Chordal graphs with bounded tree-width

joint work with Jordi Castellví, Michael Drmota and Marc Noy

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UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

Analysis of Algorithms (AofA2023@Taipei)
Academia Sinica - 29/06/2023

k-trees

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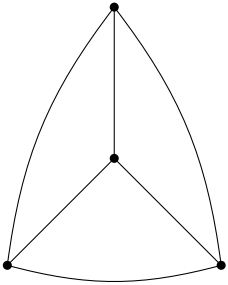
- start with K_{k+1},

Example $(k = 3)$:



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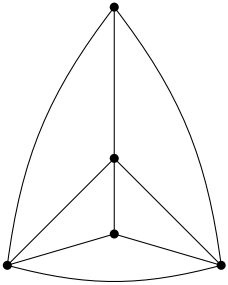

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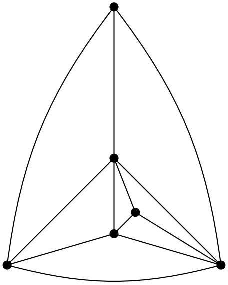

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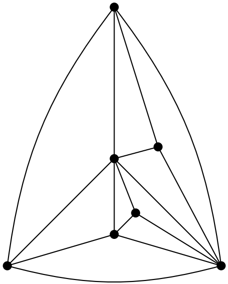

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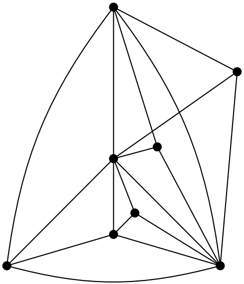

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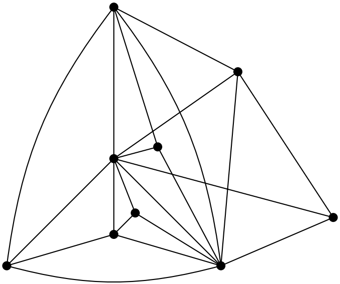

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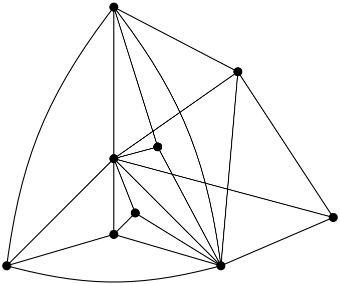

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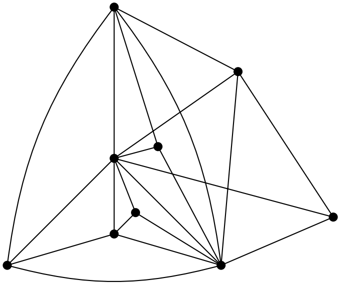

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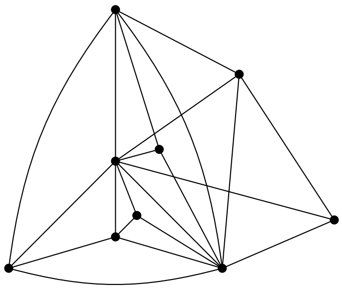

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- 2-trees are maximal series-parallel graphs.
- 3-trees with n vertices have $3n - 6$ edges, $3n - 8$ triangles and $n - 3$ K_4's.

Graphs with bounded tree-width

Graphs with tree-width at most k are exactly the subgraphs of k-trees

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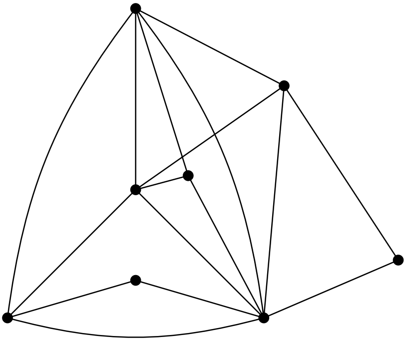
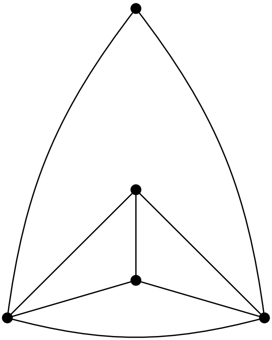
$$\left(\frac{ek}{\log k}\right)^n 2^{nk} n! \leq g_n \leq (ek)^n 2^{nk} n!$$

[Baste, Noy & Sau (2018)]: lower bound (using pathwidth).

