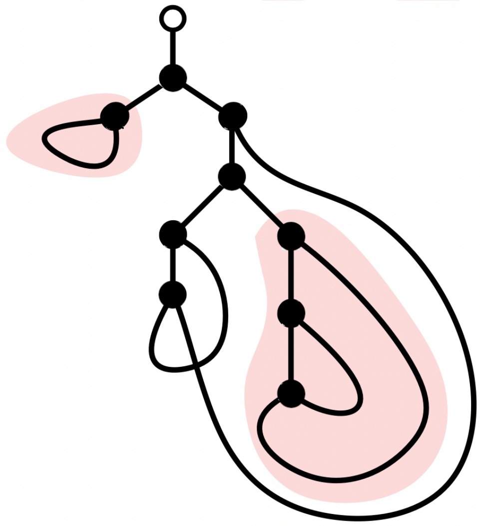
$\lambda x.t \rightarrow$



$(t\ s) \rightarrow$



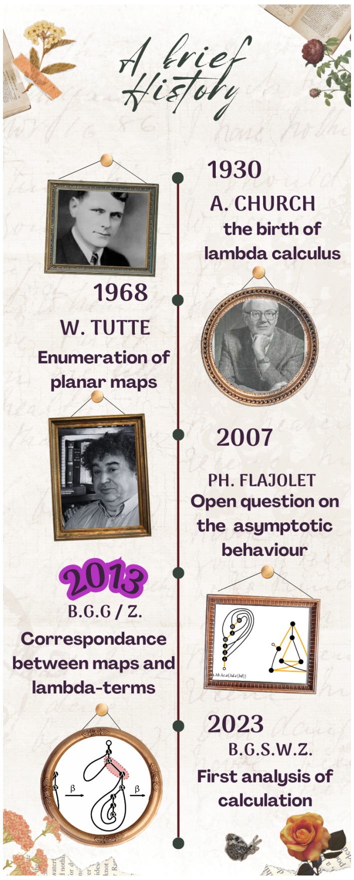
$(\lambda x.x)\ (\lambda y.(\lambda z.z\ y)\ (\lambda w.\lambda u.w\ u))$



Dictionary

- # subterms $\leftrightarrow$ # edges
- closed subterms $\leftrightarrow$ bridges
- using variables in order $\leftrightarrow$ planarity of maps

Divergent Riccati equation



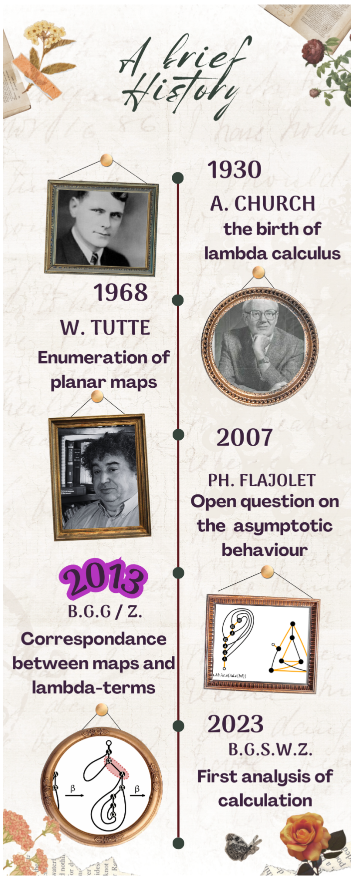
Divergent Riccati equation

Classical change of variable $T = \dfrac{U'}{U}$

Which can be seen as a combinatorial operation : Take only the connected objects + Pointing.

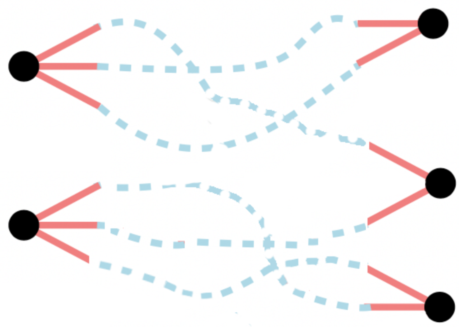
All Riccati equation arising in combinatorics (I know) come from that (irreducible diagrams, irreducible permutations,...) !

