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Asymptotic Analysis of Regular Sequences

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Prelude 1: Merge Sort

- Partition into two sets of (almost) equal size;
- Sort both parts individually & recursively;
- Merge results.

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In other words:

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$$s(2n) = s(n),$$
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Common Pattern

Recall:

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- Sequence x
- Recursively given: Express $x(2n)$ and $x(2n+1)$ ...
- ... in terms of $x(n)$, possibly $x(n+1)$, known sequences

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q-Regular Sequences

Vector-valued sequence $v\colon \mathbb{N}_0 \to \mathbb{C}^D$ with

$$v(qn + r) = A_r v(n)$$

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- $q \ge 2$, $D \ge 1$: integers;
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First component of v: *q-regular sequence* (Allouche–Shallit 1992).

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- Set $x(n) := [n \text{ is esthetic}]$.

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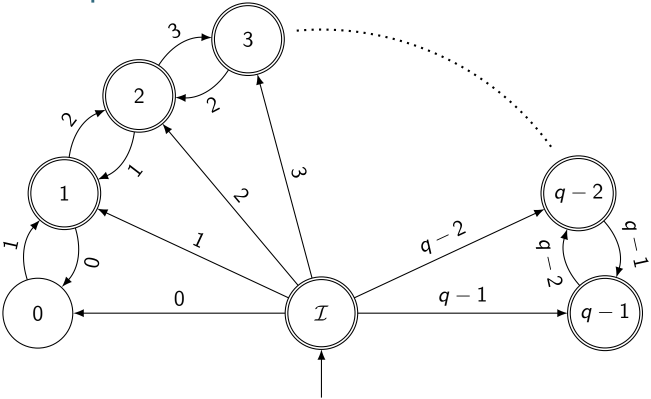
- Set $x(n) := [n \text{ is esthetic}]$.
- For $N \to \infty$, determine number $\sum_{0 \le n < N} x(n)$ of esthetic numbers up to N.

Linear Representation



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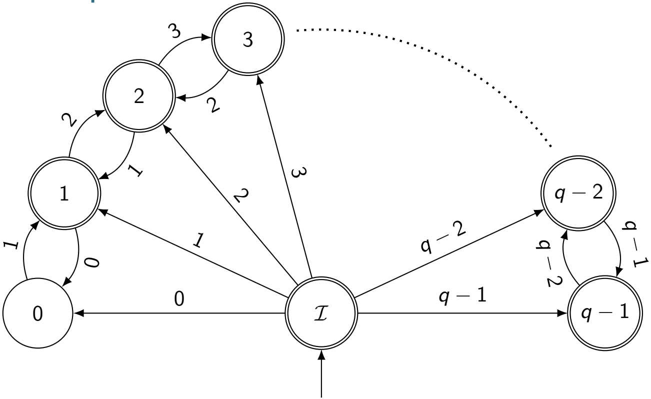

- Let $n := (n_{\ell-1} \dots n_0)_q$ and $x_{\mathcal{I}}(n) := [n \text{ is esthetic}] = [\exists \text{ path with label } n_0 n_1 \dots n_{\ell-1} \text{ from } \mathcal{I} \text{ to some final state}]$.

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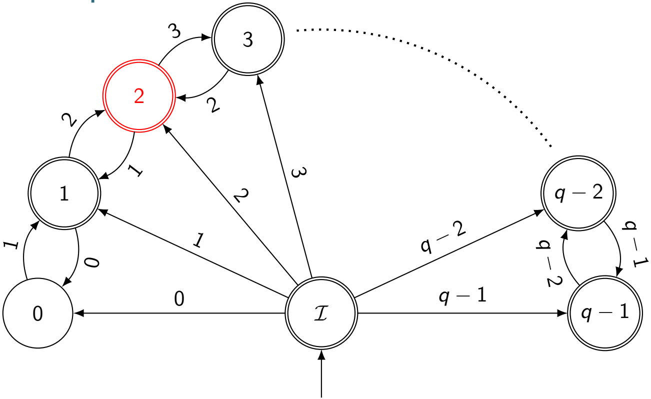

