# Bijections between lattice paths and integer compositions

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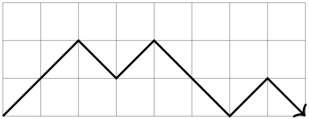
# Introduction: Dyck Paths

We consider lattice paths

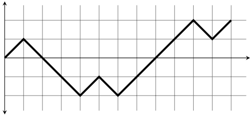
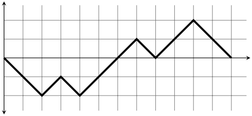
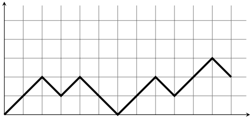
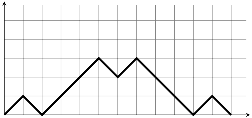
## Dyck paths

A *Dyck path* is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ with up steps $u=(1,1)$ and down steps $d=(1,-1)$, never going below the $x$-axis.



- Enumerated by the *Catalan numbers* $C_n = \frac{1}{n+1}\binom{2n}{n}$
- A *Dyck meander* allows the path to terminate at any nonnegative altitude, while never going below the $x$-axis.
- A *Dyck bridge* allows the path to traverse even below the $x$-axis but it must end on the $x$-axis.
- A *Dyck walk* relaxes the condition of a bridge terminating on the $x$-axis.

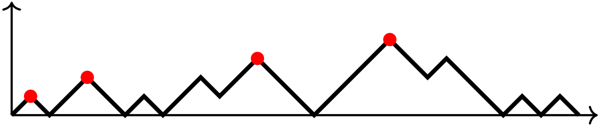
# Classification of paths

| | ending anywhere | ending at 0 |
|---|---|---|
| unconstrained (on $\mathbb{Z}$) |  Dyck walk |  Dyck bridge |
| constrained (on $\mathbb{Z}_+$) |  Dyck meander |  Dyck path |

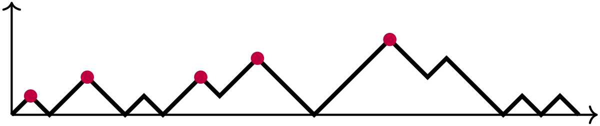
# Left-to-right Maxima

**Dyck Paths: Terminology**

- A *strict left-to-right maximum* is a peak that is greater in height than all peaks to its left.



- A *weak left-to-right maximum* is a peak that is greater than or equal to in height all peaks to its left.



# Integer compositions

## Compositions

A *composition* of $n$ is an ordered sum $n_1 + n_2 + \cdots + n_k = n$ of positive integers $n_i$, i.e., an ordered partition of $n$. The elements $n_i$ are called *parts*.

For example, there are eight compositions of 4:

| | |
|---|---|
| $4$ | $2+1+1$ |
| $3+1$ | $1+2+1$ |
| $1+3$ | $1+1+2$ |
| $2+2$ | $1+1+1+1$ |

- P. A. MacMahon noted that there are of course $2^{n-1}$ compositions of $n$.
- Since then studied extensively by various mathematicians.

# 3-compositions

## $k$-compositions

George Andrews defined *$k$-compositions* of an integer $n$ as compositions of $n$

- into parts of $k$ distinct colors,
- such that the last part has color 1.

In this talk we only consider $k = 3$, called 3-*compositions*. For example, there are 16 different 3-compositions of 3:

$$\begin{array}{llll} 3_1 & 1_1 + 2_1 & 1_1 + 1_2 + 1_1 & 1_2 + 1_3 + 1_1 \\ 2_1 + 1_1 & 1_2 + 2_1 & 1_1 + 1_3 + 1_1 & 1_3 + 1_1 + 1_1 \\ 2_2 + 1_1 & 1_3 + 2_1 & 1_2 + 1_1 + 1_1 & 1_3 + 1_2 + 1_1 \\ 2_3 + 1_1 & 1_1 + 1_1 + 1_1 & 1_2 + 1_2 + 1_1 & 1_3 + 1_3 + 1_1 \end{array}$$

