Negative Results

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If yes, is there a way to measure the problem's difficulty without solving it?

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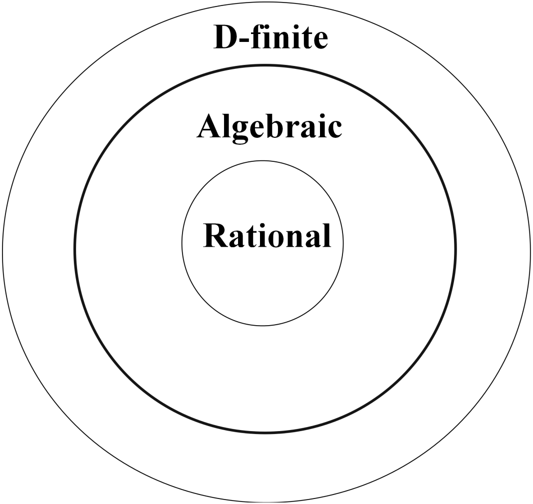

Figure: Types of power series.

Nonrationality

Back to our main question: if we do not know a power series, how do we prove that it is not rational?

We will use the main law of combinatorial asymptotics: if $F(z) = \sum_{n \geq 0} f_n z^n$ is a combinatorial generating function, then the exponential growth rate of its coefficients f_n is equal to $1/R$, where R is the radius of convergence of F around 0.

Supercriticality

Definition

Let F and G be two generating functions with nonnegative real coefficients that are analytic at 0, and let us assume that $G(0) = 0$. Then the relation

$$F(z) = \frac{1}{1 - G(z)}$$

is called *supercritical* if $G(R_G) > 1$, where R_G is the radius of convergence of G.

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Analytical meaning: As the coefficients of $G(z)$ are nonnegative, $G(R_G) > 1$ implies that $G(\alpha) = 1$ for some $\alpha \in (0, R_G)$. So, F has a singular point *before* G does, and so the exponential growth rate of the coefficients of F is larger than that of G.

Combinatorial meaning:
$F(z) = 1/(1 - G(z)) = 1 + G(z) + G(z)^2 + G(z)^3 + \cdots$, so if F counts structures that can have several components, then G is the generating function of those structures that have only one such component.

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For instance, let $F(z)$ be the generating function for the number of ways to tile a $1 \times n$ path with red and blue tiles of size 1×1 and green, white, and yellow tiles of size 1×2. Then $G(z) = 2z + 3z^2$, implying that $R_G = \infty$ and $G(R_G) = \infty > 1$, and

$$F(z) = \frac{1}{1 - 2z - 3z^2}.$$

Rational functions and supercriticality

Theorem

Let $G(z)$ be a rational power series with nonnegative real coefficients that satisfies $G(0) = 0$. Then the relation

$$F(z) = \frac{1}{1 - G(z)}$$

is supercritical.

Proof.

If $G(z)$ is a polynomial, then $R_G = \infty$, so $G(R_G) = \infty > 1$, and our claim is proved. Otherwise, $G(z)$ is a rational function that has at least one singularity, and all its singularities are poles. Let R_G be a singularity of smallest modulus. Then $G(R_G) = \infty > 1$, completing our proof. □

Our Main Application: Permutation Patterns

Permutation patterns are a good example here because

1. they are very widely studied,
2. their generating functions are mostly unknown, and
3. their is a long-standing, difficult conjecture about their generating functions.

In other words, in order to prove that some generating function F is not rational, it is sufficient to show that if

$$F(z) = \frac{1}{1 - G(z)},$$

then the relation between F and G is *not supercritical*.

And one way to do that is by showing that F and G have the *same convergence radius*, or, equivalently, their coefficients have the *same exponential order*.

Pattern Containment

We say that a permutation $p_1 p_2 \cdots p_n$ contains the shorter permutation $q_1 q_2 \cdots q_k$ as a pattern if there is a subsequence of entries in p that relate to each other as the entries of q.

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That is, p contains q as a pattern if there is a subsequence of k entries $p_{i_1} p_{i_2} \cdots p_{i_k}$ so that $p_{i_a} < p_{i_b}$ if and only if $q_a < q_b$.

