Tree Rooted Planar Graphs

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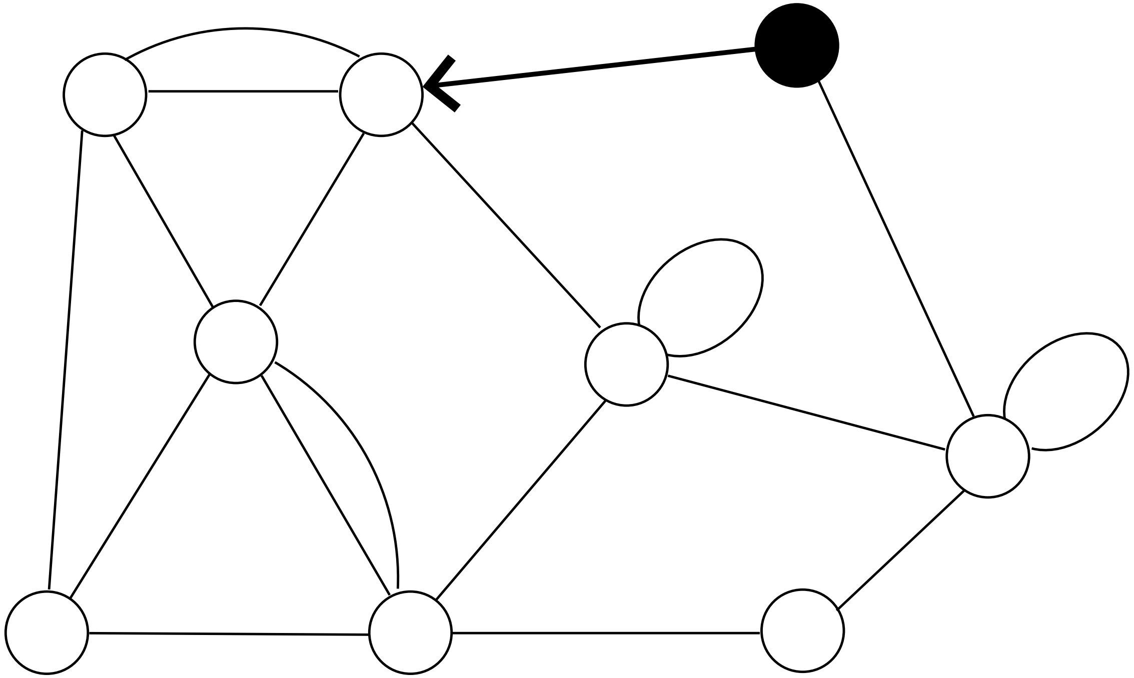
AofA 2023, Taipei, June 29, 2023.

Planar Maps



A **planar map** is a connected planar graph, possibly with loops and multiple edges, together with an embedding in the plane.
A map is **rooted** if a vertex v and an edge e incident with v are distinguished, and are called the root-vertex and root-edge, respectively. The face to the right of e is called the root-face and is usually taken as the **outer face**.

Planar Maps

m_n ... number of rooted planar maps with n edges [Tutte]

$$\boxed{m_n = \frac{2(2n)!}{(n+2)!n!}3^n}$$

The proof is given with the help of generating functions and the so-called **quadratic method**.

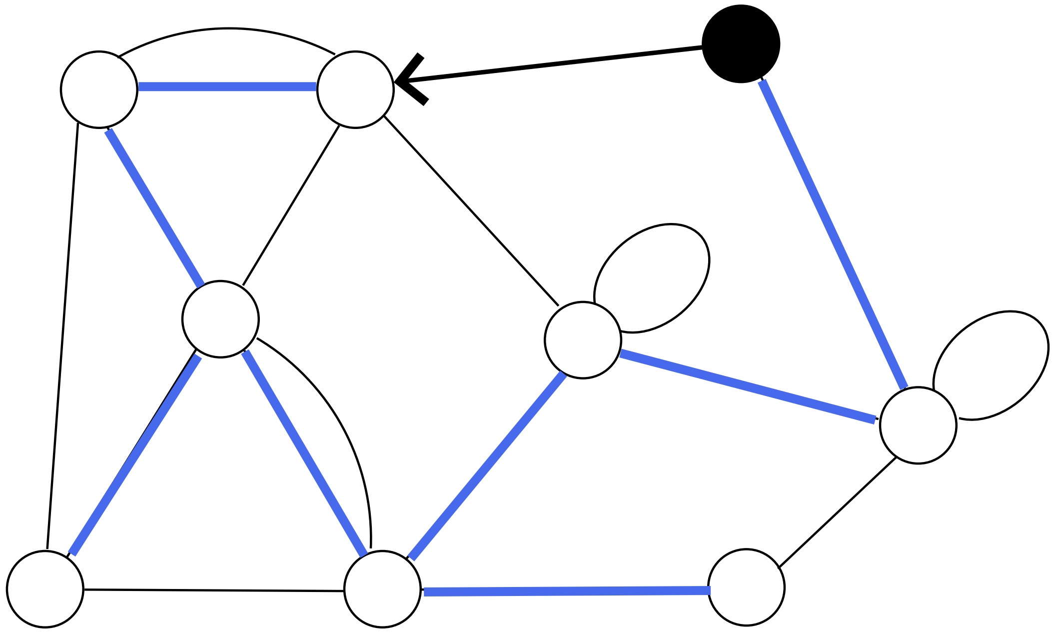
Asymptotics:

$$m_n \sim \frac{2}{\sqrt{\pi}} \cdot n^{-5/2}\, 12^n$$

Generating Function:

$$m(z) = \sum_{n\geq 0} m_n z^n = -\frac{1}{54z^2}\left(1 - 18z - \boxed{(1-12z)^{3/2}}\right)$$

Tree Rooted Planar Maps



A **tree rooted planar map** is a planar map with a **distinguished spanning tree**.

