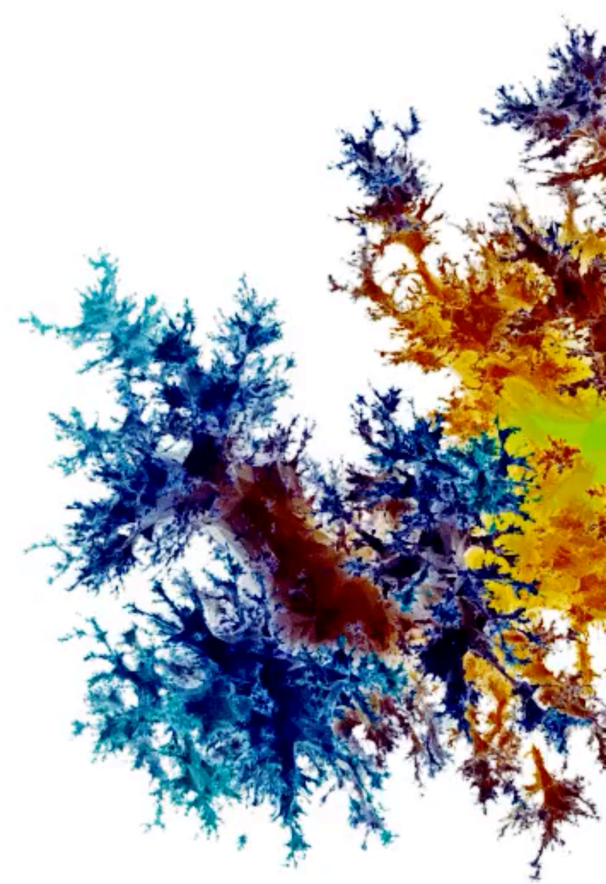
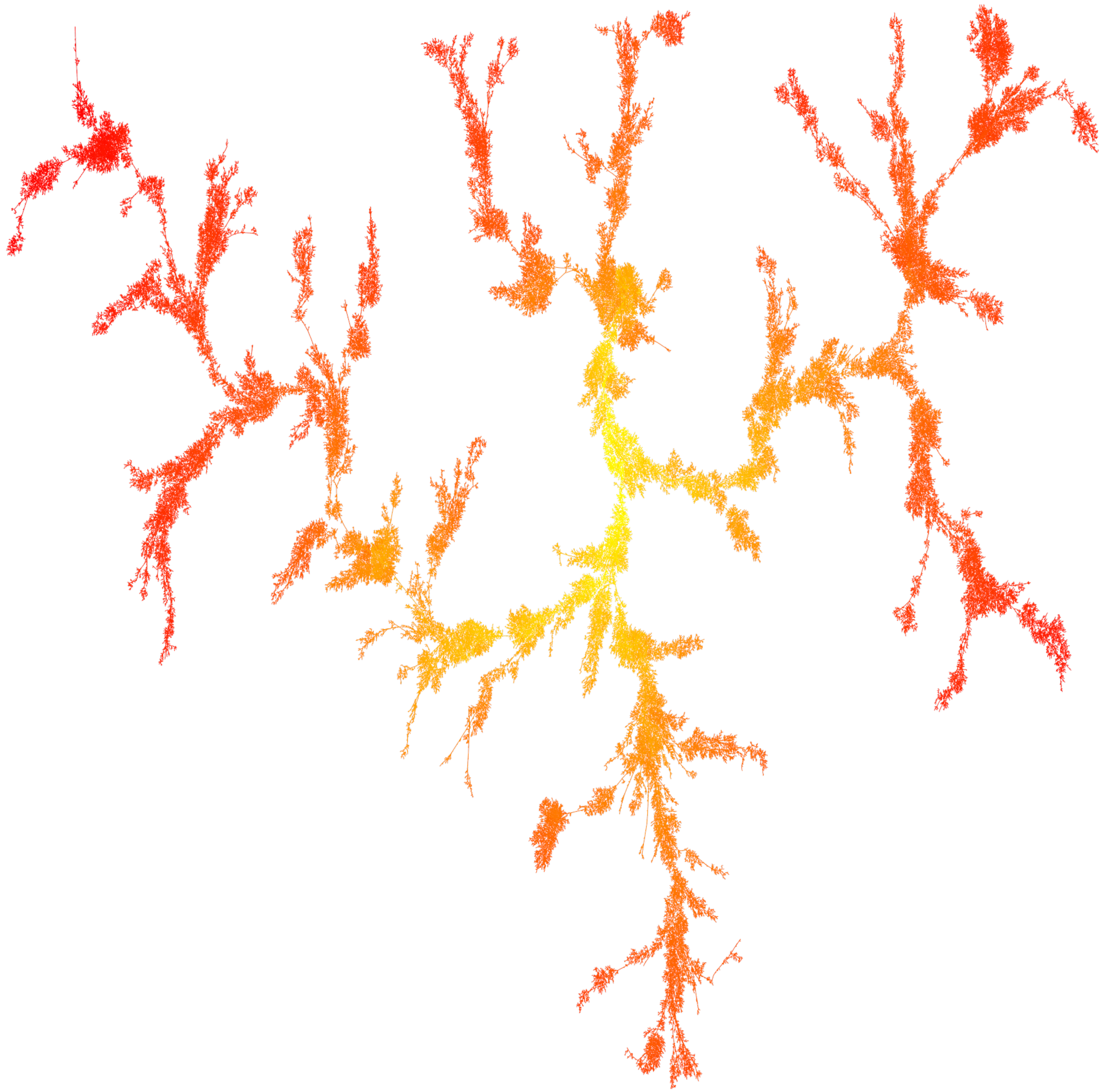
SCALING LIMITS OF RANDOM ELEMENTS OF COMBINATORIAL CLASSES



Benedikt Stufler
Technische Universität Wien
dmg.tuwien.ac.at/stufler

BROWNIAN TREE: UNIVERSAL LIMIT OF RANDOM TREES



Simulation: GRANT (Generate RANDom Trees), available here: http://github.com/BenediktStufler/grant

LIMITS OF DILUTE SUPERTREES

$$T_1(z) = z \exp(T_1(z))$$

$T_d(z) = T_1(zT_{d-1}(z))$ for $d \geq 2$.