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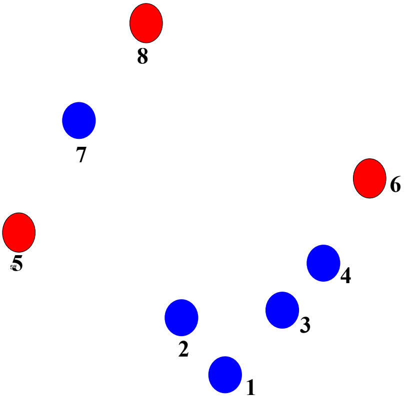

Figure: Containing the pattern 132.

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The known results are on two special cases, short patterns, and monotone patterns.

Short patterns:

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If q is of length three, then $Av_n(q) = \binom{2n}{n}/(n+1)$, so

$$A_q(z) = \frac{1-\sqrt{1-4z}}{2z},$$

which is algebraic.

If $q = 1342$, then

$$Av_q(z) = \frac{2z}{1+20z-8z^2-(1-8z)^{3/2}},$$

which is algebraic again.

Monotone patterns

If $q = 12 \cdots k$, then $A_q(z)$ is D-finite.

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If $q = 12 \cdots k$, then $A_q(z)$ is D-finite.

However, if $k > 2$ is even, then $A_q(z)$ is *not* algebraic.

Size

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Due to some trivial and some non-trivial equivalences, this means that the only patterns of length four for which an explicit formula is not known is 1324 and its reverse 4231. For that pattern, we do not know even the exponential growth rate of the sequence $Av_n(q)$, or how nice the generating function $A_q(z)$ is.

The Zeilberger-Noonan conjecture, single pattern case

Let $Av_n(q)$ be the number of permutations of length n that avoid the pattern q.

Conjecture

(Z-N, 1997) For all q, the sequence $Av_n(q)$ is polynomially recursive. Equivalently, the generating function $A_q(z)$ of the sequence is D-finite.

Note that for the case of multiple patterns, the conjecture has been disproved by Igor Pak and Scott Garrabrant.

Our result

We cannot decide the Z-N conjecture for single patterns, but we can prove, in two slightly different ways, that for most patterns, $A_q(z)$ is not rational.

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We say that a permutation p is *skew indecomposable* if it is not possible to cut p into two parts so that each entry before the cut is larger than each entry after the cut.

For instance, $p = 3142$ is skew indecomposable, but $r = 346512$ is not as we can cut it into two parts by cutting between entries 5 and 1, to obtain $3465|12$.

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If p is not skew indecomposable, then there is a unique way to cut p into **nonempty** skew indecomposable strings $s_1, s_2, \cdots, s_\ell$ of consecutive entries so that each entry of s_i is larger than each entry of s_j if $i > j$.

We call these strings s_i the *skew blocks* of p.

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For instance, $p = 67|435|2|1$ has four skew blocks, while skew indecomposable permutations have one skew block.

Theorem

Let q be a skew indecomposable pattern that does not end in its largest entry. Then $A_q(z)$ is not rational.

It is clear that p avoids q if and only if each of the skew blocks of p avoids q.

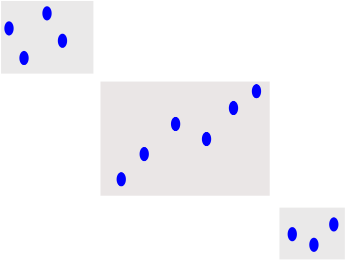

Figure: The skew blocks of p.

This means that

$$A_q(z) = \frac{1}{1 - A_{q,1}(z)}$$

holds, where $A_{q,1}(z)$ is the generating function of the skew indecomposable q-avoiders.

Remember, we have seen that if $A_q(z)$ is rational, this means that the two generating functions *cannot* have the *same* convergence radius, so their coefficients *cannot* have the *same* exponential order.

Let p be of length n, and let p avoid q. Now affix a new entry $n+1$ at the end of p. The new permutation p' still avoids q, but is also skew indecomposable.

So, if $Av_{n,1}(q)$ is the number of skew indecomposable q-avoiders, then

$$Av_{n,1}(q) \leq Av_n(q) \leq Av_{n+1,1}(q),$$

so the sequences $Av_n(q)$ and $Av_{n+1,q}$ have the same exponential order.