Tree Rooted Planar Maps

Counting Result by Mullin (1967)

$M_{n,k}$ ... number of tree-rooted maps with n edges and $k+1$ vertices:

$$\boxed{M_{n,k} = \frac{(2n)!}{k!(k+1)!(n-k)!(n-k+1)!}}$$

M_n ... number of tree-rooted maps with n edges:

$$\begin{aligned} M_n = \sum_{k=0}^{n} M_{n,k} &= \frac{(2n)!}{((n+1)!)^2} \sum_{k=0}^{n} \binom{n+1}{k}\binom{n+1}{n-k} \\ &= \frac{(2n)!}{((n+1)!)^2}\binom{2n+2}{n} = \boxed{C_n C_{n+1}} \sim \frac{4^{2n+1}}{\pi n^3} \end{aligned}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ denotes the n-th Catalan number
[Natural bijection to pairs of trees by Bernaradi 2007]

Tree Rooted Planar Maps

Expected number of spanning trees

The average number of spanning trees in planar maps of size n

$$= \frac{M_n}{m_n} = \frac{C_n C_{n+1}}{\frac{2\cdot 3^n}{n+2} C_n} \sim \boxed{\frac{2}{\sqrt{\pi n}} \left(\frac{4}{3}\right)^n}$$

Tree Rooted Planar Maps

Generating Functions

$$M(s,t) = \sum_{n,k} M_{n,k} s^n t^k = \sum_{n,k} \frac{(2n)!}{k!(k+1)!(n-k)!(n-k+1)!} s^n t^k$$

$$M(s,1) = \sum_{n\geq 0} C_n C_{n+1} s^n = {}_3F_2\left(\frac{1}{2}, 1, \frac{3}{2}; 2, 3; 16s\right).$$

(${}_3F_2$ ... hypergeometric series)

Tree Rooted Planar Maps

Generating Functions

$M_0(s,t)$ ... GF for tree-rooted maps, where the root edge is not contained in the spanning tree

$M_1(s,t)$ ... GF for tree-rooted maps, where the root edge is contained in the spanning tree

$$M_0(s,t) = 1 + \sum_{n\ge1}\sum_{k=0}^{n}\left(1-\frac{k}{n}\right)\frac{(2n)!}{k!(k+1)!(n-k)!(n-k+1)!}s^n t^k,$$

$$M_1(s,t) = \sum_{n\ge1}\sum_{k=0}^{n}\frac{k}{n}\frac{(2n)!}{k!(k+1)!(n-k)!(n-k+1)!}s^n t^k.$$

$$M(s,t) = M_0(s,t) + M_1(s,t)$$

$$M_0(s,1) = 1 + M_1(s,1).$$

Vertex Labelled Planar Graphs

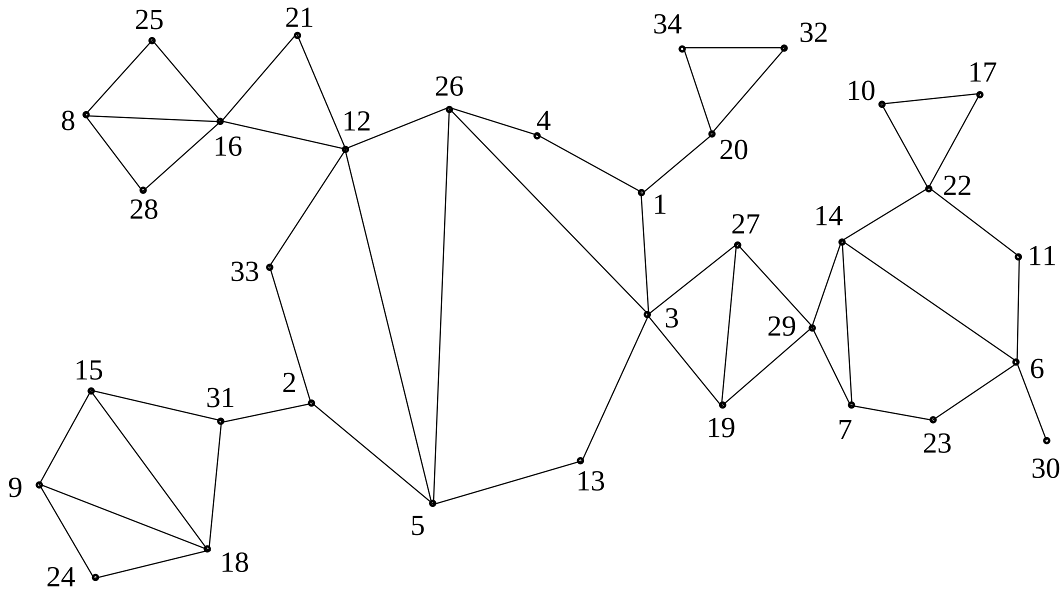
Asymptotic number of planar graphs

Theorem [Gimenez-Noy]

