Permutation statistics in conjugacy classes of the symmetric group

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[1]Based on ongoing work with Jesse Campion Loth, Michael Levet, Kevin Liu, and Sheila Sundaram.

An overview of our results

statistic	$\lambda = (1^{a_1}2^{a_2}\dots) \vdash n$	$\lambda_i \geq 3\ \forall i$	$\lambda = (1^{a_1}2^{a_2})$	$\lambda = (2^{a_2})$	All of S_n
des	$\frac{n^2-n+2a_2-a_1^2+a_1}{2n}$	$\frac{n-1}{2}$	$\frac{n^2-a_1^2}{2n}$	$\frac{n}{2}$	$\frac{n-1}{2}$
maj	$\frac{n^2-n+2a_2-a_1^2+a_1}{4}$	$\frac{n(n-1)}{4}$	$\frac{n^2-a_1^2}{4}$	$\frac{n^2}{4}$	$\frac{n^2-n}{4}$
inv	$\frac{3n^2-n+2a_2-a_1^2+a_1-2na_1}{12}$	$\frac{n(3n-1)}{12}$	$\frac{(3n+a_1)(n-a_1)}{12}$	$\frac{n^2}{4}$	$\frac{n^2-n}{4}$
baj	$\frac{(n+1)(n^2-n+2a_2-a_1^2+a_1)}{12}$	$\frac{n(n^2-1)}{12}$	$\frac{(n+1)(n^2-a_1^2)}{12}$	$\frac{n^2(n+1)}{12}$	$\frac{1}{4}\binom{n+1}{3}$
baj − inv	$\frac{(n-2)(n^2-n+2a_2-a_1^2+a_1)}{12}$	$\frac{n(n-1)(n-2)}{12}$	$\frac{(n-2)(n^2-a_1^2)}{12}$	$\frac{n^2(n-2)}{12}$	$\frac{1}{4}\binom{n}{3}$
cdes	$\frac{n^2-n+2a_2-a_1^2+3a_1-2}{2(n-1)}$	$\frac{(n+1)(n-2)}{2(n-1)}$	$\frac{n^2-a_1^2+2a_1-2}{2(n-1)}$	$\frac{n^2-2}{2(n-1)}$	$\frac{n}{2}$
$\widetilde{\mathrm{exc}}$	$\frac{n+a_1}{2}$	$\frac{n}{2}$	$\frac{n+a_1}{2} = a_1 + a_2$	$\frac{n}{2} = a_2$	$\frac{n+1}{2}$
exc, aexc	$\frac{n-a_1}{2}$	$\frac{n}{2}$	$\frac{n-a_1}{2} = a_2$	$\frac{n}{2} = a_2$	$\frac{n-1}{2}$
cdasc, cddes	$\frac{n-a_1-2a_2}{6}$	$\frac{n}{6}$	0	0	$\frac{n-2}{6}$
cval, cpk	$\frac{n-a_1+a_2}{3}$	$\frac{n}{3}$	$\frac{n-a_1}{2} = a_2$	$\frac{n}{2} = a_2$	$\frac{2n-1}{6}$

Table: Mean of various statistics in the conjugacy class C_λ and in S_n.

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Major references in this area:
Enumerative Combinatorics Volumes 1 & 2 by Richard P. Stanley
Combinatorics of Permutations by Miklós Bóna

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The symmetric group S_n

We represent elements using the following notation:

two-line	one-line	cycle
$\begin{smallmatrix}123\\123\end{smallmatrix}$	123	(1)(2)(3)
$\begin{smallmatrix}123\\213\end{smallmatrix}$	213	(12)(3)
$\begin{smallmatrix}123\\132\end{smallmatrix}$	132	(1)(23)
$\begin{smallmatrix}123\\321\end{smallmatrix}$	321	(13)(2)
$\begin{smallmatrix}123\\231\end{smallmatrix}$	231	(123)
$\begin{smallmatrix}123\\312\end{smallmatrix}$	312	(132)

Sometimes we include commas in the cycle notation or remove 1-cycles.