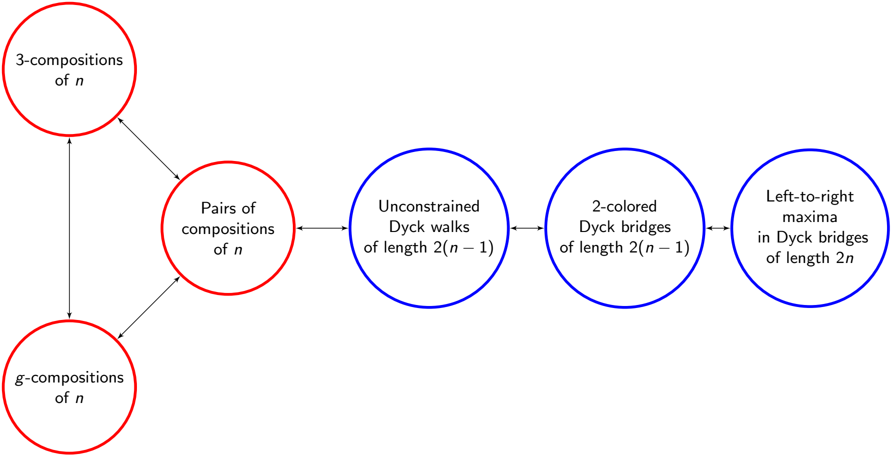
**Enumeration:**

Choose $n_1$ parts of color 1, $n_2$ of color 2, and $n_3$ of color 3.

$\Rightarrow$ Number of such choices: $\binom{n-1}{n_1, n_2, n_3, n-1-n_1-n_2-n_3}$

$\Rightarrow$ Total number of 3-compositions: $\sum_{n_1,n_2,n_3 \geq 0} \binom{n-1}{n_1, n_2, n_3, n-1-n_1-n_2-n_3} = 4^{n-1}$.

# Bijections between Lattice Paths and Integer Compositions



# Known results on pairs of compositions and lattice paths

There are instances in literature where connections between walks and compositions (including pairs of compositions) have been studied in detail. For example:

- [Bender, Lawler, Pemantle, Wilf 03] studied the probability that pairs of irreducible compositions.
- [Bóna, Knopfmacher 10] studied the probability that pairs of compositions have equal number of parts (for parts being restricted to 2 elements, and unrestricted) and also established bijections to weighted lattice paths.
- [Banderier, Hitczenko 12] generalized the previous result to any restricted subset for the parts.
- [Dunkl 07] has given relations between pairs of compositions and hook-lengths in the modified Ferrers diagram which could also be connected to walks via relations to adjusted Young tableaux.

# 3-compositions to Pairs of Compositions

**Theorem**

There exists a natural bijection between 3-compositions of $n$ and pairs of compositions of $n$.

**Sketch of Proof:**
Remove the labels of color 1, use label $L$ for color 2, and label $R$ for color 3.
Now we describe a map from 3-compositions of $n$ to pairs of compositions of $n$. First, we create two identical copies. In the first copy, we remove the labels $R$ and add the parts labeled by $L$ to the next part. If the next part has also a label $L$, then the addition continues to the next part, etc.
This gives a composition $A$ without any labels. Similarly, in the second copy, we remove the labels $L$ and add the parts labeled by $R$ to the next part.
and so on...
It can be shown that in fact is the bijective map we are looking for.

$$\begin{array}{c}
\boxed{6_1 + 1_2 + 4_3 + 2_1} \\
\downarrow \\
\boxed{(6 + 1_L + 4_R + 2), (6 + 1_L + 4_R + 2)} \\
\downarrow \\
\boxed{(6 + 1_L + 4 + 2), (6 + 1 + 4_R + 2)} \\
\downarrow \\
\boxed{(6, 1_L + 4, 2), (6, 1, 4_R + 2)} \\
\downarrow \\
\boxed{(6, 5, 2), (6, 1, 6)}
\end{array}$$

# $k$- and $g$-compositions

## $g$-compositions [Ouvry, Polychronakos 19]

A $g$-composition of $n$ is a composition of $n$ which allow both positive integers and zeros, such that at most $g-2$ consecutive zeros may appear between positive parts.

## Theorem [Hopkins, Ouvry 21]

There is a natural bijection between $k$-compositions and $g$-compositions when $g = k + 1$.

Sketch of Proof:

We look at the compositions from right to left. Take the g-composition of $n$.
If $\lambda_i$ is a part of the composition then we look at the number of 0's that have occurred after it (say m) and map that part to a $k$-composition with the label $m+1$. We transport each part of the composition in this fashion to obtain the required $k$-composition. Of course, we also note at this juncture that since we don't allow any zeros before the first part, therefore we end always with the label 1 which is required by the definition of $k$-compositions.