Let c_n denote the number of connected vertex labelled planar graphs with n vertices. Then we have

$$\boxed{c_n \sim c\, n^{-7/2}\, \gamma^n\, n!}$$

with (analytically describable) $c \approx 0.41043 \cdot 10^{-5}$ and $\gamma \approx 27.2268$.



Tree Rooted Vertex Labelled Planar Graphs

THEOREM [D+Noy+Requilé+Rué]

Let C_n denote the number of vertex-labeled connected and tree-rooted planar graphs with n vertices. Then we have

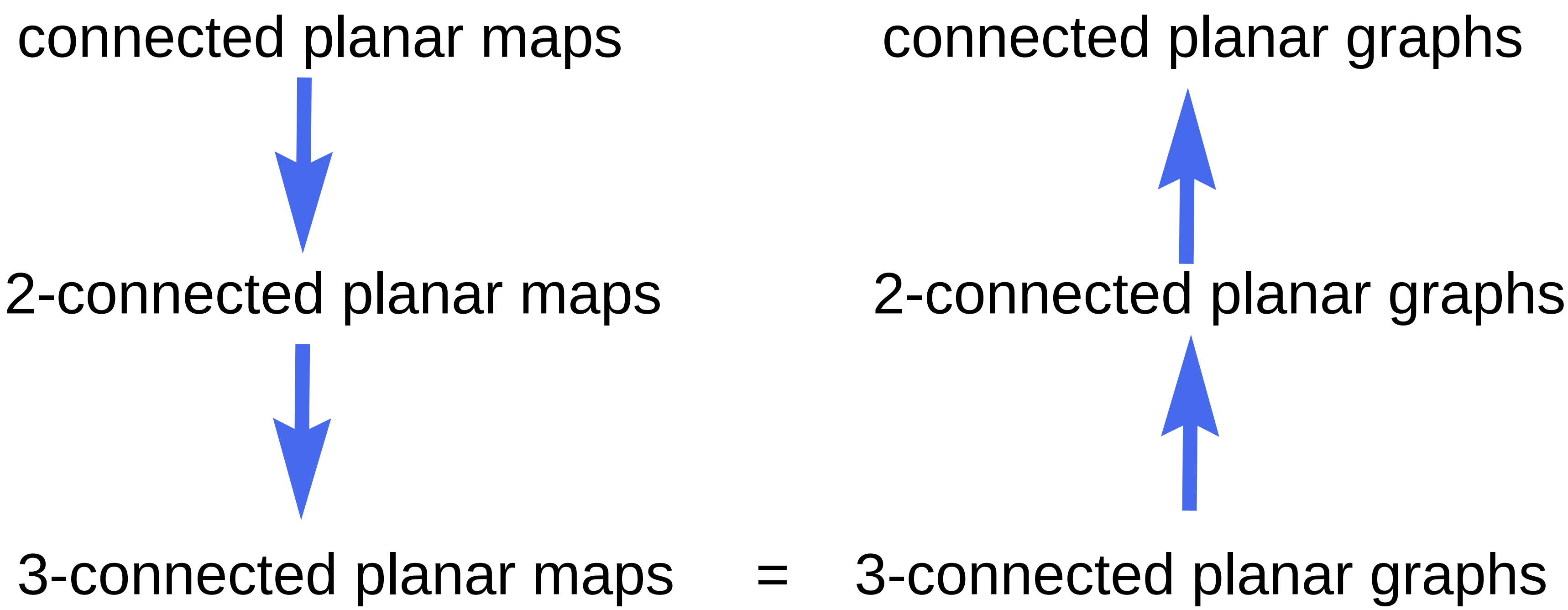
$$\boxed{C_n \sim \overline{c}\, n^{-4}\, \overline{\gamma}^n\, n!},$$

where the constants $\overline{c} > 0$ and $\overline{\gamma} > \gamma$ can be numerically calculated to any given precision.