Divergent Riccati equation



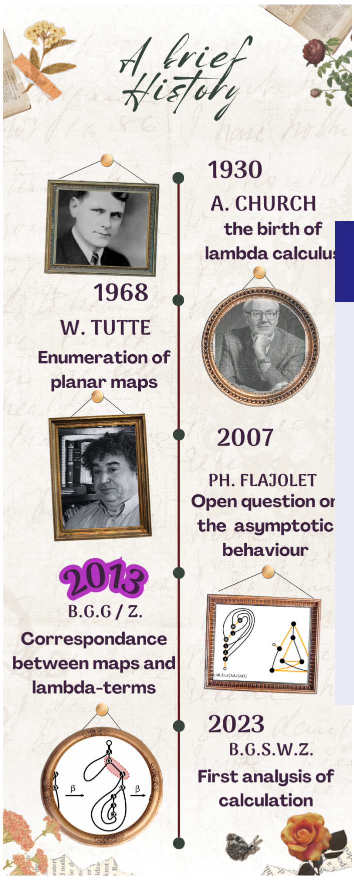
Divergent Riccati equation

$$T(z^2) = z^4 + z^5 \frac{\partial}{\partial z} \ln\left(\exp(z^3/3) \odot \exp(z^2/2)\right)$$



(rigidity)

Divergent Riccati equation



Divergent Riccati equation

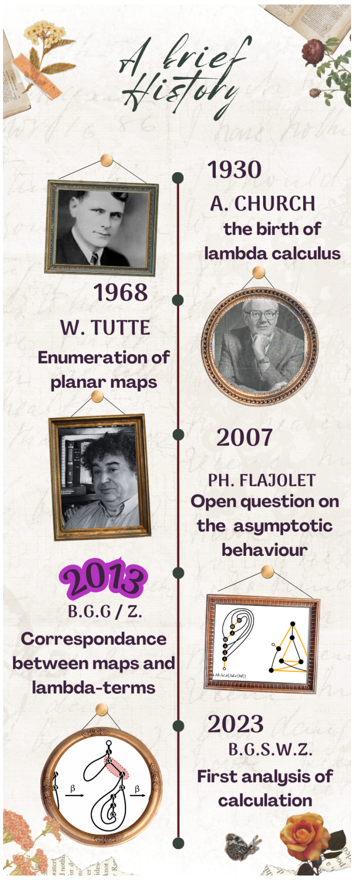
Theorem (BGJ'2013)

$$T(z^2) = z^4 + z^5 \frac{\partial}{\partial z} \ln \left(\exp(z^3/3) \odot \exp(z^2/2) \right)$$

$$\lambda_{3n+2} \sim 3/\pi \times 6^n n!$$

- Efficient random sampling.

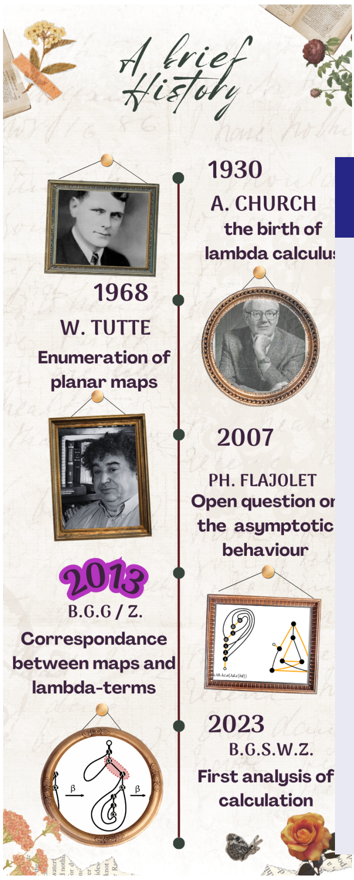
Linear λ-terms and Maps



Efficient random sampling for linear λ-terms

- Do...
- Draw a permutation of size $6n$ product of 3-cycles.
- Draw an involution without fixed point of size $6n$.
- Build a map by gluing half-edges.
- Until the map is connected (this appends with probability 1 when $n \to \infty$)
- Build the linear lambda term from it.

