Phase transitions of composition schemes and their universal limit laws

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article: arXiv:2103.03751, Annals of Applied Probability, to appear.

Part 1:
Compositions and
generalized Mittag-Leffler distributions

Ubiquity of compositions schemes in combinatorics

Combinatorial structure = assemblage of basic building blocks

- random walks
- Pólya urns
- Galton–Watson processes
- trees
- permutations
- random mappings
- set partitions
- integer partitions
- tilings
- graphs
- maps
- ...

A composition scheme for generating functions

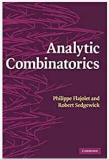
$$\sum_{n\geq 0} f_n z^n = F(z) = G\big(H(z)\big) M(z)$$

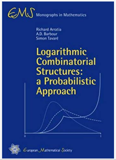
Let ρ_G and ρ_H be the radii of convergence of $G(z)$ and $H(z)$, resp. Then, the composition scheme is *critical* if $H(\rho_H) = \rho_G$ and $\rho_M \geq \rho_H$.

Examples:

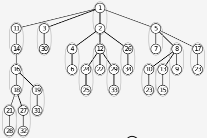
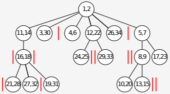
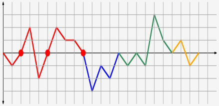
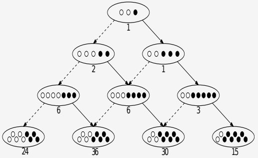
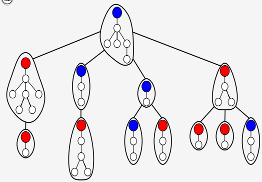
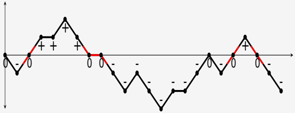
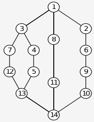
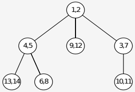
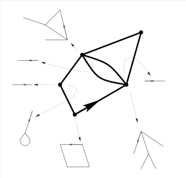
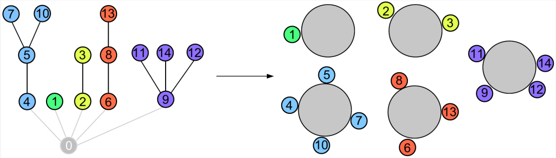
- Bicolored supertrees: $F(z) = C(2zC(z))$
- Factorization of walks: $W(z) = \frac{1}{1-H(z)} M(z)$

NB: If not critical: [Bender 1973, Gourdon 1998, Hwang 1999, ...]



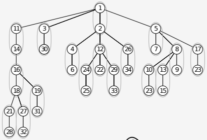
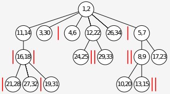
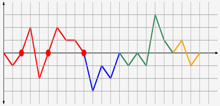
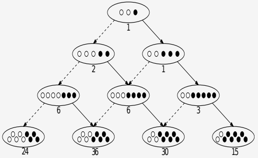
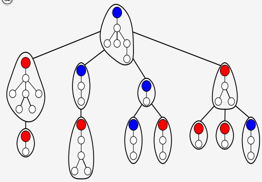
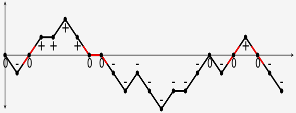
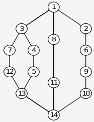
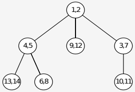
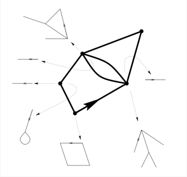
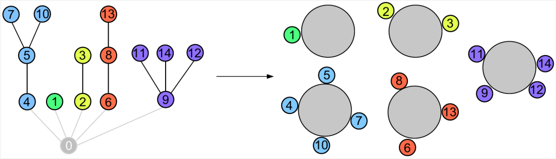


Combinatorial structures $\qquad G(H(z)) \times M(z)$

For sure, sum of almost iid $\rightsquigarrow$ asymptotics distributions are Gaussian.

Combinatorial structures $\quad G(H(z)) \times M(z)$

For sure, sum of almost iid $\rightsquigarrow$ asymptotics distributions are NON Gaussian.

Goal 1: Analyse $F(z,u) = G(uH(z))M(z)$

Number of $\mathcal{H}$-components: Define the discrete random variable X_n of the *core size*:

$$\mathbb{P}\{X_n = k\} = \frac{[z^n u^k]F(z,u)}{[z^n]F(z,1)}$$

Note that $H(z)$ has typically the following singular expansion

$$H(z) = \tau_H + c_H\left(1 - \frac{z}{\rho_H}\right)^{\lambda_H} + \dots.$$

$\Rightarrow$ the asymptotic behaviour of $\mathbb{P}\{X_n = k\}$ depends on the *singular exponent λ_H*!

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Limit law of X_n related to certain distributions:

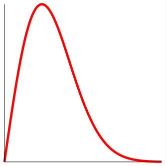
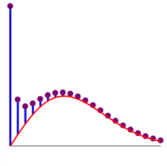
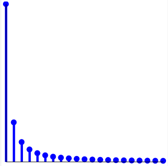
- $\lambda_H < 0$: scheme *not* critical as $H(z)$ diverges at $z = \rho_H$
  (called supercritical, typically Gaussian)
- $0 < \lambda_H < 1$: generalized Mittag-Leffler distribution (this talk!)
  ($\lambda_H = 1/2$, $M(z) = 1$: Rayleigh distribution)
- $1 < \lambda_H < 2$: related to stable laws of parameter λ_H
  ($\lambda_H = 3/2$, $M(z) = 1$: map-Airy distribution
  [Banderier, Flajolet, Schaeffer, Soria 2001])
- $\lambda_H > 2$: Gaussian