Statistics on $\omega \in S_n$

An inversion is any pair (i, j) with $i < j$ and $\omega(i) > \omega(j)$.
A descent is any pair $(i, i+1)$ with $\omega(i) > \omega(i+1)$.
Denote the number of descents and inversions as $\text{des}(\omega)$ and $\text{inv}(\omega)$.

ω	descents	$\text{des}(\omega)$	inversions	$\text{inv}(\omega)$
123	$\emptyset$	0	$\emptyset$	0
213	$\{(1,2)\}$	1	$\{(1,2)\}$	1
132	$\{(2,3)\}$	1	$\{(2,3\}$	1
321	$\{(1,2),(2,3)\}$	2	$\{(1,2),(1,3),(2,3)\}$	3
231	$\{(2,3)\}$	1	$\{(1,3),(2,3)\}$	2
312	$\{(1,2)\}$	1	$\{(1,2),(1,3)\}$	2

These permutation statistics have been studied extensively on the entire symmetric group S_n.

The generating function, expected value, and variance of des are due to Riordan, while the generating function and expected value of inv are due to Rodrigues.

The Eulerian statistics exc and des are equidistributed over S_n, with a bijective proof via the first fundamental transformation. The Mahonian statistics maj and inv are equidistributed over S_n, with a bijective proof via the second fundamental transformation.

Mean number of descents and inversions

Question

What is the mean number of descents and inversions in S_n?

Mean number of descents and inversions

Interchanging the image of i and j maps permutations that have an inversion at (i,j) to ones that do not.

Example

For $i = 1$ and $j = 3$ in S_3,

$$123 \leftrightarrow 321 \qquad 213 \leftrightarrow 312 \qquad 132 \leftrightarrow 231.$$

Conclusion: half of the elements in S_n have the inversion (i,j).

Summing over all $(i, i+1)$ or (i,j), the mean number of descents is $\frac{1}{2}(n-1)$ and the mean number of inversions is $\frac{1}{2}\binom{n}{2}$.

Mean number of descents and inversions

Question

What is the mean number of descents and inversions in a conjugacy class of S_n?

Note that the interchanges

$$123 \leftrightarrow 321 \qquad 213 \leftrightarrow 312 \qquad 132 \leftrightarrow 231$$

do not preserve conjugacy class, e.g., consider the identity permutation.

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Conjugacy class of S_n

Fact

Two elements are in the same conjugacy class if and only if they have the same cycle type. Hence, conjugacy classes C_λ are indexed by partitions $\lambda = (1^{a_1}, 2^{a_2}, \ldots, n^{a_n})$ of n.

λ	C_λ	des	inv
(1^3)	$(1)(2)(3)$	0	0
$(1^1, 2^1)$	$(12)(3), (1)(23), (13)(2)$	$1, 1, 2$	$1, 1, 3$
(3^1)	$(123), (132)$	$1, 1$	$2, 2$

Statistics in conjugacy class C_λ

Theorem (Fulman '98)

Let C_λ be the conjugacy class of S_n with cycle type $\lambda = (1^{a_1}, 2^{a_2}, \ldots, n^{a_n})$.

1. *The mean number of descents in C_λ is*

$$\frac{n-1}{2} + \frac{a_2 - \binom{a_1}{2}}{n}.$$

2. *When $a_1 = a_2 = \cdots = a_{2\ell} = 0$, the ℓth moment is independent of λ, i.e. only a function of n.*

Theorem (Hultman '14)

Let C_λ be the conjugacy class of S_n with cycle type $\lambda = (1^{a_1}, 2^{a_2}, \ldots, n^{a_n})$. The mean number of inversions in C_λ is

$$\frac{3n^2 - n + 2a_2 - a_1^2 + a_1 - 2na_1}{12}.$$

Mean number of inversions when $\lambda = (1^0, 2^0, 3^{a_3}, \ldots, n^{a_n})$

$I_{i,j} :=$ indicator function that is 1 when (i,j) is an inversion and 0 otherwise.