Hence, the **average number of spanning trees in planar graphs** of size n

$$= \frac{C_n}{c_n} \sim \boxed{\frac{\overline{c}/c}{\sqrt{n}} \left(\frac{\overline{\gamma}}{\gamma}\right)^n}$$

Strategy of the Proof

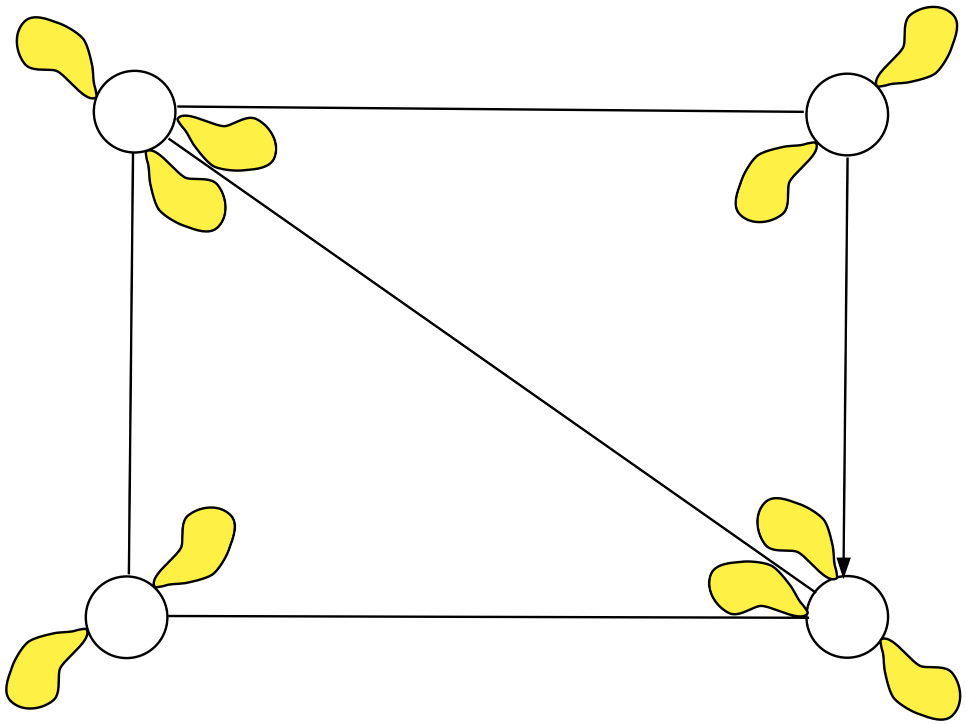


Tree Rooted 2-Connected Planar Maps

$N(z,t)$ ... GF of tree rooted 2-connected planar maps

Lemma

$$M(s,t) = 1 + s(1+t)M(s,t)^2 + \boxed{N(sM(s,t)^2, t)}$$



Tree Rooted 2-Connected Planar Maps

$N_0(z,t)$ ... GF for 2-connected tree-rooted maps, where the root edge is not contained in the spanning tree

$N_1(z,t)$ ... GF for 2-connected tree-rooted maps, where the root edge is contained in the spanning tree

Lemma

$$M_0(s,t) = 1 + sM(s,t)^2 + N_0(sM(s,t)^2, t),$$
$$M_1(s,t) = tsM(s,t)^2 + N_1(sM(s,t)^2, t).$$

Tree Rooted 3-Connected Planar Maps

$T_0(u,v)$ ... GF for 3-connected tree-rooted maps, where the root edge is not contained in the spanning tree

$T_1(u,v)$ ... GF for 3-connected tree-rooted maps, where the root edge is contained in the spanning tree

Lemma

$$D_0 = z + \frac{tD_0^2}{1+tD_0} + \frac{D_0 D_1 (D_1+2)}{(1+D_1)^2} + \frac{T_0(D_1, tD_0/D_1)}{tD_1},$$
$$D_1 = z + \frac{tD_0 D_1 (tD_0+2)}{(1+tD_0)^2} + \frac{D_1^2}{1+D_1} + \frac{T_1(D_1, tD_0/D_1)}{tD_0},$$

where $D_0(z,t) = N_0(z,t)/(tz)$ and $D_1(z,t) = N_1(z,t)/(tz)$.