Clique-sums

The class of graphs with tree-width at most k is stable by taking clique-sums:





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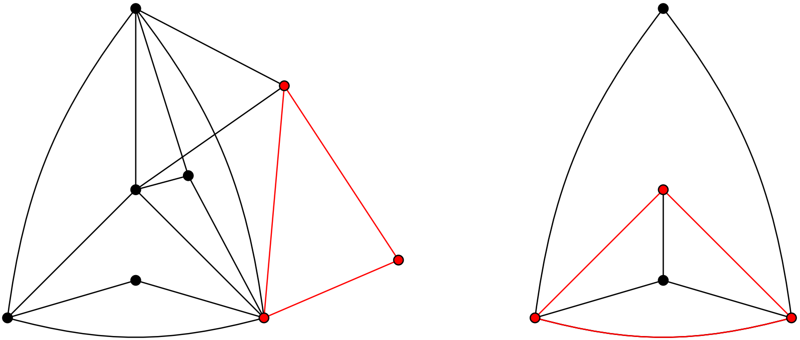

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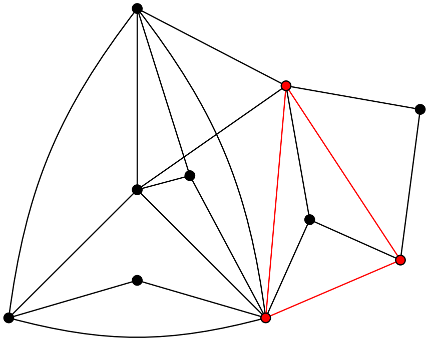

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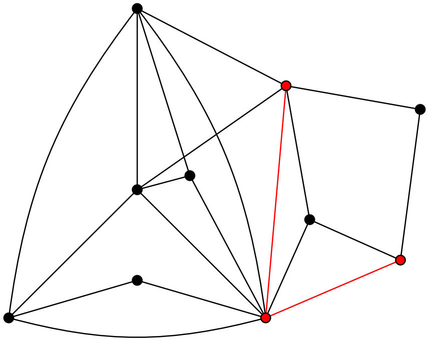

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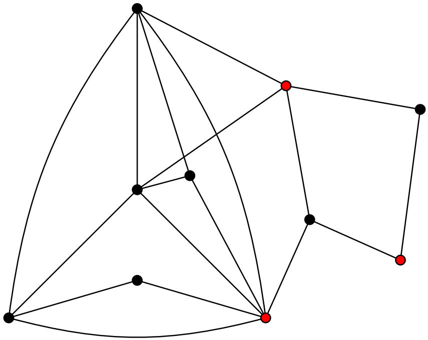
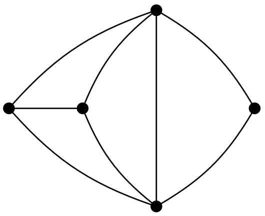
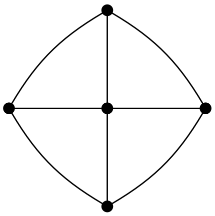


- the edge removal makes this operation seem non-tractable from the point of view of recursive enumeration.

Chordal graphs

Definition: a graph is chordal if it admits no induced cycle of length ≥ 4.





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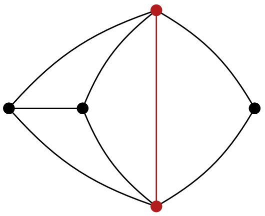

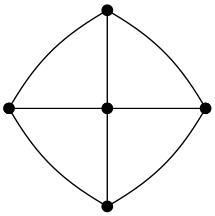

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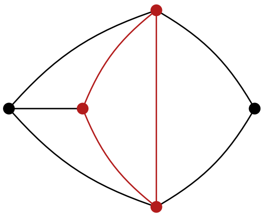

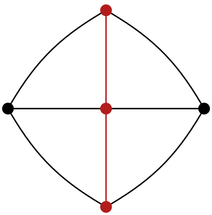

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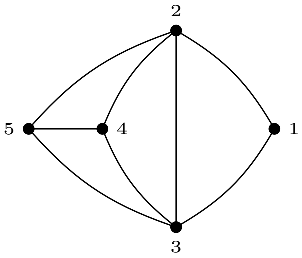