Decompose

$$\mathbb{E}_\lambda[\text{inv}] = \sum_{1 \le i < j \le n} \mathbb{E}_\lambda[I_{i,j}] = \sum_{1 \le i < j \le n} \mathbb{P}_\lambda[I_{i,j}(\omega) = 1].$$

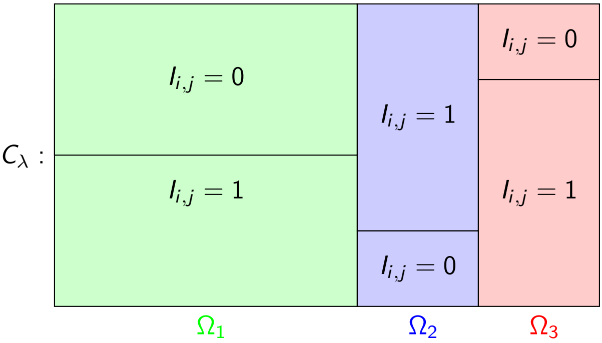
It suffices to calculate for any $i < j$, $\mathbb{P}_\lambda[I_{i,j}(\omega) = 1]$.

Mean number of inversions when $\lambda = (1^0, 2^0, 3^{a_3}, \ldots, n^{a_n})$

Partition C_λ into

$$\Omega_1 = \{\omega \in C_\lambda : \omega(i) \neq j \text{ and } \omega(j) \neq i\},$$

$$\Omega_2 = \{\omega \in C_\lambda : \omega(i) = j\}, \qquad \Omega_3 = \{\omega \in C_\lambda : \omega(j) = i\}.$$

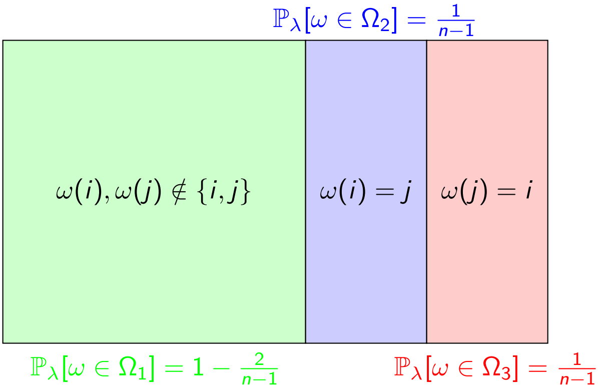


By the law of total probability,

$$\mathbb{P}_\lambda[I_{i,j}(\omega) = 1] = \sum_{k=1}^{3} \mathbb{P}_\lambda[\omega \in \Omega_k] \cdot \mathbb{P}_\lambda[I_{i,j}(\omega) = 1 | \omega \in \Omega_k].$$

Mean number of inversions when $\lambda = (1^0, 2^0, 3^{a_3}, \ldots, n^{a_n})$

Claim: C_λ :



Main idea for Ω_2 and Ω_3: group up elements in C_λ by the image of i or j.

Mean number of inversions when $\lambda = (1^0, 2^0, 3^{a_3}, \ldots, n^{a_n})$

Fact

Let σ, τ be elements in S_n and suppose σ has cycle decomposition

$$(b_1, b_2, \ldots, b_{k_1}), (c_1, c_2, \ldots, c_{k_2}), \ldots .$$

Then $\tau\sigma\tau^{-1}$ has cycle decomposition

$$(\tau(b_1), \tau(b_2), \ldots, \tau(b_{k_1})), (\tau(c_1), \tau(c_2), \ldots, \tau(c_{k_2})), \ldots .$$

Claim: $\mathbb{P}_\lambda[\omega \in \Omega_2] = \mathbb{P}_\lambda[\omega(i) = j] = \frac{1}{n-1}$.