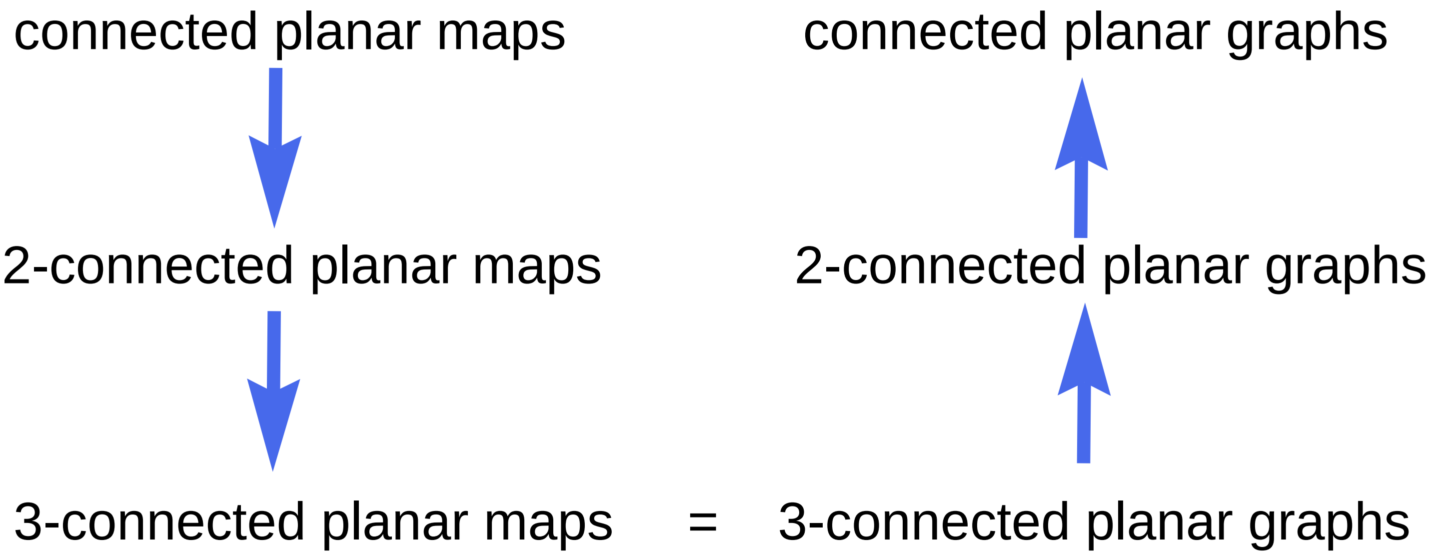
Tree Rooted 3-Connected Planar Maps

By Tutte-decomposition:

$$
\begin{aligned}
D_0 &= z + S_0 + P_0 + H_0,\\
D_1 &= z + S_1 + P_1 + H_1,\\
S_0 &= tD_0(D_0 - S_0),\\
S_1 &= tD_0(D_1 - S_1) + tD_1(D_0 - S_0),\\
P_0 &= \frac{D_0 - P_0}{(1-(D_1 - P_1))^2} - (D_0 - P_0),\\
P_1 &= \frac{(D_1 - P_1)^2}{1-(D_1 - P_1)},\\
H_0 &= \frac{T_0(D_1, tD_0/D_1)}{tD_1},\\
H_1 &= \frac{T_1(D_1, tD_0/D_1)}{tD_0}.
\end{aligned}
$$

Tree-rooted 3-connected planar graphs

By **Whitney's theorem**, tree-rooted 3-connected planar graphs have a unique embedding into the plane. Therefore they **coincide** with tree-rooted 3-connected planar maps.



Tree-rooted 2-connected planar networks

$\overline{D}_0(x,y)$ ... GF for 2-connected "tree-rooted" planar networks, where only the north pole is not contained in the spanning tree

$\overline{D}_1(x,y)$ ... GF for 2-connected tree-rooted planar networks

Lemma

$$\overline{D}_1 = (1+y)\exp\left(\frac{x\overline{D}_0\overline{D}_1(2+x\overline{D}_0)}{(1+x\overline{D}_0)^2} + \frac{T_1(\overline{D}_1, x\overline{D}_0/\overline{D}_1)}{2x\overline{D}_0}\right) - 1$$

$$\overline{D}_0 = \frac{2x^2\overline{D}_0\overline{D}_1(y+(1+y)\overline{D}_0)}{2x\overline{D}_1(1+x\overline{D}_0)(1+y)}(1+\overline{D}_1)$$
$$+ \frac{x(2y\overline{D}_1 + (1+y)(\overline{D}_0T_0(\overline{D}_1, x\overline{D}_0/\overline{D}_1) + T_1(\overline{D}_1, x\overline{D}_0/\overline{D}_1)))}{2x\overline{D}_1(1+x\overline{D}_0)(1+y)}(1+\overline{D}_1)$$

Tree-rooted 2-connected planar networks

By Tutte-decomposition:

$$\begin{aligned}
\overline{D}_0 &= y + \overline{S}_0 + \overline{P}_0 + \overline{H}_0, \\
\overline{D}_1 &= y + \overline{S}_1 + \overline{P}_1 + \overline{H}_1, \\
\overline{S}_0 &= x\overline{D}_0(\overline{D}_0 - \overline{S}_0), \\
\overline{S}_1 &= x\overline{D}_0(\overline{D}_1 - \overline{S}_1) + x\overline{D}_1(\overline{D}_0 - \overline{S}_0), \\
\overline{P}_0 &= y\left(\exp(\overline{S}_1 + \overline{H}_1) - 1\right) + (\overline{S}_0 + \overline{H}_0)\left((y+1)\exp(\overline{S}_1 + \overline{H}_1) - 1\right), \\
\overline{P}_1 &= y\left(\exp(\overline{S}_1 + \overline{H}_1) - 1\right) + \exp(\overline{S}_1 + \overline{H}_1) - 1 - (\overline{S}_1 + \overline{H}_1), \\
\overline{H}_0 &= \frac{T_0(\overline{D}_1, t\overline{D}_0/\overline{D}_1)}{2x\overline{D}_1}, \\
\overline{H}_1 &= \frac{T_1(\overline{D}_1, t\overline{D}_0/\overline{D}_1)}{2x\overline{D}_0}.
\end{aligned}$$