This contradiction completes our proof.

What are the patterns for which this does not work

The preceding argument shows that if q is skew indecomposable, and does not end in its largest entry, then $A_q(z)$ is not rational.

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It is not difficult to prove that a bit more is true.

Theorem

Let q be a skew indecomposable pattern of length k so that at least one of the following conditions hold.

1. *q does not start with the entry 1, or*
2. *q does not end with the entry k, or*
3. *q is Wilf-equivalent to a skew indecomposable pattern that satisfies at least one of the first two conditions.*

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Then $A_q(z)$ is not rational.

For instance, $q = 12 \cdots k$ does not satisfy the first two conditions, but it is Wilf-equivalent to $2134 \cdots k$, that does (if $k > 2$).

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A slightly different proof

It can be shown that for patterns as in the previous theorem, the inequality

$$Av_{n,2}(q) \leq Av_{n,1}(q)$$

holds, where $Av_{n,i}(q)$ is the number of permutations of length n avoiding q with i skew blocks.

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If $A_{q,1}(z)$ is rational, then its dominant singularity z_0 is a pole. So

$$\lim_{z \to z_0} A_{q,1}(z) = \infty.$$

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This contradicts

$$\sum_{n\geq 1} Av_{n,1}(q) z_1^n = A_{1,q}(z_1) \geq A_{2,q}(z_1) = \sum_{n\geq 2} \mathrm{Av}_{n,2}(q) z_1^n,$$

that holds since all coefficients are non-negative.

Another approach

We know that coefficients of rational generating functions satisfy the rule

$$a_n \sim b \cdot c^n \cdot n^d,$$

where c is a nonnegative real number, and d is a *natural* number.

However, for pattern avoiding permutations,

$$A_n(q)A_m(q) \leq A_{n+m}(q),$$

which implies that $d = 0$ must hold.

Non-algebraicity

A classic result of Jungen says that if a complex power series $f = \sum_{n \geq 0} f_n z^n$ is algebraic, and

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The following result is due to Amitaj Regev.

Theorem

For all $k \geq 2$, there exists a constant γ_k so that the asymptotic equality

$$Av_n(k \cdots 21) \simeq \gamma_k \frac{(k-1)^{2n}}{n^{(k^2-2k)/2}}$$

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holds.

So the generating function for such permutations is not algebraic if k is even. This is the only nonrationality result for single pattern avoidance.

A lemma by Alin Bostan

Lemma

Let $f(z) = \sum_{n \geq 0} f_n z^n$ *be a power series with nonnegative real coefficients that is analytic at the origin. Let us assume that constants* c, C, K *and* m *exist so that* $m > 1$ *is an integer, and for all positive integers* n, *the chain of inequalities*

$$c\frac{K^n}{n^m} \leq f_n \leq C\frac{K^n}{n^m} \tag{1}$$

holds. Then $f(z)$ *is not an algebraic power series.*

An application

Let $\mathfrak{S}_n(q,1)$ denote the set of permutations of length n that contain exactly one copy of the pattern q, and let $\mathfrak{S}_n(q) := \mathfrak{S}_n(q,0)$. Let $S_n(k\cdots 21,1) = |\mathfrak{S}_n(q,1)|$.

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In joint work with Alex Burstein, we have recently proved the following.

Theorem

There exists an injection

$$F_k : \mathfrak{S}_n(k\cdots21,1) \to \mathfrak{S}_{n+2}(k\cdots21).$$

Because of our injection, $S_n(k \cdots 21, 1) \leq Av_{n+2}((k-1)k \cdots 21))$, while it is obvious that

$$S_n(k \cdots 21, 1) \geq Av_{n-k}(k \cdots 21).$$

So the exponential order of the sequence $S_n(k \cdots 21, 1)$ is the same as that of the sequence $Av_n(k \cdots 21)$, that is, $(k-1)^{2n}$.

And, we proved that the conditions of Bostan's lemma hold, that is, the growth rate of the sequence $S_n(k\cdots 21, 1)$ is squeezed between two constant multiples of

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If k is even, that means that the exponent in the denominator is an integer, so the generating function of $S_n(k \cdots 21, 1)$ is not algebraic.