Stufler (2022, 2023+): *We have*

$$[z^n]T_d(z) \sim \frac{2^{1-1/2^d}}{|\Gamma(-1/2^d)|} n^{-1-2^{-d}} e^n$$

as $n \to \infty$. *The uniform* n*-vertex supertree* $\mathsf{T}_{d,n}$ *from the class* T_d *equipped with the uniform measure* $\mu_{\mathsf{T}_{d,n}}$ *on its vertex set satisfies*

$$\left(\mathsf{T}_{d,n}, \frac{1}{2\sqrt{n}} d_{\mathsf{T}_{d,n}}, \mu_{\mathsf{T}_{d,n}}\right) \xrightarrow{\mathrm{d}} \mathscr{S}(1/2, -1/2^d, 1/2, L_{\mathrm{Brownian}})$$

in the Gromov–Hausdorff–Prokhorov sense as $n \to \infty$.

COMPOSITION SCHEMES / GIBBS PARTITIONS

Consider a composition scheme $V(W(z))$ with coefficients v_n, w_n for $n \geq 0$.

- Convergent case: single giant component, rest stochastically bounded
- Dense case: linear number of components
- Mixture case: coin flip small or linear number of components
- Dilute case: polynomial (sublinear) number of components
- ...

Gourdon (1996)
Banderier, Flajolet, Schaeffer, Soria (2001)
Flajolet, Sedgewick (2009)
Addario-Berry (2019)
Stufler (2018, 2020)
Banderier, Kuba, Wallner (in print)
Stufler (in print)

DILUTE REGIME

Let $0 < \alpha, \beta < 1$ and suppose $w_n \sim c_w n^{-1-\alpha} \rho_w$ and $v_n = L_v(n) n^{-1-\beta} W(\rho_w)^{-n}$ with L_v slowly varying. Let $K_{(1)} \geq K_{(2)} \geq \ldots$ denote the ordered component sizes of the associated random Gibbs partition / composition scheme $V(W(z))$.

Stufler (2022), Gibbs partitions: a comprehensive phase diagram

Let $\Upsilon_n = \sum_{i:K_{(i)}>0} \delta_{K_{(i)}/n}$ denote the point process on $]0,1]$ with δ referring to the Dirac measure. Then $\Upsilon_n \xrightarrow{d} \Upsilon_{\alpha,\beta}$ for a point process $\Upsilon_{\alpha,\beta}$ with intensity

$$\frac{x^{-\alpha-1}(1-x)^{\alpha(1-\beta)-1}}{B(1-\alpha, \alpha(1-\beta))} \, \mathrm{d}x$$

DILUTE REGIME

also observed that for each integer $m \geq 1$ the mth correlation function of $\Upsilon_{\alpha,\beta}$ is given by

$$\mathbb{E}\left[\left(\mathrm{Poi}\left(\frac{c_w}{W(\rho_w)}Z\right)\right)_m\right] \mathbb{1}_{\substack{x_1+\ldots+x_m \leq 1 \\ x_1,\ldots,x_m \geq 0}} \frac{(1-x_1-\ldots-x_m)^{\alpha(m-\beta)-1}}{x_1^{\alpha+1}\cdots x_m^{\alpha+1}},$$

with the first factor denoting the mth factorial moment of the Poisson random variable $\mathrm{Poi}\left(\frac{c_w}{W(\rho_w)}Z\right)$ with random parameter $\frac{c_w}{W(\rho_w)}Z$, so that

$$\mathbb{P}(\mathrm{Poi}(vZ) = k) = \mathbb{E}\left[\frac{1}{k!}\left(\frac{c_w}{W(\rho_w)}Z\right)^k \exp\left(-\frac{c_w}{W(\rho_w)}Z\right)\right], \qquad k \geq 0.$$

The distribution of $\frac{c_w}{W(\rho_w)}Z$ in fact only depends on α and β, so that

$$\mathbb{E}\left[\left(\mathrm{Poi}\left(\frac{c_w}{W(\rho_w)}Z\right)\right)_m\right] = \left(\frac{\alpha}{\Gamma(1-\alpha)}\right)^m \frac{\Gamma(1-\alpha\beta)\Gamma(m+1-\beta)}{\Gamma(1+\alpha(m-\beta))\Gamma(1-\beta)}.$$

DILUTE REGIME

We let $\eta_1 \geq \eta_2 \geq \ldots > 0$ denote the ranked points of Υ on $]0,1]$. Since almost surely $\Upsilon(]0,1]) = \infty$, there are indeed infinitely many such points. For all integers $k \geq 1$ the size $K_{(k)}$ of the kth largest component of P_n satisfies the joint distributional convergence

$$K_{(k)}/n \xrightarrow{w} \eta_k, \qquad k \geq 1 \text{ (jointly)}.$$

Since the sum of the component sizes equals n we have $\eta_k \leq 1/k$.