Main results: composition scheme

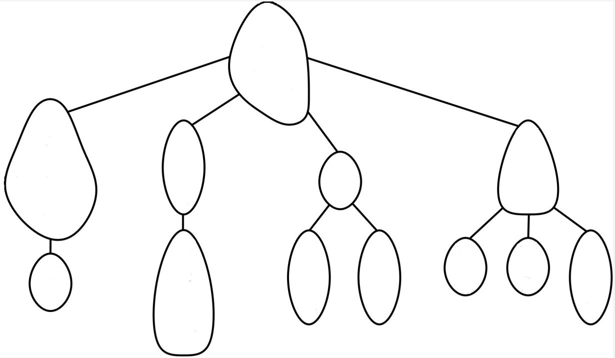
Our model: $$F(z,u) = G\big(uH(z)\big) \cdot M(z),$$

for $F/G/H/M$ analytic at the origin, with nonnegative coefficients, and singular exponents $\lambda_F/\lambda_G/\lambda_H/\lambda_M$, such that $0 < \lambda_H < 1$.

Main result 1: Limit laws of X_n are generalized Mittag-Leffler product distr.

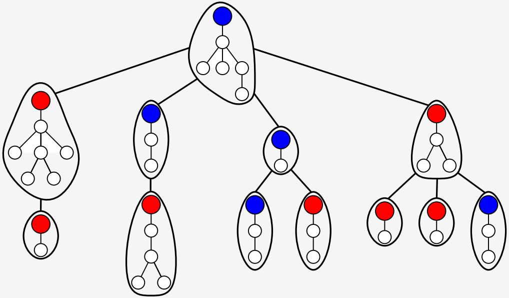
Singular exponent	$\lambda_M > \lambda_G\lambda_H$ (pure scheme)	$\lambda_M = \lambda_G\lambda_H$ (confluent scheme)	$\lambda_M < \lambda_G\lambda_H$ (degenerate scheme)
Limit law	continuous (gen. ML)	linear combination ($\text{ML} + \mathcal{B}$)	discrete (Boltzmann $\mathcal{B}$)
Example			
	$X_n \sim C n^{\lambda_H} \text{ML}$	$X_n \sim \text{LinComb}(n^{\lambda_H}\,\text{ML}, \mathcal{B})$	$\mathbb{P}\{X_n = k\} \sim \frac{g_k \rho_G^k}{G(\rho_G)}$

Example: Bicolored supertrees



plane trees: $C(z) = z\frac{1}{1-C(z)} = \frac{1-\sqrt{1-4z}}{2z}$

Example: Bicolored supertrees



supertrees trees: $F(z) = C(2zC(z))$

$$f_n = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{2^k}{n-k} \binom{2k-2}{k-1} \binom{2n-3k-1}{n-k-1} \sim \frac{4^n n^{-5/4}}{8\Gamma(3/4)}.$$

A parte for systems of equations with coefficients ≥ 0 *(*$\approx$ *context-free grammars)*
[Drmota–Lalley–Woods 1995]: if strongly connected $\rightsquigarrow$ $n^{-3/2}$ asymptotics;
if not, [Flajolet 1985] conjectured that $n^{-1/3}$ never occurs...

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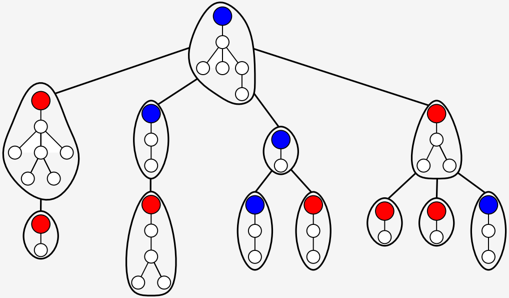

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if not, [Flajolet 1985] conjectured that $n^{-1/3}$ never occurs... proven by
[Banderier, Drmota 2015]: $n^{-1-\lambda_F}$, with $\lambda_F = 1/2^k$ or $\lambda_F = -m/2^k$ $(m, k \geq 1)$.

Example: Bicolored supertrees

Theorem (The number of patatoids)

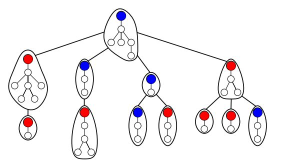
The core size X_n in supertrees of size n has factorial moments

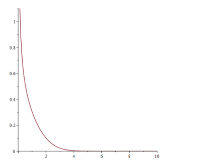
$$\mathbb{E}(X_n^{\underline{s}}) \sim n^{s/2} \cdot \mu_s, \qquad \mu_s = \frac{\Gamma(s-\frac{1}{2})\Gamma(-\frac{1}{4})}{\Gamma(-\frac{1}{2})\Gamma(\frac{s}{2}-\frac{1}{4})}.$$

The scaled random variable $X_n/n^{1/2}$ converges in distribution with convergence of all moments to a 2-parameter Mittag-Leffler distribution:

$$\frac{X_n}{n^{1/2}} \xrightarrow[m]{d} X, \qquad \text{with} \qquad X \stackrel{d}{=} \mathrm{ML}\left(\frac{1}{2}, -\frac{1}{4}\right).$$

Moreover, we have the local limit theorem $\mathbb{P}\{X_n = x \cdot n^{1/2}\} \sim n^{-1/2} f_X(x)$, with $f_X(x)$ denoting the density of the random variable X.





2-parameter Mittag-Leffler distribution

- A positive random var. S_α follows a **stable law of parameter** $\alpha \in (0,1)$ if

$$\mathbb{E}(e^{-tS_\alpha}) = e^{-t^\alpha} \quad \text{or, equivalently,} \quad f_{S_\alpha}(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\Gamma(-n\alpha)} x^{-n\alpha-1}.$$

- A random variable M_α follows a **Mittag-Leffler distribution** $\mathrm{ML}(\alpha)$ if

$$M_\alpha \stackrel{d}{=} (S_\alpha)^{-\alpha}.$$

$\Rightarrow$ Its MGF $\mathbb{E}(e^{xM_\alpha})$ is the Mittag-Leffler function $E_\alpha(x) = \sum_{k\geq 0} \frac{x^k}{\Gamma(1+\alpha k)}$.