Tree-rooted 2-connected planar graphs

$B(x,y)$ ... GF for 2-connected tree-rooted planar graphs

Lemma

$$B_y(x,y) = \frac{x^2}{2}$$
$$+ \frac{x^2}{2(1+y)} \Big(\overline{D}_0(x,y) + \overline{D}_1(x,y) - 2y - y \exp(\overline{S}_1(x,y) + \overline{H}_1(x,y)) + y \Big)$$

where

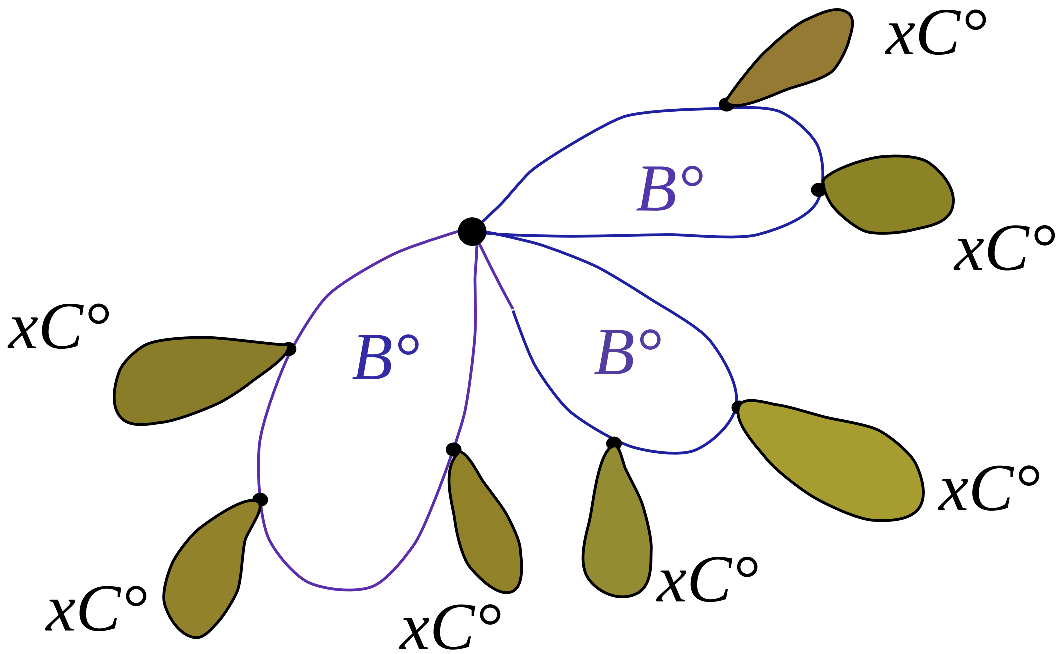
$$\overline{S}_1(x,y) = \frac{x\overline{D}_0(x,y)\overline{D}_1(x,y)(2 + x\overline{D}_0(x,y))}{(1 + x\overline{D}_0(x,y))^2},$$
$$\overline{H}_1(x,y)) = \log \frac{1 + \overline{D}_1(x,y))}{1+y} - \frac{x\overline{D}_0(x,y)\overline{D}_1(x,y)(2 + x\overline{D}_0(x,y))}{(1 + x\overline{D}_0(x,y))^2}.$$

Tree-rooted connected planar graphs

$C(x,y)$ ... GF for connected tree-rooted planar graphs

Lemma

$$xC_x(x,y) = x\,\mathsf{exp}\left(B_x(xC_x(x,y),y)\right).$$



Strategy of the Proof

connected planar maps → 2-connected planar maps → 3-connected planar maps

3-connected planar maps = 3-connected planar graphs

3-connected planar graphs → 2-connected planar graphs → connected planar graphs

Tree-rooted connected planar maps

Lemma

$$M(s,t) = M(s_0(t),t) - s_0(t)M_s(s_0(t),t)(1 - s/s_0(t))$$
$$+ \boxed{\frac{(1+\sqrt{t})^3}{4\pi t^{5/4}}(1 - s/s_0(t))^2 \log \frac{1}{1 - s/s_0(t)}} + O\left((1 - s/s_0(t))^2\right),$$

where $s_0(t) = 1/(4(1+\sqrt{t})^2)$ is the dominating singularity of the mapping $s \to M(s,t)$.

For later use we use the notation

$$\boxed{M(s,t) = M_0(t) + M_1(t)S + M_2(t)S^2 \log S + O(S^2)}$$

with $\boxed{S = 1 - s/s_0(t)}$, $M_0(t) = M(s_0(t),t)$ etc.

Remark. There are similar expansions for $M_0(s,t)$ and $M_1(s,t)$.

Tree-rooted connected planar maps

Integral representation

$$
\begin{aligned}
s\,M(s,t) &= \sum_{n,k} \frac{(2n)!}{k!(k+1)!(n-k)!(n-k+1)!} s^{n+1} t^k \\
&= \sum_{n\geq 0} \frac{C_n}{n+1} \frac{1}{2\pi i} \int_{|w|=1/\sqrt{t}} \left(\frac{(1+tw)(1+w)}{w} s \right)^{n+1} dw \\
&= \frac{1}{2\pi i} \int_{|w|=1/\sqrt{t}} \left(\log \left(1 + \sqrt{1 - 4\frac{(1+tw)(1+w)}{w} s} \right) \right. \\
&\qquad \left. - \sqrt{1 - 4\frac{(1+tw)(1+w)}{w} s} \right) dw,
\end{aligned}
$$