But what is the distribution of η_k?

DILUTE REGIME

For all $0 < x \leq 1$ and all integers $k \geq 1$ this leads to

$$\mathbb{P}(\eta_k < x) = \frac{\Gamma(1-\alpha(b-1))}{2\pi|\Gamma(-\alpha)|^{b-1}\Gamma(2-b)}$$
$$\int_0^\infty \int_{-\infty}^\infty u^{\alpha(b-1)} \exp\left(-\frac{1}{u^\alpha}\int_x^1 \frac{e^{ityu}}{y^{1+\alpha}}\,\mathrm{d}y - iut - |\Gamma(-\alpha)|(-it)^\alpha\right)$$
$$\sum_{j=0}^{k-1} \frac{1}{j!}\left(\frac{1}{u^\alpha}\int_x^1 \frac{e^{ityu}}{y^{1+\alpha}}\,\mathrm{d}y\right)^j \mathrm{d}t\,\mathrm{d}u.$$

Be aware that we may not interchange the order of integration.

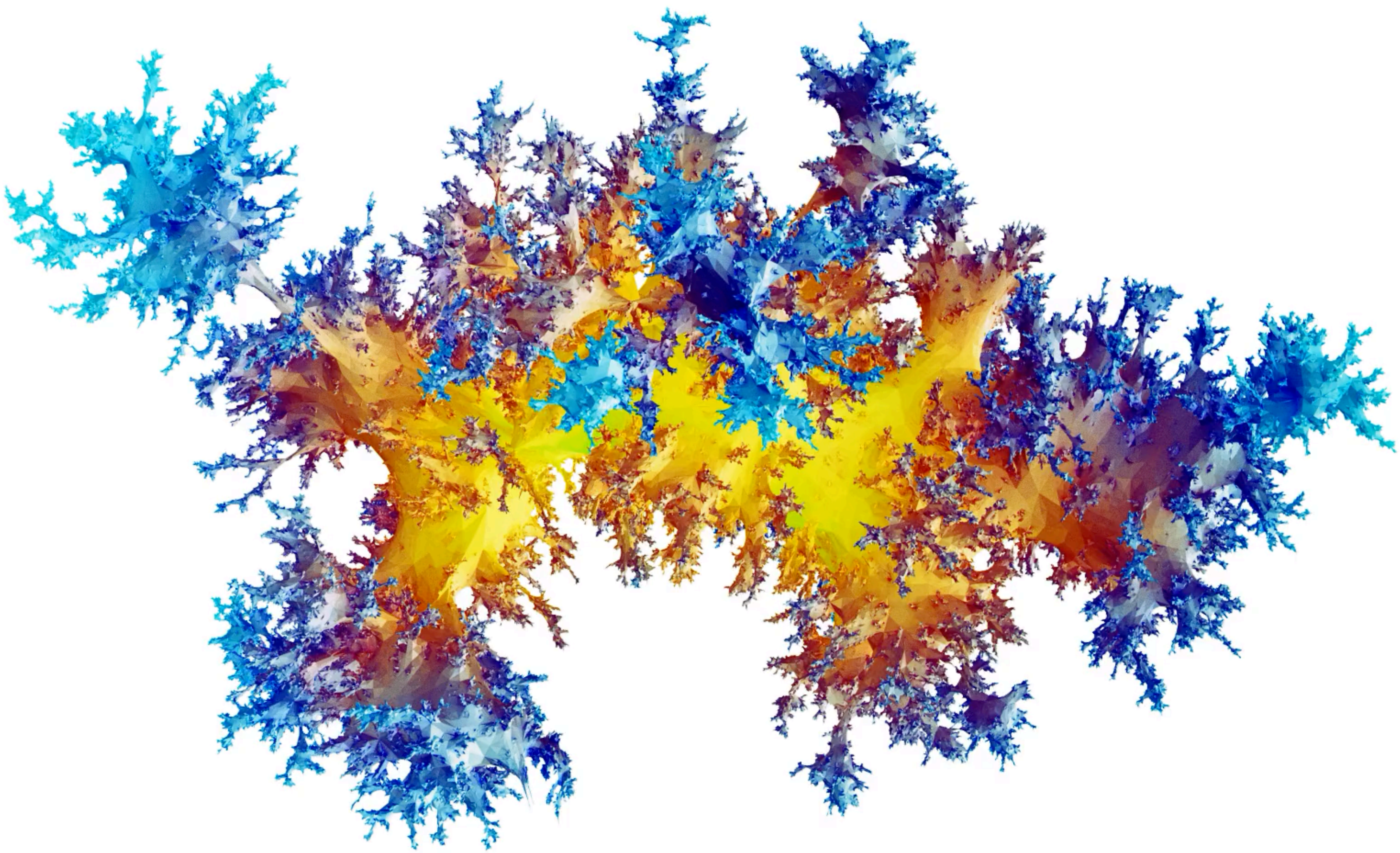
DILUTE REGIME

- Consider the two-parameter Poisson-Dirichlet process PD(α,θ) introduced by Pitman, Yor (1997)
- Handa (2009) determined its correlation functions. For $\theta = -\alpha\beta$ they agree with those of $\Upsilon_{\alpha,\beta}$.
- Using the method of moments a lemma from Kallenberg's book *Random measures* it follows that