Definition ([Pitman 2006, James 2015, Goldschmidt, Haas 2015])

Let $\alpha \in (0,1)$ and $\beta > -\alpha$. Then, the **2-parameter Mittag-Leffler distribution** $\mathrm{ML}(\alpha,\beta)$ is uniquely defined by its moments

$$\mathbb{E}(X^s) = \frac{\Gamma\left(s + \frac{\beta}{\alpha} + 1\right)\Gamma(\beta+1)}{\Gamma(\alpha s + \beta + 1)\Gamma\left(\frac{\beta}{\alpha}+1\right)} = \frac{\Gamma\left(s + \frac{\beta}{\alpha}\right)\Gamma(\beta)}{\Gamma(\alpha s + \beta)\Gamma\left(\frac{\beta}{\alpha}\right)}.$$

- $\mathrm{ML}(\alpha,0) = M_\alpha$
- $\mathrm{ML}(1/2,0)$: *half-normal distribution* $|\mathcal{N}(0,\sigma^2)|$ of parameter $\sigma = \sqrt{2}$
- $\mathrm{ML}(1/2,1/2)$: *Rayleigh distribution* of parameter $\sqrt{2}$

3-parameter Mittag-Leffler distribution

The distributions of *critical composition schemes* will be the **3-parameter Mittag-Leffler distributions** $\mathrm{ML}(\alpha, \beta, \gamma)$ defined as

$$Z \stackrel{d}{=} Y \cdot B^{\alpha}$$

where $Y \stackrel{d}{=} \mathrm{ML}(\alpha, \beta)$ and $B \stackrel{d}{=} \mathrm{Beta}(\beta, \gamma)$ are independent, such that $0 < \alpha < 1$, $\beta > 0$, and $\gamma \geq 0$.

Lemma

The 3*-parameter Mittag-Leffler distribution* $\mathrm{ML}(\alpha, \beta, \gamma)$ *has the following moments*

$$\mathbb{E}(Z^s) = \frac{\Gamma\left(s + \frac{\beta}{\alpha}\right)\Gamma\left(\beta + \gamma\right)}{\Gamma\left(\alpha s + \beta + \gamma\right)\Gamma\left(\frac{\beta}{\alpha}\right)}.$$

One has the following identity

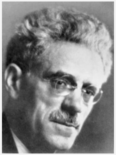
$$Z \stackrel{d}{=} \mathrm{ML}(\alpha, \beta)\,\mathrm{Beta}(\beta, \gamma)^{\alpha} \stackrel{d}{=} \mathrm{ML}(\alpha, \beta + \gamma)\,\mathrm{Beta}(\frac{\beta}{\alpha}, \frac{\gamma}{\alpha}).$$

- distribution with moments of Gamma type [Janson 2010]
- explicit representation of its density by integrals or hypergeometric functions

A moment problem

Torsten Carleman

(1892-1949)



Maurice Fréchet

(1878-1973)

Theorem

Our X_n converges to the 3-parameter Mittag-Leffler distribution, which is characterized by its moments

$$\mathbb{E}(\mathrm{ML}(\alpha,\beta,\gamma)^r) = \frac{\Gamma\left(r+\frac{\beta}{\alpha}\right)\Gamma(\beta+\gamma)}{\Gamma(\alpha r+\beta+\gamma)\,\Gamma\left(\frac{\beta}{\alpha}\right)}.$$

Proof. Set $m_r := \mathbb{E}[X^r]$ and $m_r(n) := \mathbb{E}[X_n^r]$.

[Fréchet, Shohat 1930]: if $m_r(n) \to m_r$ then $X_n \xrightarrow{d} X$

... if the moments determine X uniquely!

[Carleman 1923]: There is a unique distribution with such moments if :

– for support $[0,\infty)$ (Stieljes moment problem): $\sum 1/m_r^{1/2r} = \infty$

– for support $(-\infty,\infty)$ (Hamburger moment problem): $\sum 1/m_{2r}^{1/2r} = \infty$

– for support $[0,1]$ (Hausdorff moment problem): m_r completely monotonic. □

A tool to identify densities: tilts and shifts

Lemma (Tilted lemma)

Let X be a random variable with moment sequence $(\mu_s)_{s\geq 0}$ and density $f(x)$ of support $[0,+\infty)$.
For any $c \in \mathbb{R}^+$, we have a random variable X_c satisfying:

$$\mathbb{E}(X_c^s) = \frac{\mu_{s+c}}{\mu_c} \qquad \textit{(shifted moments)}$$

$$\iff \qquad f_c(x) = \frac{x^c}{\mu_c} \cdot f(x) \qquad \textit{(tilted density)}$$

$$\iff \qquad \mathbb{E}(e^{tX_c}) = \frac{1}{\mu_c} \partial_t^c \mathbb{E}(e^{tX}) \qquad \textit{(MGF differentiation)}$$

(with fractional calculus definition of ∂_t^c for $c \notin \mathbb{N}$).
We write $X_c = \operatorname{tilt}_c X$.