+ asymptotic analysis of the integral for $s \approx s_0(t) = 1/(4(1+\sqrt{t})^2)$

Tree-rooted 2-connected planar maps

Asymptotic Transfer

Lemma

Suppose that Y is a complex variable that varies in a region of the form

$$R := \{z \in \mathbb{C} : |z| < \eta, \, |\arg(z)| < \pi - \varphi\}$$

for some $\eta > 0$ and some φ with $0 < \varphi < \frac{\pi}{2}$ and that Y is related with X by

$$\boxed{X = aY + bY^2 \log Y + O(Y^2)} \qquad (Y \to 0, \, Y \in R),$$

where a and b are non-zero and $|\arg(a)| < \varphi$. Then there exist $\eta' > 0$ and φ' with $0 < \varphi < \frac{\pi}{2}$ such that the above relation can be uniquely inverted for $X \in R' = \{z \in \mathbb{C} : |z| < \eta', \, |\arg(z)| < \pi - \varphi'\}$ and we have

$$\boxed{Y = \frac{1}{a}X - \frac{b}{a^3}X^2 \log X + O(X^2)} \qquad (X \to 0, \, X \in R').$$

Proof by bootstrapping.

Tree-rooted 2-connected planar maps

Application to $\boxed{M(s,t) = 1 + s(1+t)M(s,t)^2 + N(sM(s,t)^2, t)}$

$M(s,t)$ gets **singular** for $\boxed{s = s_0(t) = 1/(4(1+\sqrt{t})^2)}$ and we have

$$M(s,t) = M_0(t) + M_1(t)S + M_2(t)S^2 \log S + O(S^2)$$

with $\boxed{S = 1 - s/s_0(t)}$, $M_0(t) = M(s_0(t), t)$ etc.

Hence, $N(z,t)$ gets **singular** for $\boxed{z = s_0(t)M(s_0(t), t)^2}$

Setting

$$z = sM(s,t)^2.$$

we obtain

$$\boxed{N(z,t) = -1 - (1+t)z + M(s,t)}.$$

Tree-rooted 2-connected planar maps

We set

$$\boxed{Z = 1 - \frac{z}{s_0(t)M(s_0(t),t)^2}}$$

which gives

$$z = s_0(t)M(s_0(t),t)^2(1-Z).$$

Hence we obtain

$$\begin{aligned} sM(s,t)^2 &= s_0(t)(1-S)\left(M_0^2 + 2M_0M_1S + 2M_0M_2S^2\log S + O(S^2)\right) \\ &= s_0(t)M_0^2 + s_0(t)M_0(2M_1 - M_0)S + 2s_0(t)M_0M_2S^2\log S + O(S^2) \end{aligned}$$

or

$$\boxed{Z = 1 - \frac{sM(s,t)^2}{s_0(t)M_0^2} = -\frac{2M_1 - M_0}{M_0}S - \frac{2M_2}{M_0}S^2\log S + O(S^2)}.$$

Tree-rooted 2-connected planar maps

Hence, a direct application of the previous Lemma gives

$$\boxed{S = -\frac{M_0}{2M_1 - M_0}Z - \frac{2M_0^2 M_2}{(2M_1 - M_0)^3}Z^2 \log Z + O(Z^2)}.$$

This also leads to the relation

$$\begin{aligned}
\boxed{M(s,t)} &= M_0 + M_1 S + M_2 S^2 \log S + O(S^2) \\
&= M_0 + M_1 \left(-\frac{M_0}{2M_1 - M_0}Z - \frac{2M_0^2 M_2}{(2M_1 - M_0)^3}Z^2 \log Z \right) \\
&+ M_2 \frac{M_0^2}{(2M_1 - M_0)^2} Z^2 \log Z + O(Z^2) \\
&= \boxed{M_0 - \frac{M_0 M_1}{2M_1 - M_0}Z - \frac{M_0^3 M_2}{(2M_1 - M_0)^3}Z^2 \log Z + O(Z^2)}.
\end{aligned}$$

Tree-rooted 2-connected planar maps

Summing up we have

$$\boxed{N(z,t)} = -1-(1+t)s_0(t)M_0^2(1-Z)+M(s,t)$$
$$= (M_0-1-(1+t)s_0(t)M_0^2) + \left((1+t)s_0(t)M_0^2 - \frac{M_0M_1}{2M_1-M_0}\right) Z$$
$$- \frac{M_0^3M_2}{(2M_1-M_0)^3} Z^2 \log Z + O(Z^2)$$
$$= \boxed{N_0 + N_1 Z + N_2 Z^2 \log Z + O(Z^2)}$$

with

$$N_0 = N_0(t) = M_0 - 1 - (1+t)s_0(t)M_0^2,$$
$$N_1 = N_1(t) = (1+t)s_0(t)M_0^2 - \frac{M_0M_1}{2M_1-M_0},$$
$$N_2 = N_2(t) = -\frac{M_0^3M_2}{(2M_1-M_0)^3}.$$