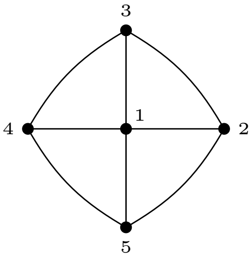

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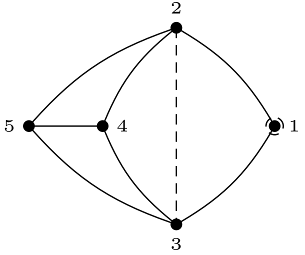

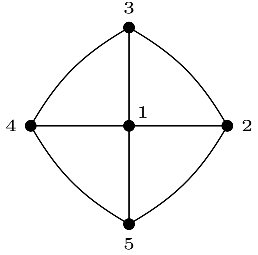

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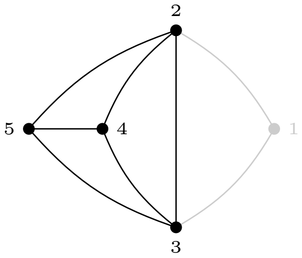

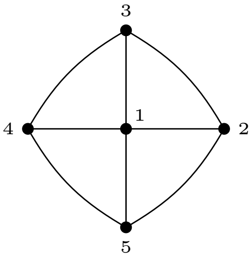

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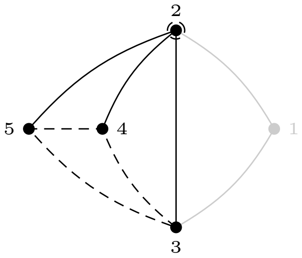

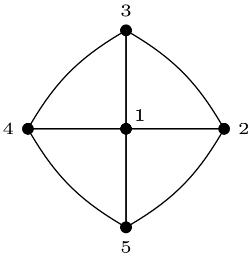

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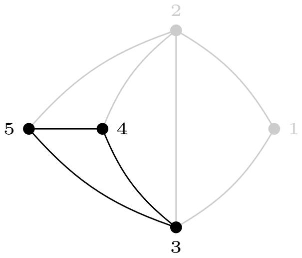

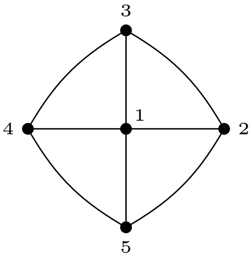

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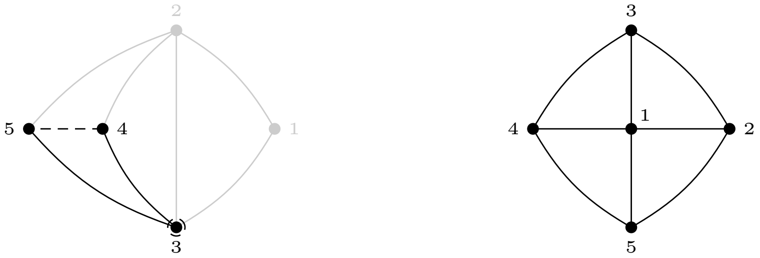

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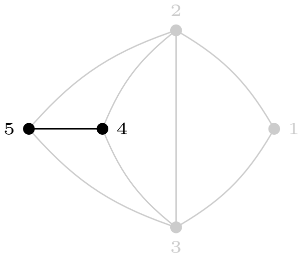

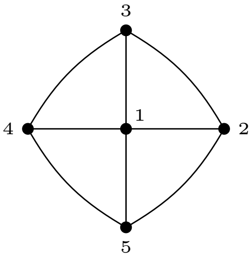

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Fix $n, k \geq 1$ and $0 \leq q \leq k$.

Let $\mathcal{G}_{k,q,n}$ be the family of q-connected chordal graphs with n labelled vertices and tree-width at most k.

[Castellví, Drmota, Noy & R. (2023+)]: $\exists\ c_{k,q} > 0$ and $\gamma_{k,q} > 1$ s.t.