$$
\begin{aligned}
3 &\rightarrow 3_1\,,\\
2+0+1 &\rightarrow 1_2+2_1\,,\\
2+0+0+1 &\rightarrow 1_3+2_1\,,\\
2+1 &\rightarrow 1_1+2_1\,,\\
1+0+0+2 &\rightarrow 2_3+1_1\,,\\
1+0+0+1+0+1 &\rightarrow 1_2+1_3+1_1\,,\\
1+0+1+0+1 &\rightarrow 1_2+1_2+1_1\,,\\
1+0+2 &\rightarrow 2_2+1_1\,,\\
1+0+1+0+0+1 &\rightarrow 1_3+1_2+1_1\,,\\
1+1+0+1 &\rightarrow 1_2+1_1+1_1\,,\\
1+1+0+0+1 &\rightarrow 1_3+1_1+1_1\,,\\
1+2 &\rightarrow 2_1+1_1\,,\\
1+1+1 &\rightarrow 1_1+1_1+1_1\,,\\
1+0+0+1+0+1 &\rightarrow 1_2+1_3+1_1\,,\\
1+0+1+0+0+1 &\rightarrow 1_3+1_2+1_1\,,\\
1+0+0+1+0+0+1 &\rightarrow 1_3+1_3+1_1\,.
\end{aligned}
$$

# Pairs of Compositions to Dyck Walks

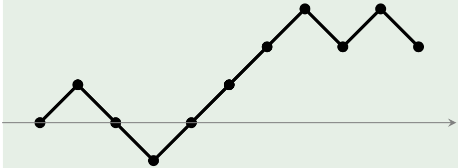
## Theorem

*Pairs of compositions of $n$ and Dyck walks of length $2n-2$ are in bijection.*

**Sketch of Proof:**

1. Convert each composition into a binary sequence: each element $k$ appends $k-1$ zeros followed by a one.
2. Delete the 1 at the end and concatenate the two sequences.
3. Replace each 0 by $\mathbf{u}=(1,1)$ and each 1 by $\mathbf{d}=(1,-1)$.

## Example: $((2,1,3),(3,2,1))$



**Composition** $(2,1,3)$
**Binary sequence:** 011001

**Composition** $(3,2,1)$
**Binary sequence:** 001011

**Comb. sequence erasing 1s at the end:**
0110000101

# Bijections between paths

**Theorem**

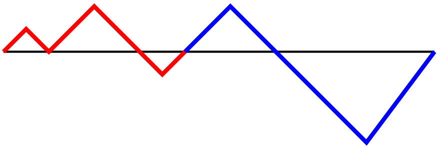
*There is an explicit bijection between 2-colored Dyck bridges and unconstrained Dyck walks of the same length.*

2-coloured Dyck bridges
have been previously used in literature (for example in Phase transitions of composition schemes: Mittag-Leffler and mixed Poisson distributions the authors have looked at the returns of coloured bridges and the interesting distributions and properties they follow.)

**Sketch of proof:** Because there is an explicit bijection between Dyck bridges and meanders of the same length, we will be transforming all bridges into meanders. So we reduce the problem into 4 cases.

# Case I: Both Bridges are Non-empty

- The change in color signifies the last crossing of the x-axis.
- Transform the second bridge into a meander or negative meander and attach it to the first bridge, ensuring the attached meander continues on the other side of the x-axis.
- Reverse this process by cutting at the last crossing of the x-axis.
- Note: All other cases will have no crossings of the x-axis.



# Case II, III and IV

- First Bridge is Non-empty, Second is Empty – Transform the first bridge into a meander.
- First Bridge is Empty, Second is Non-empty – Transform the second bridge into a negative meander.
- Both Bridges are Empty – Map them to the empty walk.

# Final Bijection for Dyck Bridges

## Proposition

There is an explicit bijection between Dyck bridges with marked left-to-right maximum of length $2n$ and 2-colored Dyck bridges of length $2(n-1)$.

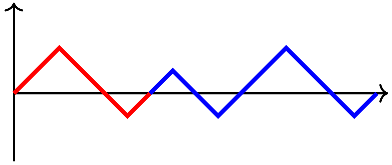


Figure: 2-Colored Dyck Bridge

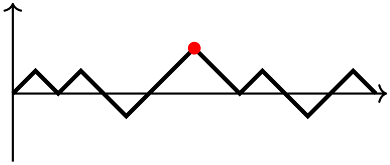


Figure: Dyck Bridge with Marked Peak

# Arithmetic Property

**Theorem**

*Let $D_r(n)$ be the number of Dyck paths with semi-length $n$ and with exactly $r$ peaks for every reached height. Then $D_r(n) \equiv 0 \mod (r+1)$ for $n > r$.*