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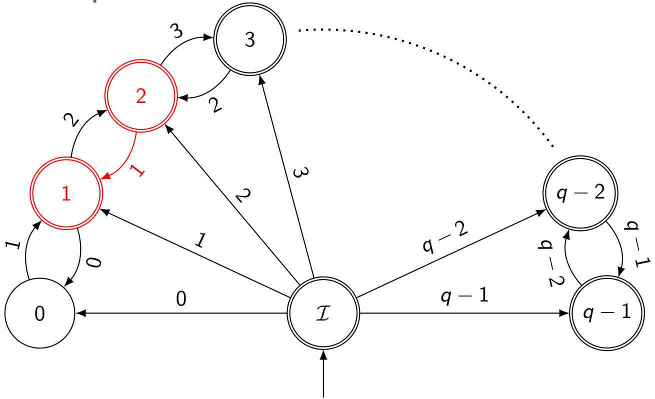

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- E.g., $x_2(n) = [n_0 = 1] x_1\left(\frac{n-n_0}{q}\right) + [n_0 = 3] x_3\left(\frac{n-n_0}{q}\right)$.

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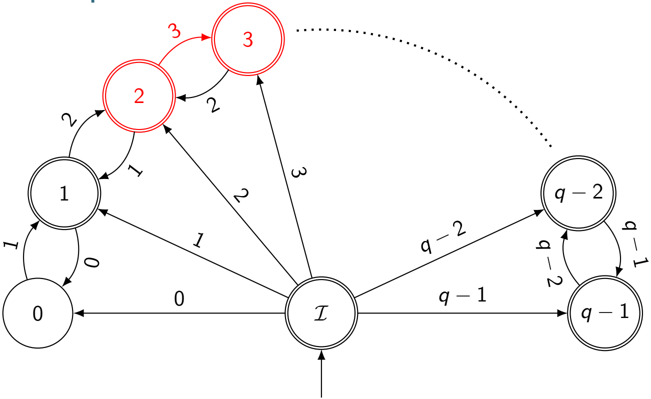

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- Then $v(qn + r) = A_r v(n)$ for suitable matrices $A_0, \ldots, A_{q-1}$.
- *Linear representation* of a *q-regular sequence*.
- Example for $q = 4$:

$$A_0 = \begin{pmatrix} 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0 \end{pmatrix}, \; A_1 = \begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&1&0&0 \end{pmatrix}, \; A_3 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&1&0 \end{pmatrix}.$$

Regular Sequences and Matrix Products

- Vector-valued sequence $v\colon \mathbb{N}_0 \to \mathbb{C}^D$ with

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- Then
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- Related to recognisable series (see Berstel and Reutenauer).

Analysis of Regular Sequences

Theorem (Dumas 2013; H–Krenn 2020)

- *$x(n)$: q-regular sequence, first component of $v(n)$*

$$\sum_{0\le n<N} x(n) = \sum_{\substack{\lambda\in\sigma(C)\\ |\lambda|>R}} N^{\log_q \lambda} \sum_{0\le k<m_C(\lambda)} \frac{(\log N)^k}{k!}\, \Phi_{\lambda k}(\{\log_q N\}) + O\Bigl(N^{\log_q R}(\log N)^{\max\{m_C(\lambda)\colon |\lambda|=R\}}\Bigr)$$

as $N\to\infty$, where $\Phi_{\lambda k}$ are suitable 1-periodic functions.

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Theorem (Dumas 2013; H–Krenn 2020)

- *$x(n)$: q-regular sequence, first component of $v(n)$*
- $C := A_0 + \cdots + A_{q-1}$
- *$\sigma(C)$: spectrum of C*

$$\sum_{0\le n<N} x(n) = \sum_{\substack{\lambda\in\sigma(C)\\ |\lambda|>R}} N^{\log_q \lambda} \sum_{0\le k<m_C(\lambda)} \frac{(\log N)^k}{k!}\,\Phi_{\lambda k}(\{\log_q N\}) + O\Big(N^{\log_q R}(\log N)^{\max\{m_C(\lambda)\colon |\lambda|=R\}}\Big)$$

as $N\to\infty$, where $\Phi_{\lambda k}$ are suitable 1-periodic functions.

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- $m_C(\lambda)$: *size of the largest Jordan block of* C *associated with* λ

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- Fourier coefficients can be computed numerically (using a functional equation for the corresponding Dirichlet series)

Analysis of Regular Sequences: Ingredients for the Proof

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- Computing Fourier coefficients extending ideas by Grabner–Hwang. Implemented in SageMath.