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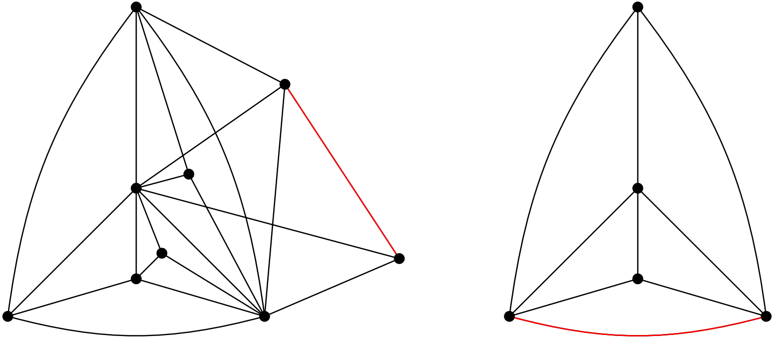

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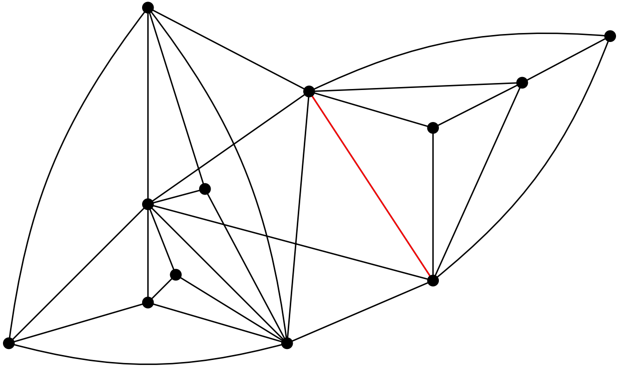
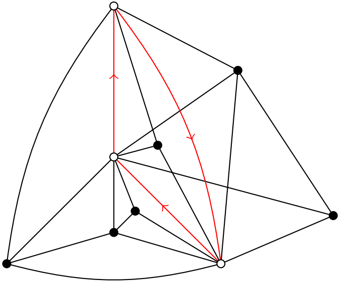


- enumeration of k-trees allows for the enumeration of $(k-1)$-connected chordal graphs with tree-width $\leq k$.

Rooted k-trees

A (labelled) k-tree is rooted when one k-clique is distinguished:

- fix a k-clique and fix an ordering of its vertices then remove their labels



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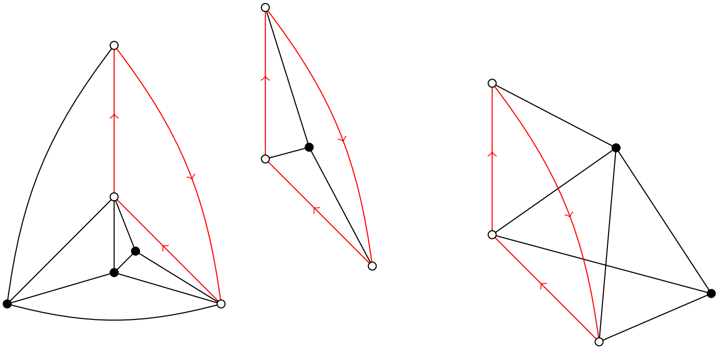

Recursive definition of the exponential generating function of rooted k-trees (variable x marks # k-cliques):

$$T_k(x) = \exp\left(xT_k(x)^k\right)$$

Analytic combinatorics: first principle

Exponential growth of the coefficients is determined by the radius of convergence.

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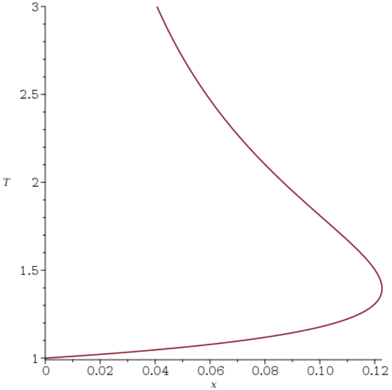
Proposition: the radius of convergence of $T_k(x)$ is a positive branch-point singularity of its implicit equation $T_k(x) = \exp\left(xT_k(x)^k\right)$