Linear λ-terms and Maps

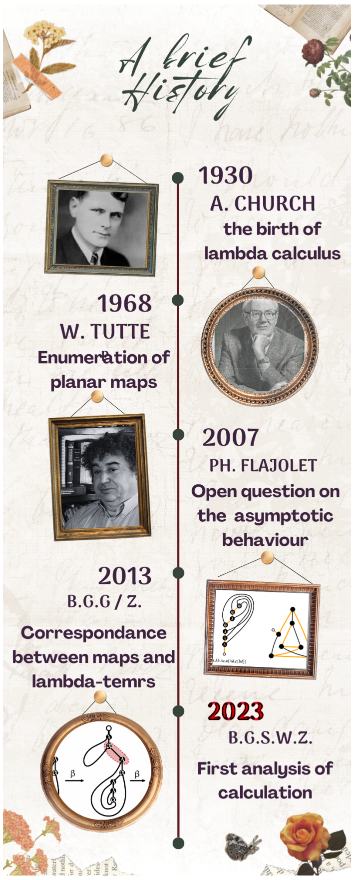


Parameters on linear λ-terms

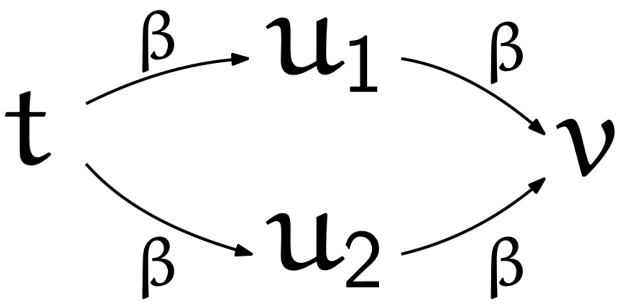
Theorem (BSZ'2021)

- *The random variable for identity-subterms in closed linear terms converges in distribution to a Poisson of parameter 1.*
- *The random variable for closed subterms in closed linear terms converges in distribution to a Poisson of parameter 1.*
- *The random variable for free variables in open linear terms converges in distribution to $\mathcal{N}((2n)^{1/3}, (2n)^{1/3})$.*

Towards calculation on linear λ-terms

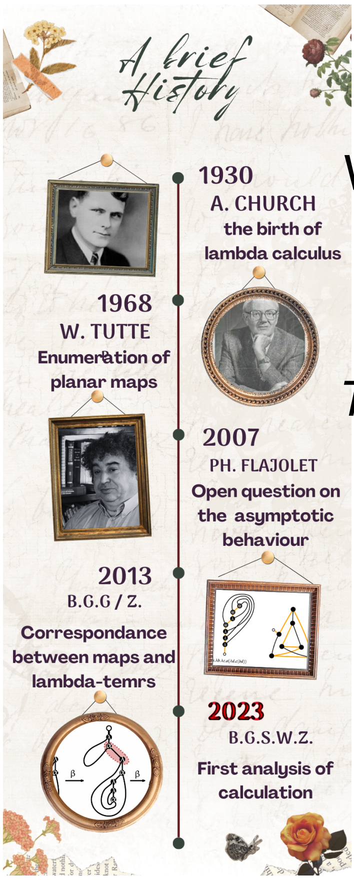


For linear terms, β-reduction possesses the strong diamond property.



All the paths of β-reduction terminates after a finite number of steps to the same λ-term with no redex (called normal form). All these paths have same number of steps.

Towards calculation on linear λ-terms

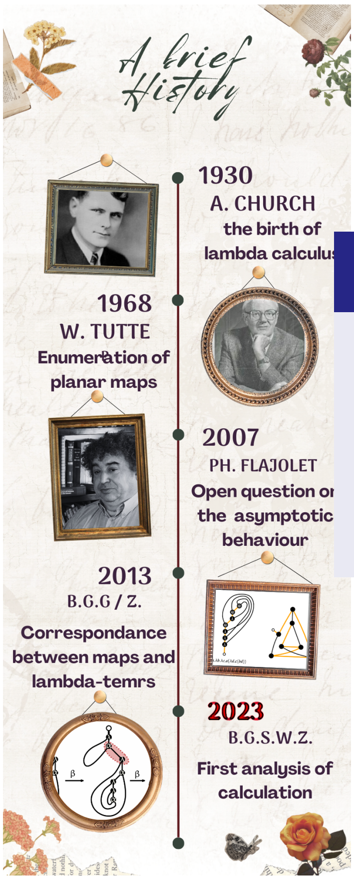


What is the mean length of full β-reduction chain ?
First step : count (mark with r) the number of redices !

$$T = z^2 + zT^2 + z^3(1+(r-1)zT)\left(\frac{z(r+5)\partial_z T}{3} - (r^2-1)\partial_r T\right) + \frac{z^4(r-1)^2T^2}{3} + \frac{4z^3(r-1)T}{3}$$

Then, pump the first two moments...