Example:

$$\operatorname{tilt}_{\beta/\alpha}(\mathrm{ML}(\alpha)) \stackrel{d}{=} \mathrm{ML}(\alpha,\beta), \qquad \text{i.e.,} \qquad \operatorname{tilt}_{\beta/\alpha}(S_\alpha^{-\alpha}) = (\operatorname{tilt}_{-\beta}(S_\alpha))^{-\alpha}.$$

Three different regimes

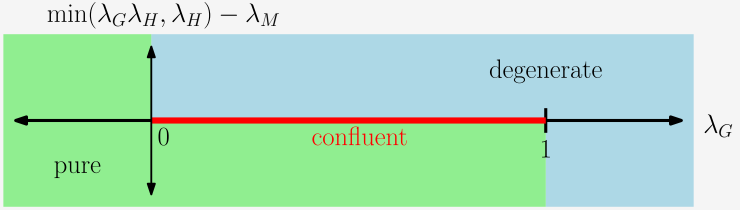
Recall: $\lambda_F/\lambda_G/\lambda_H/\lambda_M$ are the singular exponents of $F/G/H/M$

i.e., $F(z) = \underbrace{\tau_F + \dots}_{\text{initial regular part}} + c_F(1 - z/\rho_F)^{\lambda_F} + \dots$

Lemma

In a critical composition scheme $F(z) = G(H(z))M(z)$ with $0 < \lambda_H < 1$, the singular exponent λ_F of $F(z)$ satisfies

$$\lambda_F = \min(\lambda_G\lambda_H + \lambda_M, \lambda_G\lambda_H, \lambda_H, \lambda_M).$$



Composition scheme: pure case

Theorem


In a pure critical composition scheme

$$F(z,u) = G\big(uH(z)\big)M(z),$$

the core size X_n, converges in distribution and in moments to a random var. X distributed like a 3-parameter Mittag-Leffler distribution:

$$\frac{X_n}{\kappa \cdot n^{\lambda_H}} \xrightarrow[m]{d} X, \qquad \text{with} \qquad X \stackrel{d}{=} \mathrm{ML}(\alpha,\beta,\gamma),$$

where $\alpha = \lambda_H$, $\beta = -\lambda_G\lambda_H$, $\gamma = -\min(0,\lambda_M)$, and $\kappa = \frac{\tau_H}{-c_H}$.

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where $\alpha = \lambda_H$, $\beta = -\lambda_G\lambda_H$, $\gamma = -\min(0,\lambda_M)$, and $\kappa = \frac{\tau_H}{-c_H}$.
What is more, one has a local limit theorem

$$\mathbb{P}\{X_n = x \cdot \kappa n^{\lambda_H}\} \sim \frac{1}{\kappa n^{\lambda_H}} \cdot f_X(x),$$

where $f_X(x)$ is the density of X:

$$f_X(x) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta/\alpha)} \sum_{j\geq 0} \frac{(-1)^j}{j!\Gamma(\gamma - j\alpha)} x^{\beta/\alpha+j-1}.$$

Simplifications

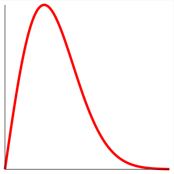
1. $\lambda_M \geq 0$ (which includes $F(z,u) = G(uH(z))$):

$$X_1 \stackrel{d}{=} \mathrm{ML}(\lambda_H, -\lambda_G \lambda_H, 0) \stackrel{d}{=} \mathrm{ML}(\lambda_H, -\lambda_G \lambda_H)$$

In particular, for $\lambda_G = -1$ and $\lambda_H = \frac{1}{2}$:

$$X_1 \stackrel{d}{=} \text{Rayleigh}$$

Sequence scheme [Drmota, Soria 1997]



$$f_{X_1}(x) = \frac{x}{2} \exp\left(-\frac{x^2}{4}\right)$$

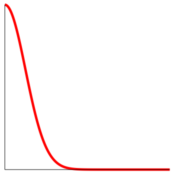
2. $\lambda_M < 0$, $\lambda_G = -1$, and $\lambda_H - \lambda_M = 1$:

$$X_2 \stackrel{d}{=} \mathrm{ML}(\lambda_H, \lambda_H, 1 - \lambda_H) \stackrel{d}{=} \mathrm{ML}(\lambda_H).$$

In particular, for $\lambda_H = \frac{1}{2}$:

$$X_2 \stackrel{d}{=} \text{Halfnormal}$$

Sequence scheme [Wallner 2020]



$$f_{X_2}(x) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{4}\right)$$

Note that $\lambda_G = -1$ "=" *sequences* of $\mathcal{H}$-components

Composition scheme: degenerate case

Theorem

In a degenerate critical composition scheme

$$F(z,u) = G\big(uH(z)\big)M(z)$$

the core size X_n converges for $0 < \lambda_G < 1$ and $\lambda_M < \lambda_G \lambda_H$ to a Boltzmann distribution:

$$\mathbb{P}\{X_n = k\} \to \mathbb{P}\{\mathcal{B}_G(\rho_G) = k\} = \frac{g_k \rho_G^k}{G(\rho_G)}.$$

The case $\lambda_G > 1$ is similar.

Definition (Boltzmann distribution $\mathcal{B}_G(x)$)

Let $G(z) = \sum_{n \geq 0} g_n z^n$ be a generating function and $x > 0$ inside the radius of convergence. Then, the *Boltzmann distribution* $\mathcal{B}_G(x)$ is defined by

$$\mathbb{P}\{X = n\} = \frac{g_n x^n}{G(x)}, \quad n \geq 0.$$

$\rightsquigarrow$ "Boltzmann method": using this Gibbs measure for each object of size n lead to a revolution for uniform random generation [Flajolet, Duchon, Louchard, Schaeffer 2001]

Composition scheme: confluent case

Theorem

In a confluent (i.e., $0 < \lambda_G < 1$ and $\lambda_M = \lambda_G \lambda_H$) ext. crit. comp. scheme

$$F(z,u) = G\big(uH(z)\big)M(z)$$

the core size X_n is a convex combination of a Boltzmann distribution $\mathcal{B}_G(\rho_G)$ and an asymptotically continuous random variable Z_n:

$$X_n \sim \mathrm{Be}(p) \cdot \mathcal{B}_G(\rho_G) + (1 - \mathrm{Be}(p)) \cdot Z_n, \qquad \frac{Z_n}{\kappa \cdot n^{\lambda_H}} \xrightarrow{d} \mathrm{ML}(\lambda_H, -\lambda_G\lambda_H),$$

where $p = \frac{c_M G(\rho_G)}{c_M G(\rho_G) + \tau_M c_G (-c_H/\rho_G)^{\lambda_G}}$, and indep. rv's $\mathrm{Be}(p)$, $\mathcal{B}_G(\rho_G)$, Z_n, and ML.