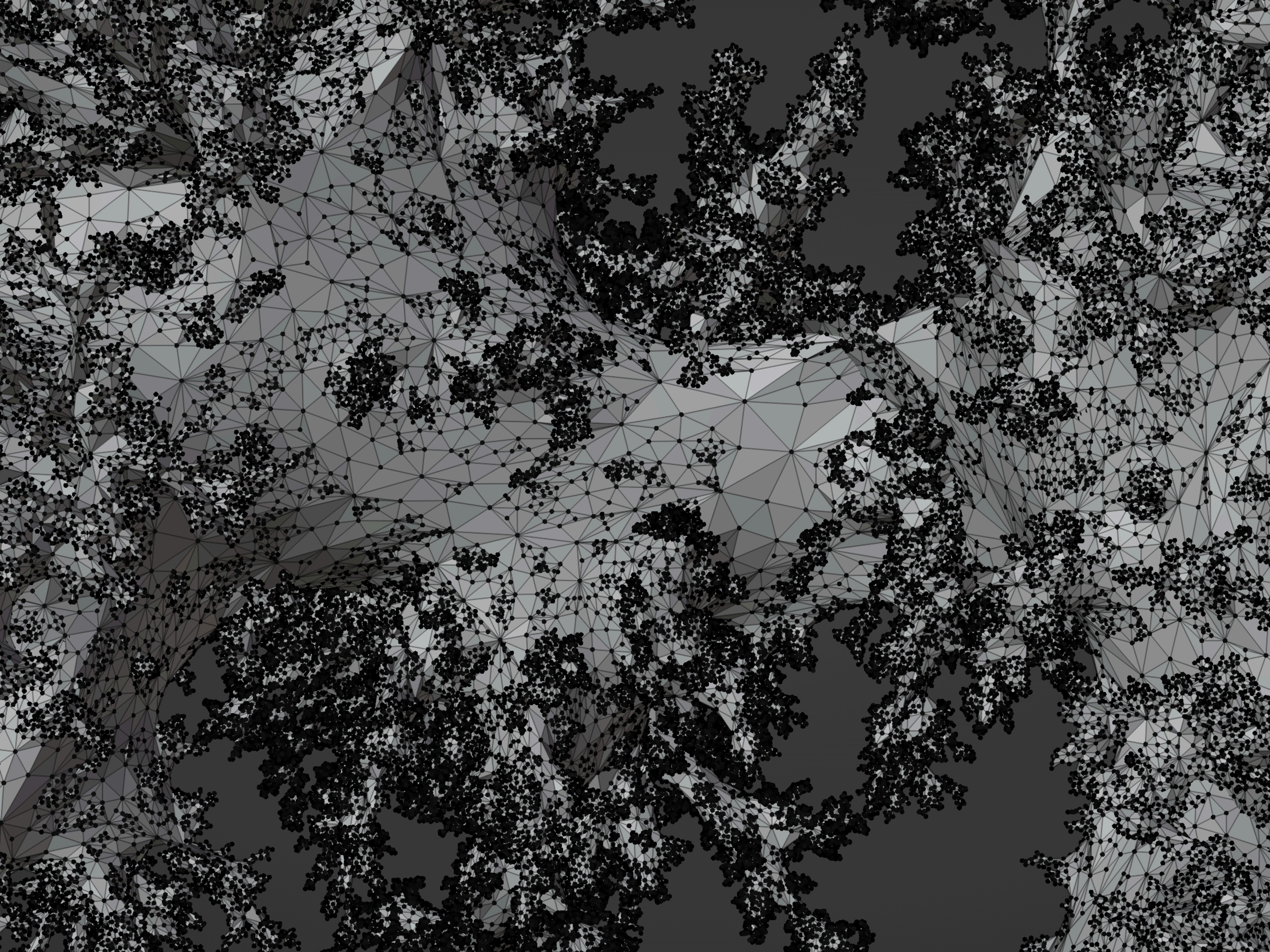
$$\Upsilon(\alpha, \beta) \stackrel{d}{=} \mathrm{PD}(\alpha, -\alpha\beta)$$

Le Gall (2013), Miermont (2013), ...

BROWNIAN SPHERE: UNIVERSAL LIMIT OF RANDOM PLANAR MAPS



Simulation: SIMTRIA (Generate SIMple TRIAngulations): http://github.com/BenediktStufler/simtria, SCENT (Calculate closeness centrality): http://github.com/BenediktStufler/scent
Mathematica, Blender



ASYMPTOTIC ENUMERATION

- **Bodirsky, Kang, Löffler, McDiarmid (2007), Noy, Requilé, Rué (2020)**: the number c_n of cubic planar graphs with n labelled vertices satisfies
$$c_n \sim c_{\text{cubic}} n^{-7/2} \rho_{\text{cubic}}^{-n} n! \quad \text{as} \quad n \to \infty$$
for some constants $c, \rho > 0$.

- **Giménez, Noy (2009)**: the number p_n of unrestricted planar graphs with n labelled vertices satisfies
$$p_n \sim c n^{-7/2} \rho^{-n} n! \quad \text{as} \quad n \to \infty$$
for some constants $c, \rho > 0$. Proof uses analytic integration and enumerative results for the 2-connected case by **Bender, Gao, Wormald (2002)**

- **Chapuy, Fusy, Kang, Shoilekova (2008):** "combinatorial integration": purely combinatorial approach to recover analytic specification by Giménez and Noy

- **S. (2023)**: recover $p_n \sim c n^{-7/2} \rho^{-n} n!$ without integration, approach based on large deviation results for random walks in the big-jump domain

WHAT ABOUT THE GLOBAL SHAPE?

- **Chapuy, Fusy, Giménez, Noy (2010):** The uniform random planar graph with n labelled vertices has diameter $n^{1/4+o_p(1)}$
- **Problem: Methods for establishing limits for planar maps break down for graphs. No known bijection to trees with label processes!**
- Asymptotic enumeration factor $n^{-7/2}$ provides a hint but has no probabilistic implications.