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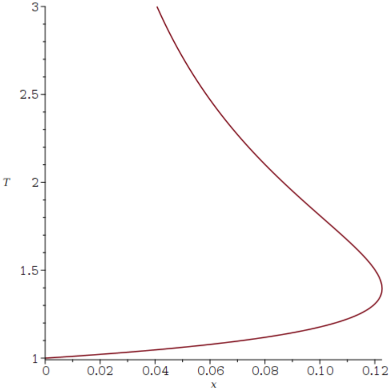

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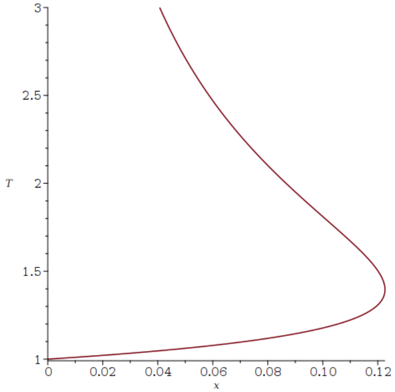

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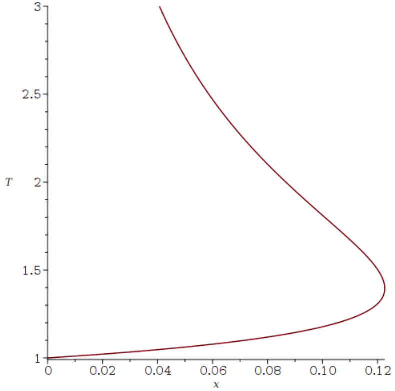

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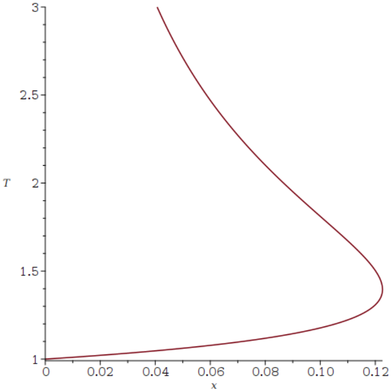
Radius of convergence of $T_k(z)$ is at $x = (ke)^{-1}$ $\rightarrow$ $((3e)^{-1} \approx 0.1226)$.

Analytic combinatorics: second principle

Sub-exponential growth of the coefficients is determined by the behaviour locally around the singularity (the singular expansion).



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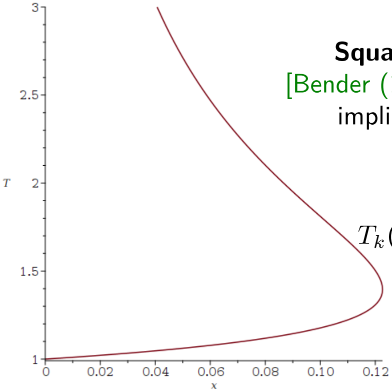

Square-root singular expansion lemma

[Bender (1974)]: Provided "nice" properties of the implicit function $T_k(x) = \exp\left(xT_k(x)^k\right)$, then as $z \sim (ke)^{-1}$

$$T_k(x) = T_0(k) - T_1(k)\left(1 - \frac{x}{(ke)^{-1}}\right)^{1/2} + O\left(1 - \frac{x}{(ke)^{-1}}\right)$$

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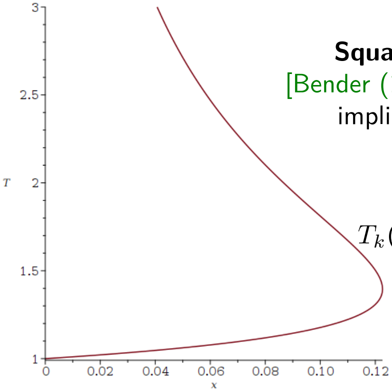

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Transfer theorem [Flajolet & Odlyzko (1982)]: as $n \to \infty$

$$[x^n]T_k(x) \sim \frac{T_1(k)}{-\Gamma(-1/2)} n^{-3/2} (ke)^n \qquad \text{where } -\Gamma(-1/2) = \sqrt{2\pi},$$

- asymptotic for unrooted k-trees → subexp. term in $n^{-5/2}$.