Towards calculation on linear λ-terms



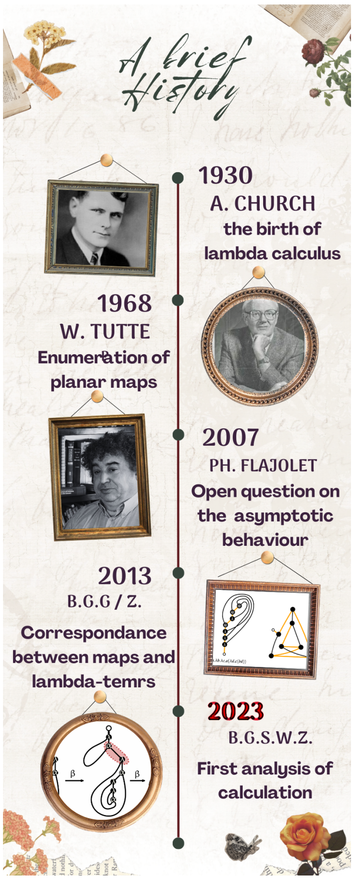
Theorem (BSZ'2021)

Mean number of redices in closed linear terms of size n is asymptotically in $\frac{n}{24}$

The variance is also asymptotically in $\frac{n}{24}$

This is of course a first lower bound for the length of the β-redex chain...

Towards calculation on linear λ-terms



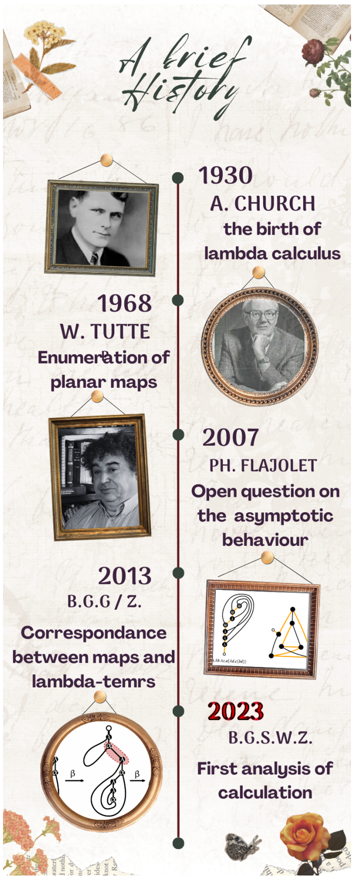
The only three patterns that "create" a new redex :
(see JJ Lévy's thesis)

$$p_1 := (\lambda x.C[(xt_1)])(\lambda y.t_2) \to_\beta C[((\lambda y.t_2)t_1)]$$

$$p_2 := (\lambda x.x)(\lambda y.t_1)t_2 \to_\beta (\lambda y.t_1)t_2$$

$$p_3 := ((\lambda x.\lambda y.t_1)t_2)t_3 \to_\beta (\lambda y.t_1[x := t_2])t_3$$

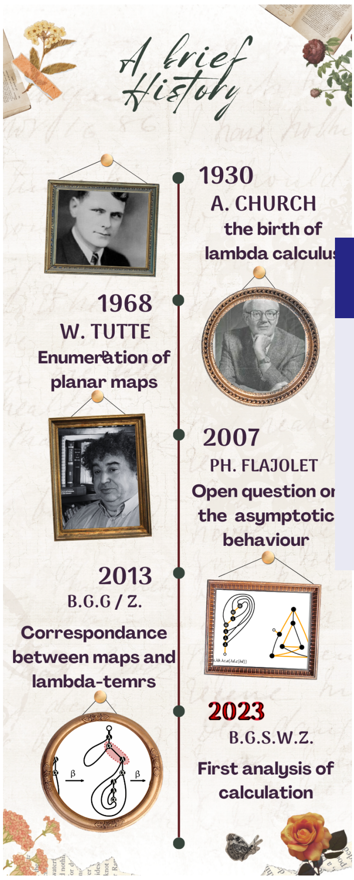
Towards calculation on linear λ-terms



The length of the β-reduction chain is larger than
$$|\beta\text{-redex}| + |p_1| + |p_2| + |p_3|$$

All that's left to do is work...