Mellin–Perron Summation Formula of Order 0

- Dirichlet series
  - sequence $(v(n))_{n\geq 1}$
  - $\mathcal{V}(s) = \sum_{n\geq 1} n^{-s} v(n)$

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- $v(qn+r) = A_r v(n)$; $\mathcal{V}_{n_0}(s) := \sum_{n\geq n_0} \frac{v(n)}{n^s}$.

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$$(I-q^{-s}C)\mathcal{V}_{n_0}(s) = \sum_{n_0\le n<qn_0}\frac{v(n)}{n^s} + \sum_{k\ge 1}\binom{-s}{k}\sum_{0\le r<q}\frac{r^k}{q^k}A_r\mathcal{V}_{n_0}(s+k)$$

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$$= N^{\kappa} \sum_{\ell\in\mathbb{Z}} \frac{\delta_\ell}{\chi_\ell(\chi_\ell+1)} \exp(2\pi i \ell \log_q N)$$

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$$+ \frac{1}{2\pi i} \int_{\Re s=\vartheta-1} \mathcal{V}(s) \frac{N^s}{s(s+1)}\, ds$$

- Fourier coefficients
  - $\delta_\ell = \operatorname{Residue}\big(\mathcal{V}(s), s = \chi_\ell\big)$
  - with $\chi_\ell = \kappa + 2\pi i\ell / \log q$,
  - κ singularity of $\mathcal{V}$ corresponding to abscissa of convergence

Pseudo-Tauberian Theorem

Theorem (H.–Krenn 2020)

Let

- $\kappa \in \mathbb{C}$, $\quad q > 1$, $\quad m \in \mathbb{Z}_+$, $\quad 0 < \beta < \alpha$;
- $\Phi_0, \ldots, \Phi_{m-1}$ *Hölder continuous* 1*-periodic functions with exponent* α

Then

$$\sum_{1 \le n < N} n^{\kappa} \sum_{\substack{j+k=m-1 \\ 0 \le j < m}} \frac{(\log n)^k}{k!} \Phi_j(\log_q n)$$

$$=$$

for integers $N \to \infty$.

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Then $\exists$ *continuously differentiable* 1*-periodic functions* $\Psi_{-1}, \Psi_0, \ldots, \Psi_{m-1}$ *and a constant* c *such that*

$$\sum_{1 \le n < N} n^\kappa \sum_{\substack{j+k=m-1 \\ 0 \le j < m}} \frac{(\log n)^k}{k!} \Phi_j(\log_q n) = c + N^{\kappa+1} \sum_{\substack{k+j=m-1 \\ -1 \le j < m}} \frac{(\log N)^k}{k!} \Psi_j(\log_q N) + O(N^{\Re\kappa+1-\beta})$$

for integers $N \to \infty$.

Pseudo-Tauberian Theorem (ctd.)

Theorem (H.–Krenn 2020, ctd.)

Let

- $\varphi_{j\ell} := \int_0^1 \Phi_j(u) \exp(-2\ell\pi iu)\,\mathrm{d}u$: *Fourier coefficients of* Φ_j
- $\psi_{j\ell} := \int_0^1 \Psi_j(u) \exp(-2\ell\pi iu)\,\mathrm{d}u$: *Fourier coefficients of* Ψ_j

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Then the corresponding generating functions fulfil

$$\sum_{0\le j<m} \varphi_{j\ell} Z^j = \Bigl(\kappa + 1 + \frac{2\ell\pi i}{\log q} + Z\Bigr) \sum_{-1\le j<m} \psi_{j\ell} Z^j + O(Z^m)$$

for $\ell \in \mathbb{Z}$ *and* $Z \to 0$.

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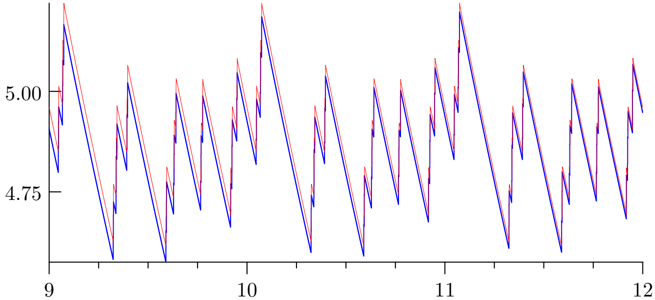
for $\ell \in \mathbb{Z}$ *and* $Z \to 0$.
If $q^{\kappa+1} \neq 1$*, then* Ψ_{-1} *vanishes.*