From $(q+1)$-connected to q-connected graphs

Multivariate GF of $\mathcal{G}_{k,q}$ the q-connected graphs (for $q \in [k]$):

$$G_q(x_1, \dots, x_k) = \sum_{A \in \mathcal{G}_{k,q}} \frac{1}{n_1(A)!} \prod_{i \in [k]} x_i^{n_i(A)} \qquad (n_i(A) = \# \; i\text{-cliques in } A).$$

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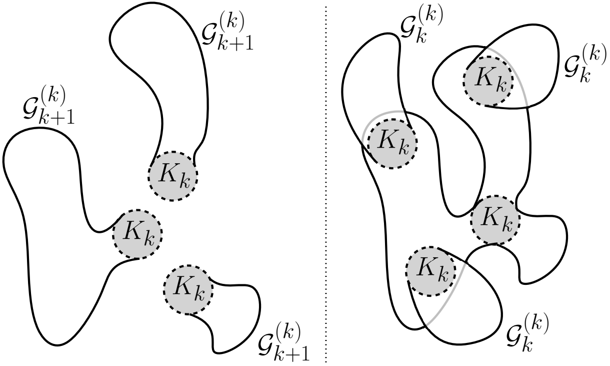
Implict equation for the GF of q-connected graphs rooted at a q-clique

$$G_q^{(q)}(x_1,\ldots,x_k) = \exp\Big(G_{q+1}^{(q)}\left(x_1,\ldots,x_{q-1},x_q G_q^{(q)}(x_1,\ldots,x_k),x_{q+1},\ldots,x_k\right)\Big)$$



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\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
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\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
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Nick Wormald already did it (in 1985)!

- *algorithm to compute the first numbers of chordal graphs with bounded clique number.*

Chordal graphs with small tree-width

k	$q = 1$	$q = 2$	$q = 3$	$q = 4$	$q = 5$	$q = 6$	$q = 7$
1	0.36788	-	-	-	-	-	-
2	0.14665	0.18394	-	-	-	-	-
3	0.07703	0.08421	0.12263	-	-	-	-
4	0.04444	0.04662	0.05664	0.09197	-	-	-
5	0.02657	0.02732	0.03092	0.04152	0.07358	-	-
6	0.01608	0.01635	0.01773	0.02184	0.03214	0.06131	-
7	0.00974	0.00984	0.01038	0.01204	0.01614	0.02583	0.05255

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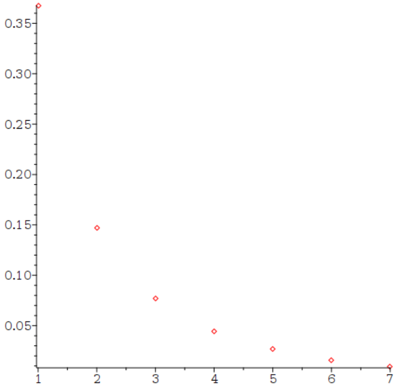

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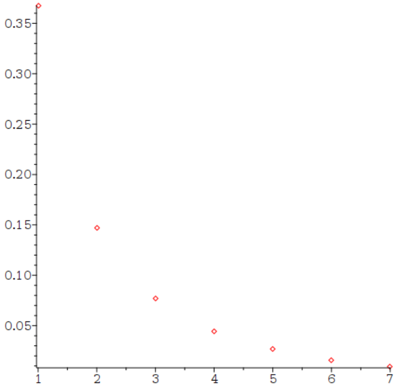

[Bender, Richmond & Wormald (1985)]:
almost all chordal graphs are split.

$\Rightarrow$ the number of chordal graphs with n labelled vertices is

$$\sim \binom{n}{n/2} 2^{n^2/4}$$

Conclusion

For $i \in \{2, \ldots, k\}$, let $X_i = \#$ i-cliques in a uniform random graph in $\mathcal{G}_{k,q,n}$.

[Castellví, Drmota, Noy & R. (2023+)]: $\exists\, \alpha, \sigma \in (0,1)$ s.t. as $n \to \infty$

$$\frac{X_i - \mathbb{E}X_i}{\sqrt{\mathbb{V}X_i}} \xrightarrow{d} N(0,1), \qquad \text{with} \quad \mathbb{E}X_i \sim \alpha n \quad \text{and} \quad \mathbb{V}X_i \sim \beta n.$$

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!開飯 !謝謝