Towards calculation on linear λ-terms

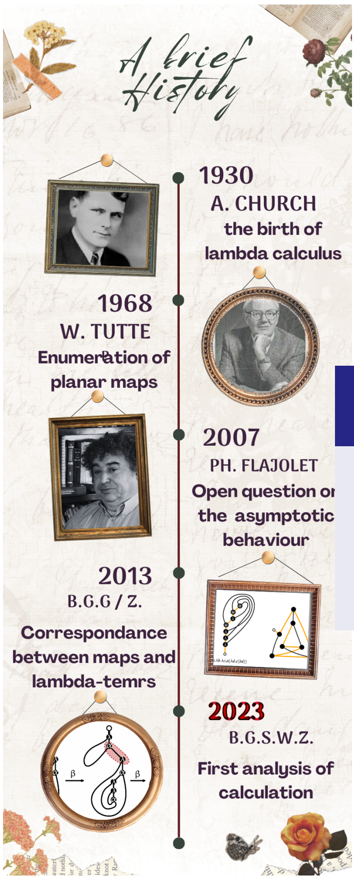


Theorem (BGSWZ'2023)

The mean number of pattern p_1 (resp. p_2) in closed linear terms of size n is asymptotically a constant $\mu_1 = \dfrac{1}{6}$ *(reps.* $\mu_2 = \dfrac{1}{48}$*)*

Essentially, we still are able to specify these patterns (but that needs some care !)

Towards calculation on linear λ-terms



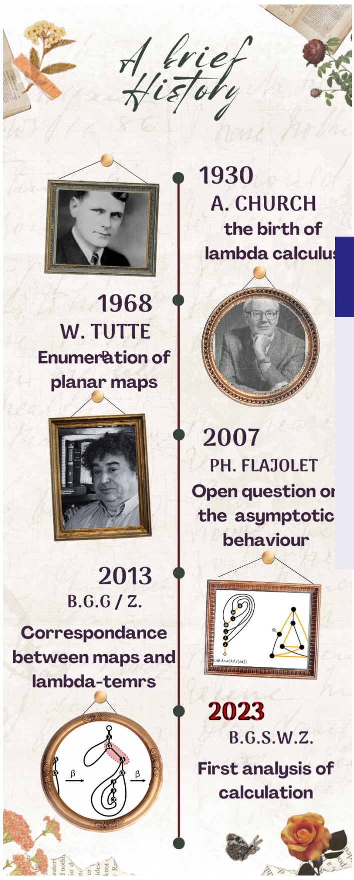
The pattern p_3 is very hard to specify (perhaps impossible due to auto-correlation). But Alexandros Singh has proposed an elegant probabilistic method to get around this difficulty !

Theorem (BGSWZ'2023)

The mean number μ_3 of pattern p_3 in closed linear terms of size n verifies asymptotically $\mu_3 \geq \frac{n}{240}$

In fact, we can have the equality, but this needs very sophisticated guess and proof approach based on differential algebra.

Towards calculation on linear λ-terms



Theorem (BGSWZ'2023)

The mean length ℓ_n of the β-reduction chain for a closed linear terms of size n verifies asymptotically

$$\ell_n \geq \frac{11n}{240}$$

Very closed to Noam's conjecture of $n/21$
$(1/21 - 11/240 = 0.0017...)$

Perspectives and (still) open questions

- Find the asymptotic of general lambda-terms (work in progress with H.K.)
- Almost nothing is known about parameters of general lambda-terms
- Prove or disprove Noam's conjecture on *beta*-reduction (In particular, we currenlty only have a lower bound
- Have more informations on the size of the terms during the *beta*-reduction process...

MERCI

THANK YOU

謝謝