SCALING LIMIT OF CUBIC PLANAR GRAPHS

Thm. (S., 2022+) Let C_n denote the uniform connected cubic planar graph with n labelled vertices. Let d_{C_n} denote the graph distance and μ_{C_n} the uniform measure on its vertex set. There is a constant $\gamma > 0$ such that in distribution

$$(C_n, \gamma n^{-1/4} d_{C_n}, \mu_{C_n}) \to (M, d_M, \mu_M)$$

as $n \in 2\mathbb{N}$ tends to infinity. Convergence is in distribution in the Gromov-Hausdorff-Prokhorov sense, with (M, d_M, μ_M) denoting the Brownian sphere.

Independent parallel proof: Albenque, Fusy, Lehericy (2022+)

Unrestricted planar graphs: ongoing joint work with Addario-Berry, Albenque, and Fusy

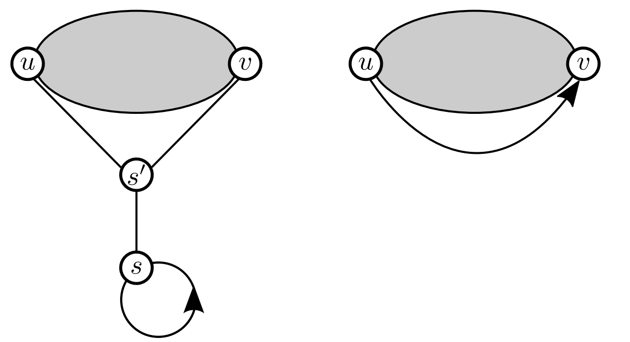
NETWORK DECOMPOSITION

Cubic network: oriented root edge that is allowed to be a loop or double edge

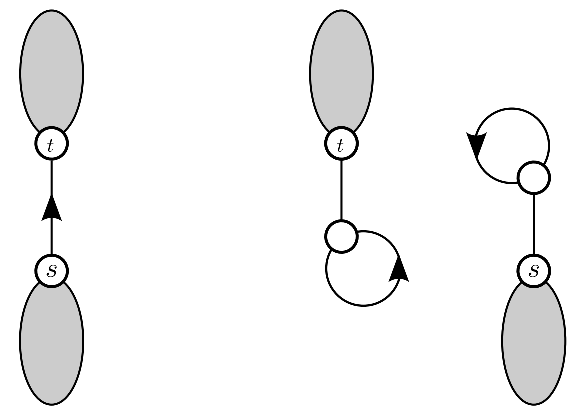
Asymptotic enumeration of cubic planar graphs via cubic network decomposition:

Bodirsky, Kang, Löffler, McDiarmid (2006)
Noy, Requilé, Rué (2019)

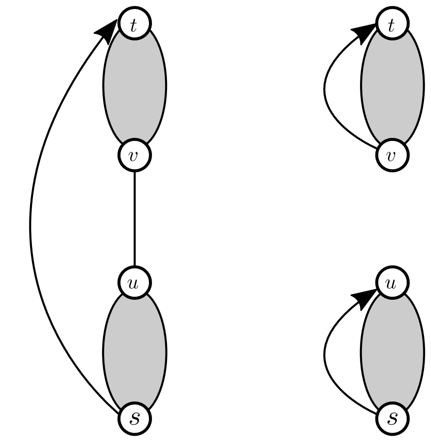
NETWORK DECOMPOSITION



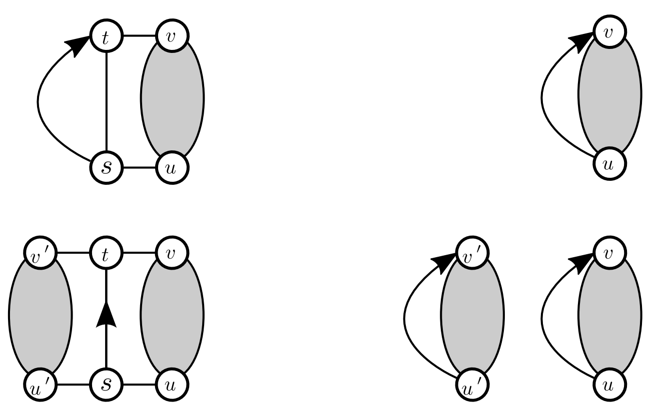
Loop Networks



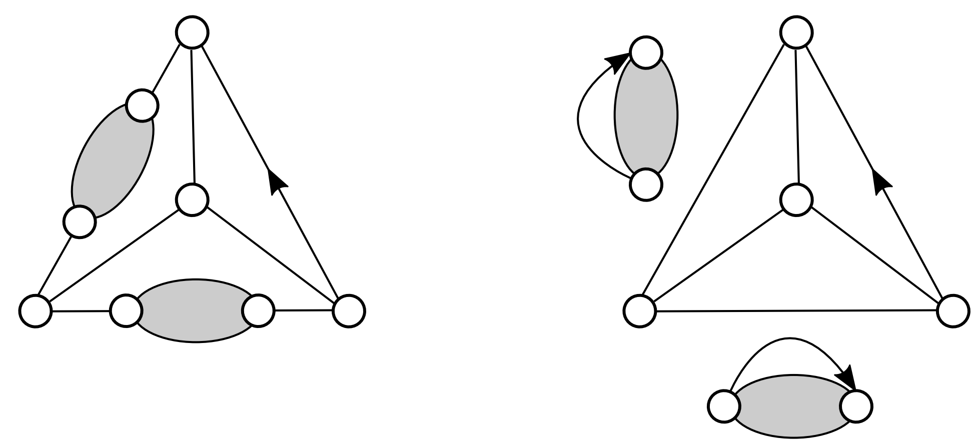
Isthmus Networks



Series Networks



Parallel Networks



Polyhedral Networks

PROOF STRATEGY

1. Study first-passage percolation (FPP) on random simple triangulations. Their duals correspond to 3-connected cubic planar graphs.

