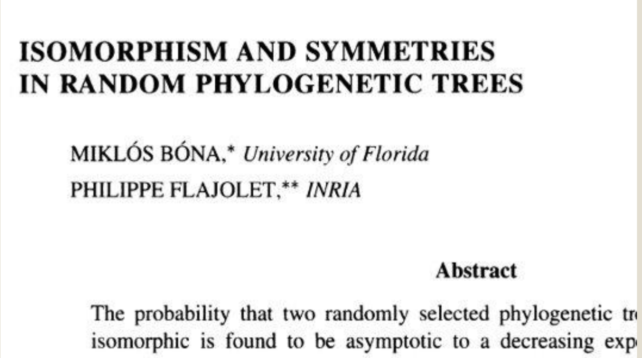
THE PROBABILITY OF RANDOM TREES BEING ISOMORPHIC

Background

- Previously studied for binary trees.
- *Simply generated* or *(Biyenaymé-)Galton–Watson trees.*
  - *Conditioned.*
  - Examples: plane trees, labeled trees, binary trees etc.
  - (For Jim Fill: ORTs, LURTs, binary trees etc.)
- *Pólya trees.*
  - *(UURTs)*
- Trees with restricted outdegrees.
- Isomorphisms and isomorphism classes.



ISOMORPHISM AND SYMMETRIES IN RANDOM PHYLOGENETIC TREES

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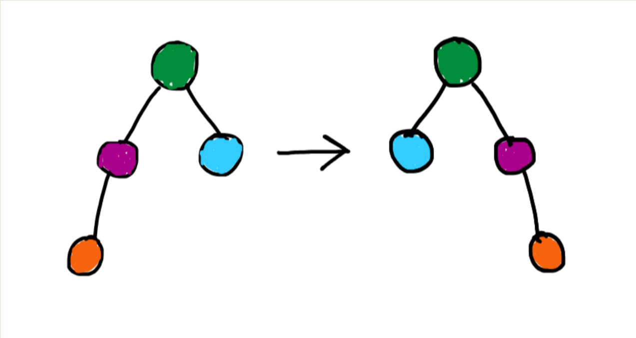
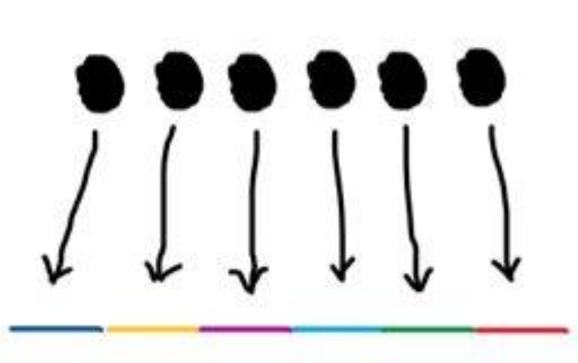
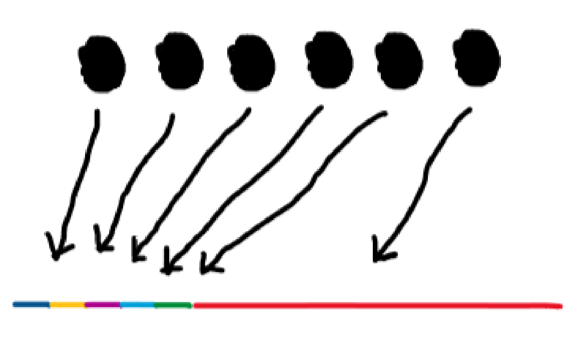
Abstract

The probability that two randomly selected phylogenetic tr
isomorphic is found to be asymptotic to a decreasing exp



Exponential decay of probabilities?





Probability of **labeled** trees being isomorphic

Theorem 1. *The probability p_n that two labeled rooted trees are isomorphic has the full asymptotic expansion*

$$p_n \sim A n^{3/2} c_l^n \left(1 + \sum_{k=1}^{\infty} \frac{e_k}{n^k}\right),$$

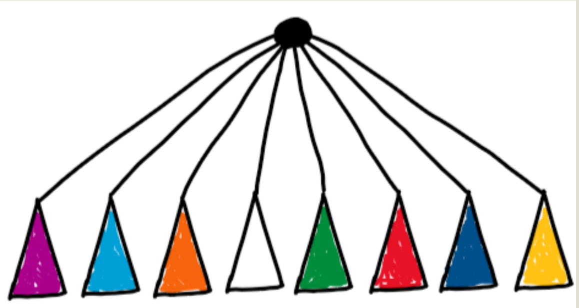
where $A \approx 2.397678$, $c_l \approx 0.354379$ and the e_k are constants that can be calculated.