Conjugating by $\tau = (i)(1, 2, \ldots, i-1, i+1, \ldots, n)$ induces bijections between

$$\underbrace{\{\omega \in C_\lambda : \omega(i) = 1\}}_{\text{contains } (i,1,\ldots)} \leftrightarrow \underbrace{\{\omega \in C_\lambda : \omega(i) = 2\}}_{\text{contains } (i,2,\ldots)} \leftrightarrow \ldots \leftrightarrow \underbrace{\{\omega \in C_\lambda : \omega(i) = n\}}_{\text{contains } (i,n,\ldots)}.$$

Mean number of inversions when $\lambda = (1^0, 2^0, 3^{a_3}, \ldots, n^{a_n})$

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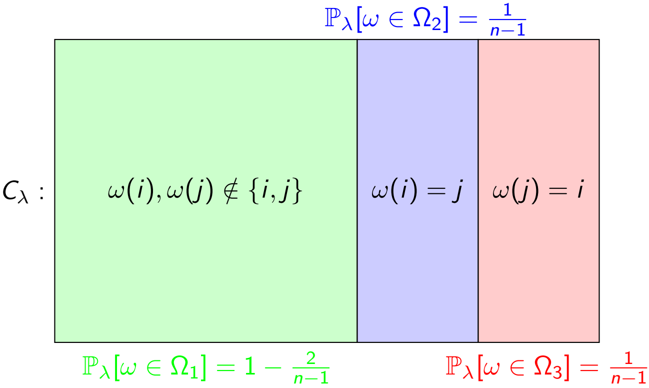
Mean number of inversions when $\lambda = (1^0, 2^0, 3^{a_3}, \ldots, n^{a_n})$



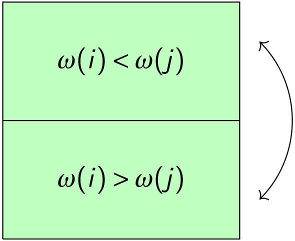
$$\mathbb{P}_\lambda[I_{i,j}(\omega) = 1] = \sum_{k=1}^{3} \underbrace{\mathbb{P}_\lambda[\omega \in \Omega_k]}_{\text{know}} \cdot \underbrace{\mathbb{P}_\lambda[I_{i,j}(\omega) = 1 | \omega \in \Omega_k]}_{?????}.$$

Mean number of inversions when $\lambda = (1^0, 2^0, 3^{a_3}, \ldots, n^{a_n})$

When $\omega(i), \omega(j) \notin \{i, j\}$, conjugation by (ij) interchanges the images of i and j:

$$(i, b_2, \ldots, j, \ldots, b_{k_1}) \ldots \leftrightarrow (j, b_2, \ldots, i, \ldots, b_{k_1}) \ldots .$$

$$(i, b_2, \ldots, b_{k_1})(j, c_2, \ldots, c_{k_2}) \ldots \leftrightarrow (j, b_2, \ldots, b_{k_1})(i, c_2, \ldots, c_{k_2}) \ldots .$$



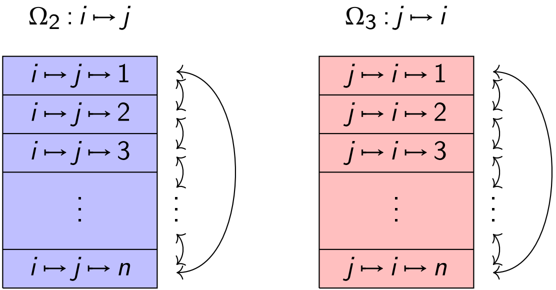
$\Omega_1 : \omega(i), \omega(j) \notin \{i, j\}$

We conclude that $\mathbb{P}_\lambda[I_{i,j}(\omega) = 1 | \omega \in \Omega_1] = \frac{1}{2}$.

Mean number of inversions when $\lambda = (1^0, 2^0, 3^{a_3}, \ldots, n^{a_n})$

For Ω_2 and Ω_3, use conjugation by

$$(i)(j)(1, 2, \ldots, i-1, i+1, \ldots, j-1, j+1, \ldots, n).$$



Consequently,

$$\mathbb{P}_\lambda[I_{i,j}(\omega) = 1 | \omega \in \Omega_2] = \frac{j-2}{n-2}$$

and

$$\mathbb{P}_\lambda[I_{i,j}(\omega) = 1 | \omega \in \Omega_3] = \frac{n-i-1}{n-2}.$$

Mean number of inversions when $\lambda = (1^0, 2^0, 3^{a_3}, \ldots, n^{a_n})$

Combined,

$$\mathbb{P}_\lambda[I_{i,j}(\omega) = 1] = \frac{1}{2} + \frac{j - i - 1}{(n-1)(n-2)},$$

and summing over all $1 \leq i < j \leq n$,

$$\mathbb{E}_\lambda[\text{inv}] = \frac{3n^2 - n}{12}.$$

Mean number of inversions for any C_λ

Observation

A generalization of this technique allows us to calculate $\mathbb{P}_\lambda[I_{i,j}(\omega) = 1]$ *for any* λ.