Esthetic Numbers—Result

Theorem (H–Krenn 2020)

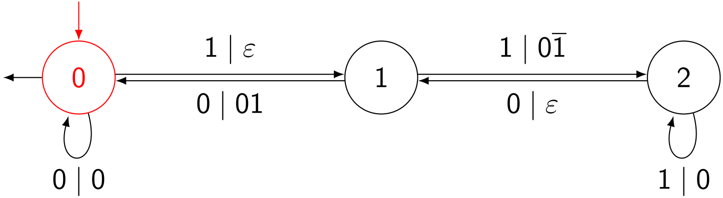
$$\sum_{0\le n<N} x(n) = \sum_{j\in\{1,2,\ldots,\lceil\frac{q-2}{3}\rceil\}} N^{\log_q(2\cos(j\pi/(q+1)))}\Phi_{qj}(2\log_{q^2} N) + O\big((\log N)^{[q\equiv -1 \pmod 3]}\big)$$

with 2-periodic continuous functions Φ_{qj}.
If q is even, then the functions Φ_{qj} are actually 1-periodic.



Transducer

Write labels from right to left (least significant to most significant digit). Empty word: ε



$27 = (0011011)_2 = (\qquad\qquad)_2$

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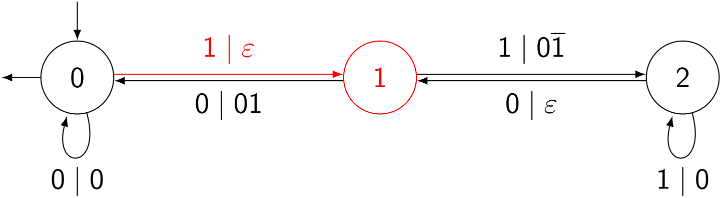

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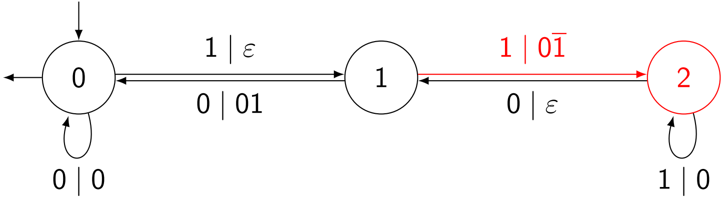

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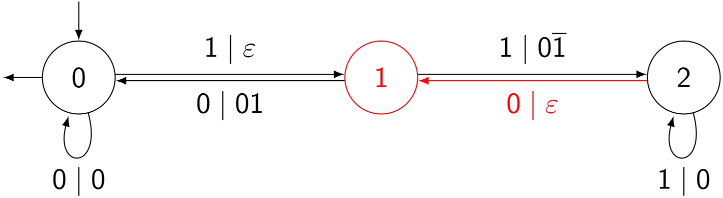

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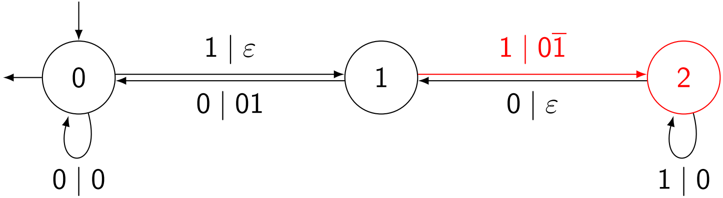

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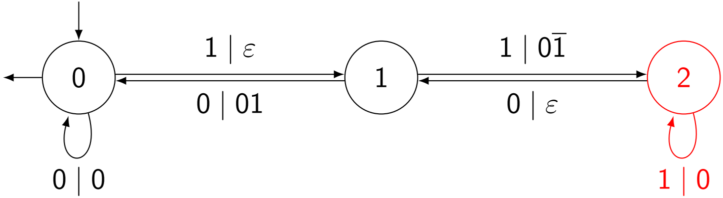

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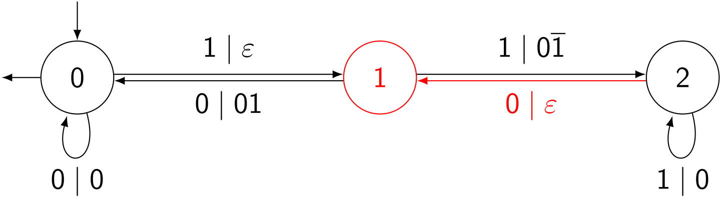