Tree-rooted 2-connected planar maps

Similarly we obtain

$$N_0(z,t) = N_{00} + N_{01}Z + N_{02}Z^2 \log Z + O(Z^2),$$

$$N_1(z,t) = N_{10} + N_{11}Z + N_{12}Z^2 \log Z + O(Z^2)$$

with $N_{ij} = N_{ij}(t)$ and

$$Z = 1 - \frac{z}{s_0(t)M(s_0(t),t)^2}$$

Tree-rooted 3-connected planar maps

With the relations (where $D_0 = N_0/(zt)$ and $D_1 = N_1/(tz)$)

$$D_0 = z + \frac{xD_0^2}{1+xD_0} + \frac{D_0D_1(D_1+2)}{(1+D_1)^2} + \frac{T_0(D_1, xD_0/D_1)}{xD_1},$$
$$D_1 = z + \frac{xD_0D_1(xD_0+2)}{(1+xD_0)^2} + \frac{D_1^2}{1+D_1} + \frac{T_1(D_1, xD_0/D_1)}{xD_0}.$$

we obtain

$$T_0(u,v) = T_{00}(v) + T_{01}(v)U + T_{02}(v)U^2 \log U + O(U^2),$$
$$T_1(u,v) = T_{10}(v) + T_{11}(v)U + T_{12}(v)U^2 \log U + O(U^2)$$

with

$$U = 1 - \frac{u}{u_0(v)}$$

and a proper function $u_0(v)$.

Tree-rooted 2-connected planar networks

With the relations

$$\overline{D}_1 = (1+y)\exp\left(\frac{x\overline{D}_0\overline{D}_1(2+x\overline{D}_0)}{(1+x\overline{D}_0)^2} + \frac{T_1(\overline{D}_1, x\overline{D}_0/\overline{D}_1)}{2x\overline{D}_0}\right) - 1$$

$$\overline{D}_0 = \frac{2x^2\overline{D}_0\overline{D}_1(y+(1+y)\overline{D}_0)}{2x\overline{D}_1(1+x\overline{D}_0)(1+y)}(1+\overline{D}_1)$$
$$+ \frac{x(2y\overline{D}_1 + (1+y)(\overline{D}_0 T_0(\overline{D}_1, x\overline{D}_0/\overline{D}_1) + T_1(\overline{D}_1, x\overline{D}_0/\overline{D}_1)))}{2x\overline{D}_1(1+x\overline{D}_0)(1+y)}(1+\overline{D}_1)$$

we obtain

$$\overline{D}_0(x,y) = \overline{D}_{00}(x) + \overline{D}_{01}(x)\overline{Y} + \overline{D}_{02}(x)\overline{Y}^2\log\overline{Y} + O(\overline{Y}^2),$$
$$\overline{D}_1(x,y) = \overline{D}_{10}(x) + \overline{D}_{11}(x)\overline{Y} + \overline{D}_{12}(x)\overline{Y}^2\log\overline{Y} + O(\overline{Y}^2),$$

with

$$\overline{Y} = 1 - \frac{y}{\overline{y}_0(x)}$$

and a proper function $\overline{y}_0(x)$.

Tree-rooted 2-connected planar graphs

With the relation

$$B_y(x,y) = \frac{x^2}{2} + \frac{x^2}{2(1+y)} \left(\overline{D}_0(x,y) + \overline{D}_1(x,y) - 2y - y\exp(\overline{S}_1(x,y) + \overline{H}_1(x,y)) + y\right)$$

we obtain

$$B_y(x,y) = \overline{B}_0(x) + \overline{B}_1(x)\overline{Y} + \overline{B}_2(x)\overline{Y}^2 \log \overline{Y} + O(\overline{Y}^2),$$

with $\overline{Y} = 1 - y/\overline{y}(x)$ and (after integrating, differentiating and switching between the expansions with respect to y and x)

$$B_x(x,y) = B_0^*(y) + B_1^*(y)\overline{X} + B_2^*(y)\overline{X}^2 \log \overline{X} + O(\overline{X}^2),$$

with

$$\overline{X} = 1 - \frac{x}{\overline{x}_0(y)}$$

Tree-rooted connected planar graphs

With the relation

$$xC_x(x,y) = x\exp\left(B_x(xC_x(x,y),y)\right)$$

we obein

$$C_x(x,y) = C_0^*(y) + C_1^*(y)\tilde{X} + C_2^*(y)\tilde{X}^2\log\tilde{X} + O(\tilde{X}^2),$$

where

$$\tilde{X} = 1 - \frac{x}{\tilde{x}_0(y)}$$

and $\tilde{x}_0(y)$ satisfies

$$\tilde{x}_0(y)C'(x_0(y),y) = \overline{x}_0(y).$$

Tree-rooted connected planar graphs

This leads finally to

$$\boxed{C(x,y) = C_0(y) + C_1(y)\tilde{X} + C_2(y)\tilde{X}^2 + C_3(y)\tilde{X}^3 \log \tilde{X} + O(\tilde{X}^3)}$$

and consequently to

$$\boxed{C_n = n![x^n]C(x,1) \sim \bar{c}\, n^{-4}\, x_0(1)^{-n}\, n!}$$

Thank You!