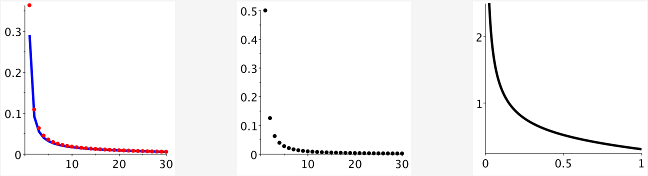
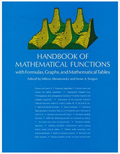
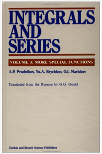
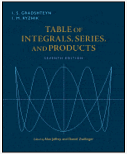


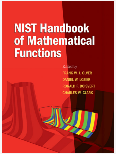
Figure: Core size in first part of pairs of supertrees: $\frac{1}{2}\,\mathcal{B}_C\left(\frac{1}{4}\right) + \frac{1}{2}\sqrt{n}\,\mathrm{ML}\left(\frac{1}{2}, -\frac{1}{4}\right)$.

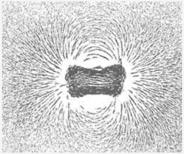
A link with electromagnetism and special functions











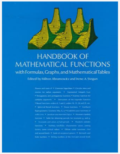
Mittag-Leffler function: 1903, extended to two parameters by Wiman in 1905, and to three parameters by Prabhakar in 1971:

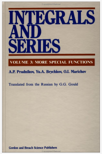
$$E_{\alpha,\beta'}^{\gamma'}(t) := \sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma')}{\Gamma(\alpha k+\beta')\Gamma(\gamma')} \frac{t^k}{k!}.$$

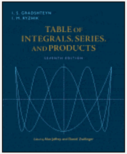
It is a special case of Fox–Wright function ($\approx$ quotients of gamma $\approx$ Wright's generalized hypergeometric $\approx$ Mellin–Barnes integral).

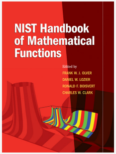
There are many articles on this function... in physics!

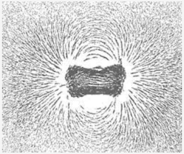
A link with electromagnetism and special functions











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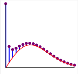
$$E^{\gamma'}_{\alpha,\beta'}(t) := \sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma')}{\Gamma(\alpha k+\beta')\Gamma(\gamma')} \frac{t^k}{k!}.$$

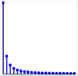
For $\beta' = \alpha\gamma'$, this "is" the inverse Laplace transform of $1/(1+s^\alpha)^{\gamma'}$, the right part of the Havriliak–Negami generalization of the Debye and Cole–Cole equations, which are classical models of dielectric relaxation in electromagnetism.
[Capelas Mainardi Vaz 2011, Garra(ppa) 2018, Górska Horzela Bratek Dattoli Penson 2018].
totally monotone: $(-1)^n \partial_t^n g(t) > 0$ for $t \in \mathbb{R}^+$. Bernstein's theorem $\Rightarrow$ density!?

The Mittag-Leffler distributions

Distribution	Moment gen. function	History of the MGF
Mittag-Leffler ML(α)	$\mathbb{E}(e^{tX}) = E_\alpha(t) = E^1_{\alpha,1}(t)$	Laplace transform of stable distributions, subordinators [Feller 1949], MGF of local time of Markov processes [Darling, Kac 1957].
two-parameter Mittag-Leffler ML(α, β)	$\mathbb{E}(e^{tX}) = \Gamma(\beta')E^{\gamma'}_{\alpha,\beta'}(t)$ $(\beta', \gamma') = (\beta, \frac{\beta}{\alpha})$	Chinese restaurant [Pitman 2002], line-breaking construction of stable trees [Goldschmidt, Haas 2015], triangular Pólya urns [Flajolet, Dumas, Puyhaubert 2006], [Janson 2006 & 2010].
three-parameter Mittag-Leffler ML(α, β, γ)	$\mathbb{E}(e^{tX}) = \Gamma(\beta')E^{\gamma'}_{\alpha,\beta'}(t)$ $(\beta', \gamma') = (\beta + \gamma, \frac{\beta}{\alpha})$	critical composition schemes [Banderier, Kuba, Wallner 2021], Pólya urns [Goldschmidt, Haas, Sénizergues 2022]

This concludes our analysis of the size of the core in $F(z, u) = G(uH(z)) \times M(z)$.





Part 2:
Compositions capturing size/distance and mixed Poisson distributions

Goal 2: Analyse $F_j(z,v) = G(H(z) - (1-v)h_j z^j)M(z)$

Profile: Number of $\mathcal{H}$-components of given size j

Let $H(z) = \sum_{n\geq 0} h_j z^n$ and define the discrete random variable $X_{n,j}$:

$$\mathbb{P}\{X_{n,j} = k\} = \frac{[z^n v^k] F_j(z,v)}{[z^n] F_j(z,1)}$$

- $X_{n,j}$ naturally refines X_n:

$$\sum_{j\in\mathbb{N}} X_{n,j} = X_n.$$

- We now show that the limit laws of $X_{n,j}$ involve *mixed Poisson distributions*.