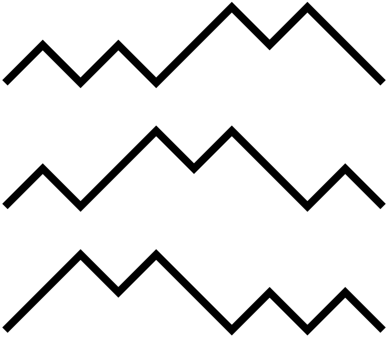


Figure: All 3 Dyck paths of semi length 5 with exactly 2 peaks for every given level

# Arithmetic Property

**Theorem**

*Let $D_r(n)$ be the number of Dyck paths with semi-length $n$ and with exactly $r$ peaks for every reached height. Then $D_r(n) \equiv 0 \mod (r+1)$ for $n > r$.*

Sketch of Proof:

- We will construct the paths recursively **height-by-height**, by lifting all possible paths of a given height by one.
- Let such a Dyck path $\mathcal{D}$ with $k$ returns to zero. Thus,

$$\mathcal{D} = \mathcal{A}_1 \mathcal{A}_2 \dots \mathcal{A}_k.$$

- Let $\mathcal{S}$ be the set of $k-1$ returns inside the path. Now we will cut at a subset $\mathcal{S}$ and lift the full paths by one:
  - Attach a step **u** at the beginning and **d** at the end.
  - Insert at each chosen contact a valley **du**.
  - Height of this new path $\widetilde{\mathcal{D}}$ increases by one and its length by $i+1$.
  
  $\Rightarrow$ New decomposition: $\widetilde{\mathcal{D}} = \widetilde{\mathcal{A}}_1 \widetilde{\mathcal{A}}_2 \dots \widetilde{\mathcal{A}}_{i+1}$.

# Sketch of the Proof (continued)

- Note that this new path $\widetilde{\mathcal{D}} = \widetilde{\mathcal{A}}_1 \widetilde{\mathcal{A}}_2 \ldots \widetilde{\mathcal{A}}_{i+1}$ still satisfies the constraint of having exactly $r$ peaks for each reached height **greater than one**.
- It remains to insert $r$ hills of the form **ud** at ground level.
  - Possible positions: at the beginning, at the end, or between $\widetilde{\mathcal{A}}_j$.
  - Choose a multiset of $r$ of these $i+2$ positions: This gives $\binom{r+i+1}{r}$ possibilities.
- Let us now define the generating function $F(z,u) = \sum_{n,k} f_{n,k} z^n u^k$, where $f_{n,k}$ gives the number of Dyck paths with exactly $r$ peaks for each reached height of semi-length $n$ with $k$ returns to zero.
- Then, we define a linear operator $L_r$ that implements the construction above:

$$L_r(u^k) = \sum_{i=0}^{k-1} \binom{r+i+1}{r} \binom{k-1}{i} (zu)^{r+i+1}$$

# ctd

- This gives the functional equation

$$F(z,u) = (zu)^r + L_r(F(z,u)).$$

The term $(zu)^r$ corresponds to the unique path of height 1 that consists of $r$ peaks:

$$(\mathbf{ud})^r.$$

- Note that this functional equation has a unique formal power series solution, as the length in $z$ increases by at least one in each iteration.
- Finishing proof of divisibility of $D_r(n)$ by $r+1$ for $n > r$:
  - We will show that all terms of height 2 fulfil this property
  - Therefore, by **linearity**, all terms in the sequence.
- Remains to lift paths of height 1 to height 2:

$$L_r((zu)^r) = z^r \sum_{i=1}^{r} \binom{r+i}{i} \binom{r-1}{i-1} (zu)^{r+i}$$

- **Key observation:** $\frac{1}{r+1}\binom{r+i}{i}\binom{r-1}{i-1}$ enumerates little Schröder paths ending at $(0,2r)$ with $i$ up steps $\mathbf{u}$. □

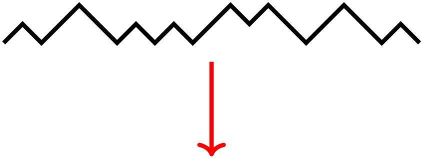
# Back to compositions and Illustration

**Theorem**

Dyck paths of height 2 with exactly $r$ peaks at each height are in bijection with $(r+1)$-tuples of integer compositions with grand total sum $r$.

**Sketch of Proof:**

- First, we decompose the path into an $r+1$ tuple by removing the $r$ hills **ud**.
- The first part is the subpath before the first hill, the second part is the subpath between the first and second hill, and so on. These parts are either empty or contain only peaks at height 2.
- Second, we map each part to a composition.
  - If the part is empty, it is mapped to 0.
  - Otherwise, we cut at the returns to zero and map each subpart to the number of peaks at height 2.
- As the total sum of peaks at height 2 is $r$, the grand total sum of the compositions is also $r$.



$$(0, 1, 0, 2+1, 0)$$

[Flajolet, Noy 99] observed the “combinatorial curiosity” that such $(r+1)$-tuples are equinumerous to the dissections of convex $(r+1)$-gons.