The number of labelings of a Pólya tree P is |P|!/|Aut(P)|.

$$P(x,t) = \sum_{P \in \mathcal{P}} \frac{x^{|P|}}{|\operatorname{Aut} P|^t}.$$

$$p_n = \frac{1}{n^{2(n-1)}} \sum_{P \in \mathcal{P}_n} \frac{n!^2}{|\operatorname{Aut} P|^2} = \left(\frac{n!}{n^{n-1}}\right)^2 [x^n] P(x,2).$$

Labeled vs general Galton–Watson trees



- **Not** true for all Galton–Watson trees

- … but it is for Galton–Watson trees with restricted degrees.

Theorem 2. *The probability g_n that two Galton–Watson trees with degrees in a finite set D are isomorphic satisfies*

$$g_n \leq B c_g^n,$$

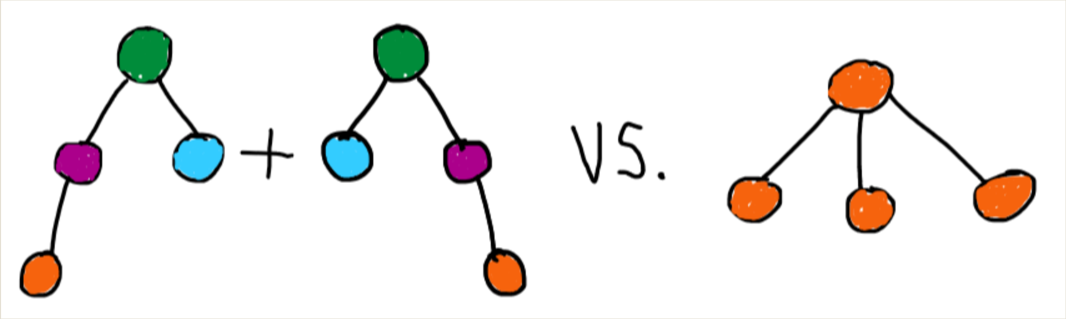
for some constants B and $c_g < 1$.

$$\mathcal{PR}(P) = \frac{\prod_{v \in V(P)} \deg(v)!}{|\operatorname{Aut} P|}.$$

$$W(P) = w(T)\mathcal{PR}(P)$$

What do random isomorphic pairs look like?

- Do pairs of **labeled** trees conditioned on being isomorphic look different than regular labeled trees?



- A multivariate central limit theorem for vertices of given degrees.

- We can e.g. show that the average number of leaves in the trees differ:
  - Labeled trees: ≈ 0.367879 n
  - Isomorphic pairs: ≈ 0.340252 n

The number of labelings of Pólya trees

- How large is the isomorphism class corresponding to a random Pólya tree?

Theorem 6. *Let $\mathcal{P}_n$ be a random Pólya tree of size n, then the number of labelings $\mathrm{L}(\mathcal{P}_n)$ of $\mathcal{P}_n$ has expected value and variance*

$$\mathrm{E}[\log \mathrm{L}(\mathcal{P}_n)] = n \log n - (\mu + 1)n + \frac{\log n}{2} + O(1),$$
$$\mathrm{Var}[\log \mathrm{L}(\mathcal{P}_n)] = \sigma^2 n + O(1),$$

for numerical constants $\mu \approx 0.137342$ and $\sigma^2 \approx 0.196770$ and satisfies

$$\frac{\log \mathrm{L}(\mathcal{P}_n) - \mathrm{E}[\log \mathrm{L}(\mathcal{P}_n)]}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2).$$

$$\log n! - \log |\operatorname{Aut} P|.$$

$$P(x,t) = \sum_{P \in \mathcal{P}} \frac{x^{|P|}}{|\operatorname{Aut} P|^t}.$$

Number of plane representations of Galton–Watson trees

- Can prove similar results for weights of isomorphism classes of Galton–Watson trees with restricted degrees.
  - Labelings, plane representations.
- Need to pick a type of Galton–Watson tree and the corresponding type of Pólya tree.
- Now, both the mean and variance is of order n.

	μ	σ^2
Binary trees	0.444518	0.072413
Unary-binary trees	0.176278	0.025865