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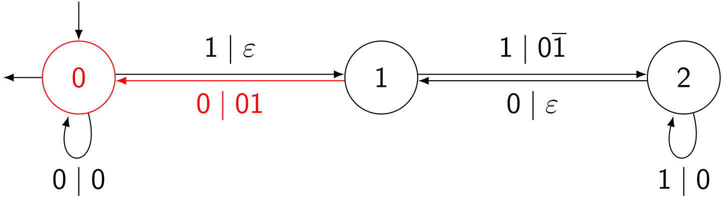

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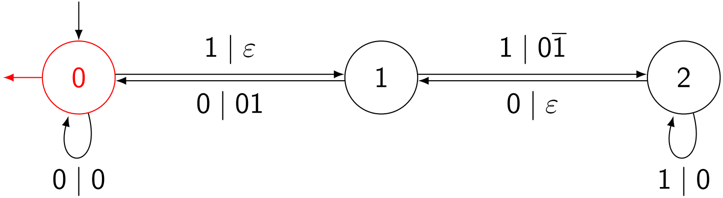

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Transducer: Central Limit Theorem

Theorem (H–Kropf–Prodinger 2015)

Let $\mathcal{T}$ be complete, deterministic transducer with input alphabet $\{0, 1, \ldots, q-1\}$ and final period p. For fixed N, use equidistribution on $\{0, 1, \ldots, N-1\}$.

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Then $\mathcal{T}(n)$ has the expected value

$$\mathbb{E}(\mathcal{T}(n)) = e_{\mathcal{T}} \log_q N + \Psi_1(\log_q N) + o(1)$$

with $e_{\mathcal{T}} \in \mathbb{R}$ and a p-periodic, continuous function Ψ_1.

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If $\mathcal{T}$ is finally connected, then the variance is

$$\mathrm{Var}(\mathcal{T}(n)) = v_{\mathcal{T}} \log_q N + \Psi_2(\log_q N) + o(1)$$

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Then $\mathcal{T}(n)$ has the expected value

$$\mathbb{E}(\mathcal{T}(n)) = e_{\mathcal{T}} \log_q N + \Psi_1(\log_q N) + o(1)$$

with $e_{\mathcal{T}} \in \mathbb{R}$ and a p-periodic, continuous function Ψ_1.
If $\mathcal{T}$ is finally connected, then the variance is

$$\operatorname{Var}(\mathcal{T}(n)) = v_{\mathcal{T}} \log_q N + \Psi_2(\log_q N) + o(1)$$

with $v_{\mathcal{T}} \in \mathbb{R}$ and a p-periodic, continuous function Ψ_2.
If $v_{\mathcal{T}} \neq 0$, then $\mathcal{T}(n)$ is asymptotically normally distributed.

Transducer: Central Limit Theorem

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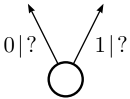
Also possible for higher dimensional input.

Connectivity Properties of the Transducer

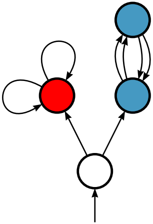
- complete and deterministic



for $q = 2$

Connectivity Properties of the Transducer

- complete and deterministic
- final component



Connectivity Properties of the Transducer

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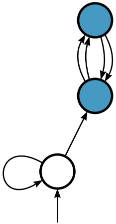
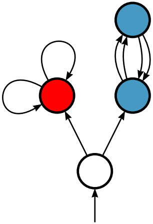


Connectivity Properties of the Transducer

- complete and deterministic
- final component
- finally connected
- period = greatest common divisor of all lengths of cycles
- final period p = least common multiple of the periods
- finally aperiodic if $p = 1$



period: 1
period: 2
final period: $p = 2$

Bounded Variance

$$\mathrm{Var}(\mathcal{T}(n)) = v_{\mathcal{T}} \log_q N + \Psi_2(\log_q N) + o(1)$$

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Theorem (H–Kropf–Wagner 2015)

Let $\mathcal{T}$ be finally connected. Then the following assertions are equivalent:

1. *The constant $v_{\mathcal{T}}$ of the variance is 0.*
2. *There is a constant k such that the average output of every cycle in the final component is k.*
3. *Length and output sum are linearly dependent (up to $O(1)$).*

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Also possible for higher dimensional output (Kropf 2016).