2. Relate FPP-distances to subspace distances on the 3-connected core $\mathscr{M}(\mathsf{C}_n)$. Hardest part of the proof. Requires new technique!

3. **Approximate C_n by its 3-connected core $\mathscr{M}(\mathsf{C}_n)$.**

PROOF STRATEGY: PART III

THEOREM — *Let V_n denote the number of vertices in the largest 3-connected component of the uniform random n-vertex cubic planar graph C_n. Let*

$$h(t) = \frac{1}{\pi t} \sum_{n \geq 1} (-t 3^{2/3})^n \frac{\Gamma(2n/3+1)}{n!} \sin(-2n\pi/3), \qquad t \in \mathbb{R}$$

denote the density of the map type Airy distribution. There are algebraic constants

$$\kappa = 0.850853090058314333870385348879612617197477\ldots$$

$$c_v = 1.205660773457703954344217302817493214574105\ldots$$

such that for any constant $M > 0$

$$\mathbb{P}(V_n = \kappa n + t n^{2/3}) = n^{-2/3}(2 c_v h(c_v t) + o(1)) \tag{1.3}$$

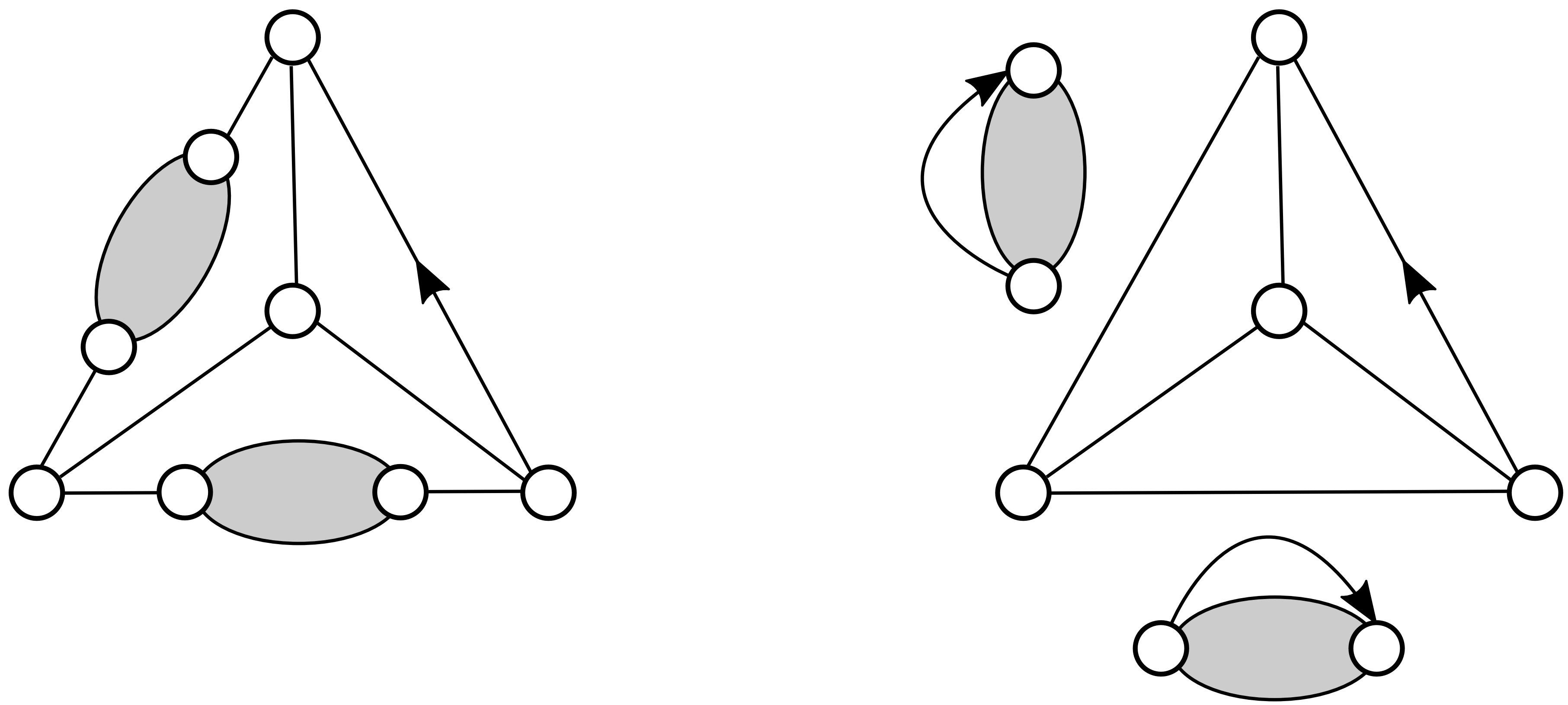
uniformly for all $t \in [-M, M]$ satisfying $\kappa n + t n^{2/3} \in 2\mathbb{N}$. Consequently,

$$\frac{V_n - \kappa n}{n^{2/3}} \xrightarrow{d} V_{3/2} \tag{1.4}$$

for a 3/2-stable random variable $V_{3/2}$ with density $c_v h(c_v t)$.