# Connections between Compositions and Paths

Why are we looking at connections between compositions and paths?

- By comparing similar structures, we can give rise to new theorems and new ideas in an analogous fashion and also link together apparently dissimilar combinatorial structures.
- For example, the arithmetic properties of partitions are famous, Ramanujan's congruences etc. We have shown here how similarly we can show congruences for special classes of Paths.

# Multiplicity

- Previously, we have discussed what happens to Dyck Paths when **all peaks have the same multiplicity**. If the multiplicity is $r$, then we find that

$$D_r(n) \equiv 0 \mod (r+1).$$

  This gives us a nice connection to little Schröder paths.
- But what if we were to ask what happens if the **multiplicity of parts were different**? It turns out to be quite difficult to answer this question. What would the same question mean for compositions or partitions for that matter?

# Connection to Wilf's Sixth Unsolved Problem

- Herb Wilf's famous sixth unsolved problem on integer partitions:

  *"Find any interesting theorem about the number $f(n)$ of partitions of $n$, where the multiplicities of its parts are all different."*

- To this day, not much known:
  - [Fill, Janson, Ward 12] and, independently, [Kane, Rhoades 12] found the lead-order asymptotics for $\log(f(n))$ as $n \to \infty$:

    $$\log f(n) \sim \frac{6^{1/3}}{3} n^{1/3} \log(n).$$

  - Note that on such partition, interchanging parts and multiplicities is an involution (initial motivation of Wilf). Let $F(n)$ be the number of fixed points on partitions of $n$. [Wagner 13] showed that

    $$\log F(n) \sim \frac{1}{2} \log(f(n)).$$

- Can we answer this question on "related objects", like **compositions** or **paths**?

# Different Multiplicites in Compositions

When we transition from partitions to **compositions**, we find it is also difficult to say anything non-trivial about the number of compositions where the multiplicities of parts are all different.

**Theorem**

For $n > 2$, the number of compositions $a(n)$ of $n$ such that all parts have different multiplicities is always even.

For example $a(4) = 6$:

| | |
|---|---|
| $4$ | $1+1+2$ |
| $2+2$ | $1+2+1$ |
| $1+1+1+1$ | $2+1+1$ |

# Parity Bias in Partitions

- Recently, the topic of parity bias in partitions has gained a lot of attention.
- [Kim, Kim, Lovejoy 20] proved that $p_o(n) > p_e(n)$, meaning that partitions with more odd parts than even parts are greater in number that partitions with more even parts than odd parts.
- This result can be shown to be true if we consider compositions instead of partitions.

# Recent Work and Conjecture

- Recently, [Kim, Kim, Lovejoy 20] had a conjecture that we [Banerjee, Bhattacharjee, MGD, Mahanta, Saikia 22] proved to be true: partitions with more odd parts than even parts are greater in number than partitions with more even parts than odd parts where all parts are different.
- We hypothesize that a similiar fact is true for Dyck paths.
- Let an **odd peak** be a peak at an odd height, an **even peak** be a peak at an even height.

**Conjecture**

Dyck paths with more number of odd peaks than even peaks are greater in number than Dyck paths with more number of even peaks than odd peaks.

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**Conjecture**

Dyck paths with more number of odd peaks than even peaks are greater in number than Dyck paths with more number of even peaks than odd peaks.

# Thank you!

# Backup

# Final Bijection for Dyck Bridges

**Proposition**

There is an explicit bijection between Dyck bridges with marked left-to-right maximum of length $2n$ and 2-colored Dyck bridges of length $2(n-1)$.

Sketch of Proof:

- Start with a Dyck bridge with marked left-to-right maximum of length $2n$.
- Cut the bridge at the first return to the $x$-axis after this maximum.
- Assign color 2 to the second part to the right.

# Bijections ctd.

- Decompose the first part into LudR, where ud is the marked left-to-right maximum or peak.
- Further decompose it into:

where $h$ is the height of the peak, the $L_i$ are negative excursions, and the $R_i$ are excursions.

# Bijections ctd.

- Now we form Dyck paths $D_i = u\phi(L_i)R_i$ for $i = 2, \ldots h$, but we need to flip the parts $L_i$ here.
- Remove the two steps of the marked peaks to get the required bridges with 2 steps less.
- We remember the height where we have labelled the maximum peak by decomposing it into that many parts.