Mixed Poisson distribution

- First introduced for actuarial math./insurance modelling [Dubourdieu 1939]
- studied by Lundberg under the name "compound Poisson processes"
- used in bacteriology [Neyman 1939]
- unimodality properties [Masse, Theodorescu 2005]
- tail asymptotics [Willmot, Lin 2001]
- combinatorics [Kuba, Panholzer 2016]

Definition

Let X be a nonneg. random variable with cumulative distribution function U. Then, Y has a **mixed Poisson distribution with mixing distribution U** and scale parameter $\xi \geq 0$, if its probability mass function is given for $\ell \geq 0$ by

$$\mathbb{P}\{Y=\ell\} = \frac{\xi^\ell}{\ell!}\int_{\mathbb{R}^+} X^\ell e^{-\xi X}\, dU = \frac{\xi^\ell}{\ell!}\mathbb{E}(X^\ell e^{-\xi X}).$$

Notation: $Y \stackrel{d}{=} \mathrm{MPo}(\xi U)$ or $Y \stackrel{d}{=} \mathrm{MPo}(\xi X)$.

Important: $\mathbb{E}(Y^{\underline{s}}) = \xi^s \mathbb{E}(X^s), \quad s \geq 1.$

Refined scheme

Theorem

Consider a size-refined pure critical composition scheme

$$F_j(z,v) = G\big(H(z) - (1-v)h_j z^j\big)M(z),$$

with $j \in \mathbb{N}$. *Let* $\xi_{n,j} = \frac{\rho_H^j}{-c_H} h_j n^{\lambda_H}$. *Then,*

1. $j \ll n^{\frac{\lambda_H}{1+\lambda_H}}$ *: we have* $\xi_{n,j} \to +\infty$ *and*

$$\frac{X_{n,j}}{\xi_{n,j}} \xrightarrow[m]{d} \mathrm{ML}(\alpha,\beta,\gamma)$$

the 3-param. Mittag-Leffler with $\alpha = \lambda_H, \;\; \beta = -\lambda_G\lambda_H, \;\; \gamma = -\min(0,\lambda_M)$.

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2. $j \sim r \cdot n^{\frac{\lambda_H}{1+\lambda_H}}$, $r \in (0,\infty)$: *we have* $\xi_{n,j} \to \xi$ *with* $\xi = r^{-\frac{\lambda_H}{1+\lambda_H}} \cdot \frac{1}{-\Gamma(-\lambda_H)}$ *and*

$$X_{n,j} \xrightarrow[m]{d} MPo(\xi X),$$

the mixed Poisson distribution with mixing distribution $X \stackrel{d}{=} \mathrm{ML}(\alpha,\beta,\gamma)$.

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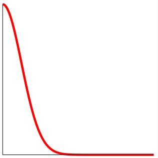
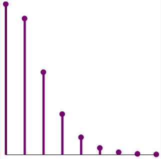
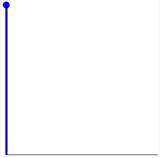
3. $j \gg n^{\frac{\lambda_H}{1+\lambda_H}}$: *we have* $\xi_{n,j} \to 0$ *and* $X_{n,j}$ *converges to a Dirac distr. at* 0.

Phase transitions for the profile $F(z,v) = G\big(H(z) - (1-v)h_j z^j\big)M(z)$

Scale	$j \ll n^{\frac{\lambda_H}{1+\lambda_H}}$	$j = \Theta\left(n^{\frac{\lambda_H}{1+\lambda_H}}\right)$	$j \gg n^{\frac{\lambda_H}{1+\lambda_H}}$
Limit law	Mittag-Leffler $\mathrm{ML}(\alpha,\beta,\gamma)$	mixed Poisson $\mathrm{MPo}(\xi\,\mathrm{ML})$	Dirac
Example			
	$X_{n,j} \sim C\,h_j\rho_H^j n^{\lambda_H}\,\mathrm{ML}$	$X_{n,j} \sim \mathrm{MPo}(\xi\,\mathrm{ML})$	$\mathbb{P}\{X_{n,j} \geq 1\} \sim 0$

1. For large n there are typically many small ($j \ll n^{\frac{\lambda_H}{1+\lambda_H}}$), some giant ($j \sim rn^{\frac{\lambda_H}{1+\lambda_H}}$), and no super-giant ($j \gg n^{\frac{\lambda_H}{1+\lambda_H}}$) $\mathcal{H}$-components of size j.
2. Conditioning on super-giant case a point process appears; see [Stufler 2022].
3. Universality of the window $\Theta(n^{1/3})$: ubiquitous square-root behaviour ($\lambda_H = \frac{1}{2}$) $\Rightarrow$ universality of the window $j = \Theta\left(n^{\frac{\lambda_H}{1+\lambda_H}}\right) = \Theta(n^{1/3})$.

Applications

In our paper:

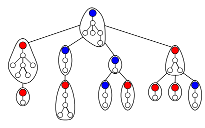
1. Core size of **supertrees**
2. Root degree and branching structure in **bilabelled increasing trees**
3. Returns to zero in **walks and bridges** with drift zero
4. Initial returns in **coloured bridges**
5. Sign changes in **Motzkin walks**
6. Table sizes in the **Chinese restaurant process**
7. Compositions in **balanced triangular urn models**

+ cycle compositions, multivariate extensions.

Example: Bicolored supertrees refined

Refined scheme: $F_j(z,v) = C\big(2zC(z) + (v-1)2c_{j-1}z^j\big)$

where $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ is the generating function of plane trees.



Theorem (Number of patatoids of size j)

The number of coloured trees of size j in supertrees of size n has factorial moments of mixed Poisson type given by

$$\mathbb{E}(X_{n,j}^{\underline{s}}) = \xi_{n,j}^s \cdot \mu_s(1+o(1)),$$

with $\xi_{n,j} = 2(\frac{1}{4})^{j-1}c_{j-1}n^{1/2}$ *and mixing distribution* $X = \mathrm{ML}\left(\frac{1}{2}, -\frac{1}{4}\right)$ *with*

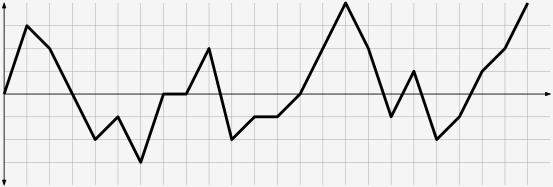
$$\mathbb{E}(X^s) = \mu_s = \frac{\Gamma(s-\frac{1}{2})\Gamma(-\frac{1}{4})}{\Gamma(-\frac{1}{2})\Gamma(\frac{s}{2}-\frac{1}{4})}.$$

Furthermore, the random variable $X_{n,j}$ possesses the three previous distinct asymptotic régimes, with a phase transition at $j = \Theta(n^{1/3})$.