Related parallel work: Drmota, Noy, Rué (2022+) on cubic planar maps. Warm thanks for comments.

PROOF STRATEGY: PART III



C_n *roughly* corresponds to 3-connected core $\mathscr{M}(\mathsf{C}_n)$ and components $(\mathscr{D}_i(\mathsf{C}_n))_{1 \le i \le 3V_n/2}$ in breadt-first-search order

PROOF STRATEGY: PART III

LEMMA — *The numbers* $|\mathcal{D}_i(\mathsf{C}_n)|$, $1 \leq i \leq 3V_n/2$, *of vertices in the network components attached to* $\mathcal{M}(\mathsf{C}_n)$ *satisfy*

$$\max_{1 \leq i \leq 3V_n/2} |\mathcal{D}_i(\mathsf{C}_n)| = O_p(n^{2/3}).$$

LEMMA — *In a random network* D_n *of size* n, *for any* $\epsilon > 0$ *there exist constants* $C, c, \delta > 0$ *such that*

$$\mathbb{P}(\mathrm{D}(\mathsf{D}_n) \geq n^{1/4+\epsilon}) \leq C \exp(-cn^{\delta})$$

for all even integers $n \geq 4$.

LEMMA — *For any* $\epsilon > 0$ *it holds that*

$$\max_{1 \leq i \leq 3V_n/2} \mathrm{Diam}(\mathcal{D}_i(\mathsf{C}_n)) = o_p(n^{1/6+\epsilon}).$$

This shows that the subspace metric $\mathscr{M}(\mathsf{C}_n)$ approximates C_n in the Gromov-Hausdorff sense with $o_p(n^{1/4})$ fluctuations.

Prokhorov distance: linear scaling of mass measure due to exchangeability of components. Argument analogous to Addario-Berry and Wen (2017)

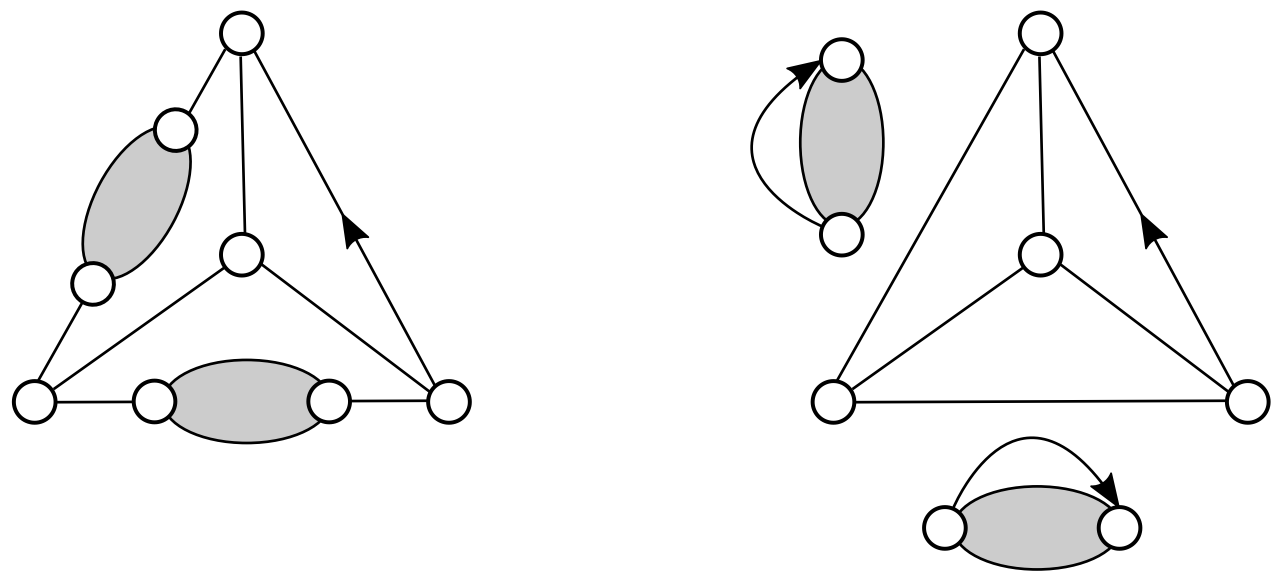
PROOF STRATEGY

1. Study first-passage percolation (FPP) on random simple triangulations. Their duals correspond to 3-connected cubic planar graphs.

2. **Relate FPP-distances to subspace distances on the 3-connected core $\mathscr{M}(\mathsf{C}_n)$. Hardest part of the proof. Requires new technique!**