Theorem (Campion Loth, Levet, Liu, Sundaram, Y. '23)

The mean of $\sum_{1\le i<j\le n} c_{i,j} I_{i,j}$ *on* C_λ *is given by:*

$$\left(\frac{n^2 - n + 2a_2 - a_1^2 + a_1}{2n(n-1)}\right) \sum_{1\le i<j\le n} c_{i,j} + \left(\frac{n - (n+1)a_1 + a_1^2 - 2a_2}{n(n-1)(n-2)}\right) \sum_{1\le i<j\le n} (j-i-1)c_{i,j}.$$

The signed symmetric group B_n

The signed symmetric group (or hyperoctahedral group) B_n is the set of permutations of $\{\pm 1, \pm 2, \ldots, \pm n\}$ satisfying $\omega(-i) = -\omega(i)$.

Example

Consider the following element of B_8, written in two-line, one-line and cycle notation.

$$\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ +2 & +7 & -1 & -5 & +8 & +3 & +6 & -4 \end{pmatrix}$$
$$= [2, 7, -1, -5, 8, 3, 6, -4] = (-5, 8, -4)(2, 7, 6, 3, -1).$$

Its cycles are then $(-5, 8, -4)$ and $(2, 7, 6, 3, -1)$, which are respectively an even cycle of length 3 and an odd cycle of length 5. Thus, its cycle type is $(\lambda, \mu) = ((3), (5))$. We have $m_3(\lambda) = m_5(\mu) = 1$, and all other $m_i(\lambda)$ and $m_i(\mu)$ are 0.

Descents over B_n

The descent statistic (see Björner and Brenti) may be calculated by

$$\mathrm{des}_B(\omega) = |\{i \in \{0\} \cup [n-1] \mid \omega(i) > \omega(i+1)\}|,$$

with the convention that $\omega(0) = 0$.

Theorem (Chow and Mansour '12)

Let X_n be des_B defined on B_n. Then X_n has mean $n/2$ and variance $(n+1)/12$, and as $n \to \infty$, the standardized random variable $(X_n - n/2)/\sqrt{(n+1)/12}$ converges to a standard normal distribution.

Descents in conjugacy class $C_{\lambda,\mu}$

The following results are from Campion Loth, Levet, Liu, Sundaram, Y. '23.

Theorem

Let $\Delta^2(\lambda,\mu) = m_1(\lambda)^2 - m_1(\mu)^2$. *Let* $B_n(t) = \sum_{\omega \in B_n} t^{\mathrm{des}_B(\omega)+1}$ *and* $B_{\lambda,\mu}(t) = \sum_{\omega \in C_{\lambda,\mu}} t^{\mathrm{des}_B(\omega)+1}$. *Then*

$$\frac{B_{\lambda,\mu}(t)}{|C_{\lambda,\mu}|} = \frac{B_n(t)}{2^n n!} + \frac{1-t}{2n} \frac{B_{n-1}(t)}{2^{n-1}(n-1)!} \Delta^2(\lambda,\mu) + (1-t)^2 g(t),$$

where $g(t)$ *is some polynomial in* t.

Corollary

Differentiating with respect to t *and setting* $t=1$, *we obtain the first moment of descents in the conjugacy class* $C_{\lambda,\mu}$:

$$\mathbb{E}_{\lambda,\mu}[\mathrm{des}_B] = \frac{n}{2} - \frac{\Delta^2(\lambda,\mu)}{2n}.$$

Theorem

Fix $\ell \geq 0$. *Suppose that all cycles in* $C_{\lambda,\mu}$ *have length greater than* 2ℓ. *Then*

$$\frac{B_{\lambda,\mu}(t)}{|C_{\lambda,\mu}|} = \frac{B_n(t)}{2^n n!} + (1-t)^{\ell+1} g(t),$$

where $g(t)$ *is some polynomial in* t.