Walks with zero drift: Returns to zero

Walk: Sequence of vectors $(v_1, \ldots, v_n) \in \mathcal{S}^n$

- Step set: $\mathcal{S} = \{s_1, \ldots, s_m\} \subset \mathbb{Z}$ with weights $\{p_1, \ldots, p_m\}$
- Step polynomial $P(u) = \sum_{i=1}^{m} p_i u^{s_i} \Rightarrow$ drift 0: $P'(1) = 0$



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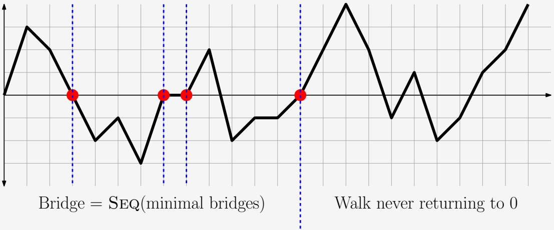


- Walk "=" initial bridge $B(z)$ + final walk $M(z) = \frac{W(z)}{B(z)}$ (not returning to 0)
- The bridge part contains all the returns to zero
- Decomposing this bridge into a sequence of "minimal bridges" $B(z) = \frac{1}{1-A(z)}$

$$\Rightarrow \qquad W(z,u) = \frac{1}{1 - uA(z)} \frac{W(z)}{B(z)}$$

Profile of returns to zero

Corollary

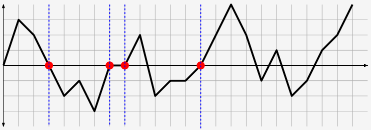
Let $X_{n,j}$ be the number of distance-j-zeroes in walks (bridges) with zero drift of length n. Then, $X_{n,j}$ has factorial moments of mixed Poisson type

$$\mathbb{E}(X_{n,j}^{\underline{s}}) = \xi_{n,j}^{s} \cdot \mathbb{E}(X^{s})\,(1 + o(1))\,,$$

with $\xi_{n,j} = \sqrt{\frac{P(1)}{2P''(1)}}\frac{h_j}{P(1)^j} \cdot n^{1/2}$, *where X is given by*

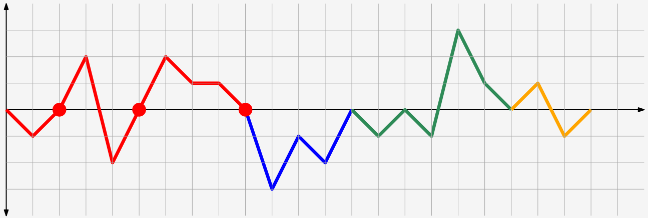
$$X = \begin{cases} HN(\sigma) & \text{for walks,} \\ Rayleigh(\sigma) & \text{for bridges,} \end{cases} \qquad \sigma = \sqrt{\frac{P(1)}{P''(1)}}.$$

Furthermore, the random variable $X_{n,j}$ possesses our three distinct asymptotic régimes, with a phase transition at $j = \Theta(n^{1/3})$.



Initial returns in coloured walks with zero drift

A 4-coloured bridge, with all its initial returns to zero marked by red dots:



Generating functions for m-colored bridges and walks:

$$B_m(z,u) = \left(\frac{1}{1-uA(z)} - 1\right)(B(z)-1)^{m-1}$$

$$W_m(z,u) = (1 + B_m(z,u))\frac{W(z)}{B(z)}$$

⇒ apply our blackbox theorems!

Corollary

The random variable X_n counting the number of initial returns in a m-coloured walk (resp. bridge) of length n satisfies

$$\mathbb{E}(X_n^{\underline{s}}) \sim n^{s/2} \left(\frac{\sigma}{\sqrt{2}}\right)^n \mu_s, \quad \sigma = \sqrt{\frac{P(1)}{P''(1)}}, \quad \mu_s = \begin{cases} \frac{\Gamma(s+1)\Gamma((m+1)/2)}{\Gamma((m+s+1)/2)}, & \textit{for walks,} \\ \frac{\Gamma(s+1)\Gamma(m/2)}{\Gamma((m+s)/2)}, & \textit{for bridges.} \end{cases}$$

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The random variable $X_n/n^{1/2}$ converges in distribution with convergence of all moments to the product of a Rayleigh and a scaled beta distribution:

$$\frac{X_n}{n^{1/2}} \xrightarrow[m]{d} X, \qquad X \stackrel{d}{=} \mathit{Rayleigh}(\sigma) \cdot B^{1/2},$$

with independent random variables

$$\mathit{Rayleigh}(\sigma) \quad \text{and} \quad B = \begin{cases} \mathrm{Beta}\left(\frac{1}{2}, \frac{m}{2}\right), & \text{for walks,} \\ \mathrm{Beta}\left(\frac{1}{2}, \frac{m-1}{2}\right), & \text{for bridges.} \end{cases}$$

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We have the local limit theorem $\mathbb{P}\{X_n = x\cdot n^{1/2}\} \sim n^{-1/2}\cdot f_X(x)$, where, for bridges

$$f_X(x) = \sqrt{\frac{2}{\pi\sigma^2}}\,\Gamma\left(\frac{m}{2}\right) e^{-\frac{x^2}{2\sigma^2}}\, U\left(\frac{m}{2}-1, \frac{1}{2}, \frac{x^2}{2\sigma^2}\right),$$

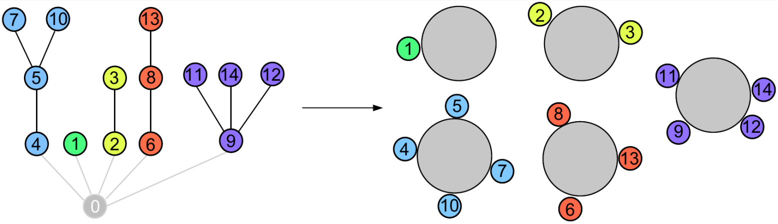
where $U(a,b,x)$ is the confluent hypergeometric function of the second kind. For walks, one replaces m by $m+1$.