Theorem (Central Limit Theorem)

For every $n \geq 1$, *pick a conjugacy class* $C_{(\lambda_n,\mu_n)}$ *in* B_n *indexed by the bi-partition* (λ_n, μ_n) *of* n. *Let* X_n *be* des_B *defined on* $C_{(\lambda_n,\mu_n)}$. *Suppose that for all* i, $m_i(\lambda_n) \to 0$ *and* $m_i(\mu_n) \to 0$ *as* $n \to \infty$. *Then for sufficiently large* n, X_n *has mean* $n/2$ *and variance* $(n+1)/12$. *Furthermore, as* $n \to \infty$, *the standardized random variable* $(X_n - n/2)/\sqrt{(n+1)/12}$ *converges to a standard normal distribution.*

Outline of proof

1. Recall a generating function from Fulman, Kim, Lee, Petersen '12:

$$\sum_{n\geq 0} \frac{u^n}{(1-t)^{n+1}} \left(\sum_{\omega\in B_n} t^{\mathrm{des}_B(\omega)+1} \prod_i x_i^{m_i(\lambda(\omega))} y_i^{m_i(\mu(\omega))} \right)$$

$$= 1 + \sum_{k\geq 1} t^k \frac{1}{1-x_1 u} \prod_{m\geq 1} \left(\frac{1+y_m u^m}{1-x_m u^m} \right)^{N(2k-1,2m)} .$$

Here

$$N(2k-1,2m) = \frac{1}{2m} \sum_{\substack{d|m \\ d \text{ odd}}} \mu(d) \left((2k-1)^{m/d} - 1 \right),$$

and $\mu(d)$ is the number-theoretic Möbius function, with $\mu(1)=1$, $\mu(n)=(-1)^k$ if n is the product of k different primes; otherwise $\mu(n)=0$.

2. We set $u = 1$ in the generating function and extract the coefficients of x_i and y_i for all i. This gives

$$B_{\lambda,\mu}(t) = \sum_{\omega \in C_{\lambda,\mu}} t^{\mathrm{des}_B(\omega)+1}$$
$$= \sum_{k \geq 1} t^k \frac{m_1(\lambda) + k - 1}{k - 1} \prod_{i \geq 1} \binom{m_i(\lambda) + N(2k-1, 2i) - 1}{m_i(\lambda)} \binom{N(2k-1, 2i)}{m_i(\mu)}.$$

3. Recall an expansion result from Steingrímsson '94:

$$B_n(t) = (1-t)^{n+1} \sum_{k \geq 1} (2k-1)^n t^k.$$

Performing asymptotic expansion on $B_{\lambda,\mu}(t)$ thus involves checking different powers of $(2k - 1)$ in the expression.

4. The argument is technical and requires use of Stirling numbers of the first kind s_n^i, whose absolute value is the number of permutations of n elements with i disjoint cycles. Showing one step here:

$$\binom{N(2k-1,2i)+m_i(\lambda)-1}{m_i(\lambda)}\binom{N(2k-1,2i)}{m_i(\mu)}$$
$$= \frac{\sum_{a=1}^{m_i(\lambda)} |s^a_{m_i(\lambda)}|(N(2k-1,2i))^a}{m_i(\lambda)!}\frac{\sum_{b=1}^{m_i(\mu)} s^b_{m_i(\mu)}(N(2k-1,2i))^b}{m_i(\mu)!}.$$

5. When all cycles have length greater than 2ℓ, as a polynomial in $(2k-1)$, the leading term remains to be $(2k-1)^n$, but the second highest-degree term is at most $(2k-1)^{n-\ell-1}$. This is because a lower order term must have either some $a \neq m_i(\lambda)$ or some $b \neq m_i(\mu)$ or some $d \neq 1$ in the expansion above. This implies that the ℓth moment of des_B in $C_{\lambda,\mu}$ is equal to the ℓth moment of des_B in B_n when all cycles in $C_{\lambda,\mu}$ have length greater than 2ℓ. Asymptotic normality readily follows from the method of moments.

Thank You! Questions?