Tables in the Chinese restaurant process

- Studied by Aldous, Pitman, Yor
- Links with fragmentation/stick breaking/Poisson-Dirichlet processes
- Discrete-time stochastic process: at time n a set partition of $\{1, \ldots, n\}$
  - Start at time $n = 1$ with the partition $\{\{1\}\}$
  - Given partition $T = \{t_1, \ldots, t_k\}$ of $[n]$ either add $n + 1$ to $t_i \in T$ with prob.

$$\mathbb{P}\{n + 1 \hookrightarrow t_i\} = \frac{|t_i| - \alpha}{n + \beta}, \quad 1 \leq i \leq k,$$

  - or as a new singleton block with remaining probability.



Embedding into plane-oriented recursive trees [Kuba, Panholzer 2016]

$\Rightarrow$ Number of tables with j customers $\stackrel{d}{=}$ branches of size j

Theorem

Let $a > 0$, $b > -1$. The random variable $X_{n,j}$ counting the number of tables with j customers in a Chinese restaurant process of parameter

$$\alpha = \frac{1}{1+a} \qquad \beta = \frac{b}{1+a},$$

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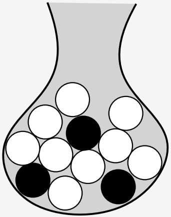
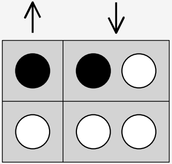
Balanced triangular Pólya urns

Replacement matrix

Let $a, b > 0$, $\sigma := a + b = d$.

$$\begin{matrix} & \circ & \bullet \\ \circ & a & b \\ \bullet & 0 & d \end{matrix} \qquad \begin{matrix} & \circ & \bullet \\ \circ & 1 & 1 \\ \bullet & 0 & 2 \end{matrix}$$





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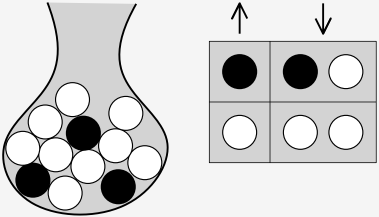

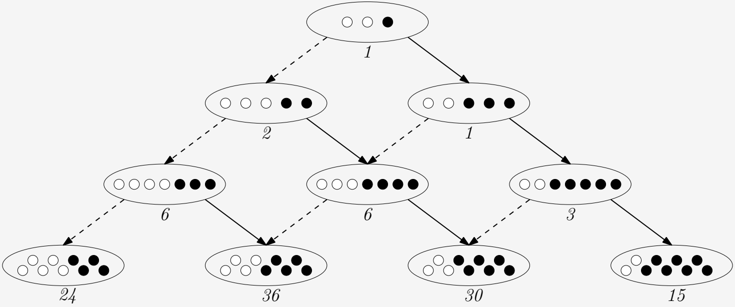

Limit law for balanced triangular Pólya urns

Problem 1.15. [Janson 2006]

Find better descriptions of the limits of triangular Pólya urns.

- Closed form of the moments known [Theorem 1.7, Janson 2006]
- For $b_0 > 0$ and $w_0 = 0$ or $w_0 = b$ Janson observed a moment-tilted stable law

History generating function [Flajolet, Dumas, Puyhaubert 2006]:

$$F(z,u) = u^{w_0}(1-\sigma z)^{-b_0/\sigma}\left(1 - u^a\left(1-(1-\sigma z)^{a/\sigma}\right)\right)^{-w_0/a}.$$

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Corollary

Let $\mathcal{W}_n$ be the rv for the number of white balls in a balanced triangular urn with initially $w_0 > 0$ white and $b_0 \geq 0$ black balls. Then, we have a convergence in distr., with convergence of all moments, to a 3-parameter Mittag-Leffler distr.

$$\frac{\mathcal{W}_n}{a n^{a/\sigma}} \xrightarrow[m]{d} \mathrm{ML}\left(\frac{a}{\sigma}, \frac{w_0}{a}, \frac{b_0}{a}\right).$$

Same limit for urns with noninteger weights [Goldschmidt, Haas, Sénizergues 2022]

Summary and extensions: a unified and generic approach!

Composition scheme	**Symbolic form**	**Limit law**
Ordinary	$G\left(uH(z)\right)$	Mittag-Leffler $\mathrm{ML}(\alpha,\gamma)$
Extended	$M(z)G\left(uH(z)\right)$	Mittag-Leffler $\mathrm{ML}(\alpha,\beta,\gamma)$ and Boltzmann distribution
Cyclic	$-\log\left(1-uH(z)\right)$	Mittag-Leffler $\mathrm{ML}(\alpha)$
Multivariate extended	$M(z)\prod_{\ell=1}^{m} G_\ell\left(u_\ell H_\ell(z)\right)$	multivariate product distribution
Refined	$M(z)G\left(H(z)-z^j h_j(1-v)\right)$	mixed Poisson type phase transition
Refined cyclic	$-\log\left(1-\left(H(z)-(1-v)h_j z^j/j!\right)\right)$	mixed Poisson type phase transition
Multivariate size-refined	$M(z)\prod_{\ell=1}^{m} G_\ell\left(H_\ell(z)-z^{j_\ell}h_{\ell,j_\ell}(1-v_\ell)\right)$	mv. mixed Poisson type phase transition

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Thanks(Thanks)! 🙂