3. Approximate C_n by its 3-connected core $\mathscr{M}(\mathsf{C}_n)$.

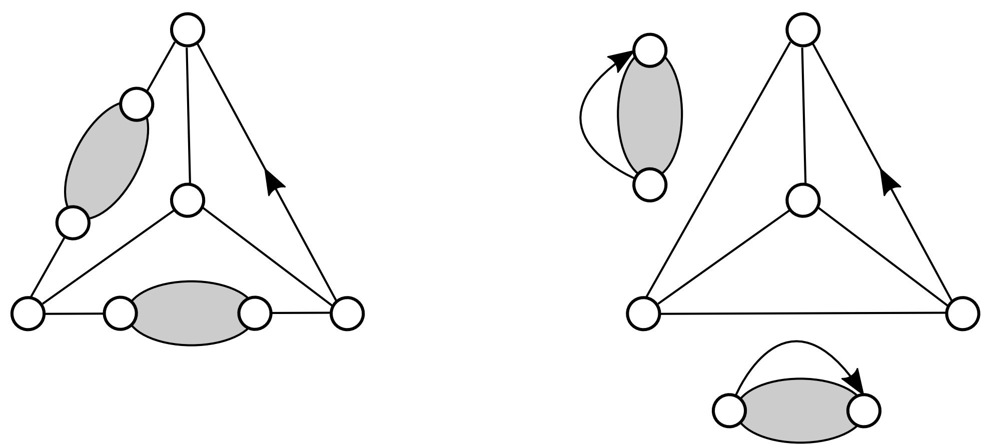
PROOF STRATEGY: PART II



3 DIFFERENT METRICS ON 3-CONNECTED CORE $\mathscr{M}(\mathsf{C}_n)$:

1. Graph distance.
2. First-passage percolation distance
3. Subspace distance $\mathscr{M}(\mathsf{C}_n) \subset \mathsf{C}_n$

PROOF STRATEGY: PART II



C_n corresponds to 3-connected core $\mathscr{M}(\mathsf{C}_n)$ and components $(\mathscr{D}_i(\mathsf{C}_n))_{1 \le i \le 3V_n/2}$ in breadt-first-search order

Let D denote the Boltzmann network model for the class $\mathscr{D} + 1$ of non-isthmus networks + a single edge

Lemma — *Let* $(\mathsf{D}(i))_{i \ge 1}$ *denote independent copies of the Boltzmann network* D. *For any* $\epsilon > 0$ *and* $0 < \delta < 3\kappa/2$ *there exist constants* $0 < c < C$ *and* $N > 0$ *and sets* $(\mathcal{E}_n)_{n \ge N}$ *such that for all* $n \in 2\mathbb{N}$ *with* $n \ge N$

$$\mathbb{P}\left((\mathcal{M}(\mathsf{C}_n), (\mathcal{D}_i(\mathsf{C}_n))_{\delta n \le i \le 3V_n/2} \notin \mathcal{E}_n\right) < \epsilon$$

and

$$\mathbb{P}\left((\mathcal{M}(\mathsf{C}_n), (\mathsf{D}(i))_{\delta n \le i \le 3V_n/2}) \notin \mathcal{E}_n\right) < \epsilon$$

and for all elements $E \in \mathcal{E}_n$

$$c < \frac{\mathbb{P}\left((\mathcal{M}(\mathsf{C}_n), (\mathcal{D}_i(\mathsf{C}_n))_{\delta n \le i \le 3V_n/2}) = E\right)}{\mathbb{P}\left((\mathcal{M}(\mathsf{C}_n), (\mathsf{D}(i))_{\delta n \le i \le 3V_n/2}) = E\right)} < C.$$

PROOF STRATEGY: PART II

For any $\epsilon > 0$ and any element $(X, d_X, \nu_X) \in \mathfrak{K}$ we define

$$\rho_\epsilon^X := \inf_{x \in X} \nu_X(C_\epsilon^X(x)) \qquad \lambda_\epsilon^X := \sup_{x \in X} \nu_X(B_\epsilon^X(x))$$

$$C_\epsilon^X(x) = \{y \in X \mid d_X(x,y) \leq \epsilon\} \qquad B_\epsilon^X(x) = \{y \in X \mid d_X(x,y) < \epsilon\}$$

COROLLARY — *Let* $(\mathsf{X}, d_{\mathsf{X}}, \nu_{\mathsf{X}})$ *and* $(\mathsf{X}_n, d_{\mathsf{X}_n}, \nu_{\mathsf{X}_n})_{n \geq 1}$ *be random elements of* $\mathfrak{K}$ *satisfying*

$$(\mathsf{X}_n, d_{\mathsf{X}_n}, \nu_{\mathsf{X}_n}) \xrightarrow{d} (\mathsf{X}, d_{\mathsf{X}}, \nu_{\mathsf{X}}).$$

Then the following statements are equivalent:

1. *Almost surely there is no* $x \in \mathsf{X}$ *with* $\nu_{\mathsf{X}}(\{x\}) > 0$.
2. *For all* $\epsilon, \epsilon' > 0$ *there exist* $\delta > 0$ *and* $N > 0$ *such that for all* $n \geq N$: $\mathbb{P}(\lambda_\delta^{\mathsf{X}_n} > \epsilon) < \epsilon'$.

Then the following statements are equivalent:

1. ν_{X} *has almost surely full support.*
2. *For all* $\epsilon, \epsilon' > 0$ *there are* $\delta, N > 0$ *such that for all* $n \geq N$: $\mathbb{P}(\rho_\epsilon^{\mathsf{X}_n} < \delta) < \epsilon'$.

In other words: if the limit is almost surely diffuse, then for large n each open ball in X_n with small radius is likely to have small mass. Conversely, if the limit is almost surely fully supported, then each closed ball in X_n with small mass is likely to have small radius.

PROOF STRATEGY: PART II

LEMMA — *Let* $\Omega_\delta \subset \mathcal{M}(\mathsf{C}_n)$ *denote the subset of vertices consisting of the endpoints of the first* $\lfloor \delta n \rfloor$ *edges. For all* $\epsilon > 0$ *we may select* $\delta > 0$ *small enough, such that*

$$\mathbb{P}(\mathrm{Diam}(\Omega_\delta) \leq \epsilon n^{1/4}) \geq 1 - \epsilon$$

for all sufficiently large $n \in 2\mathbb{N}$.

Using contiguity + time-reversal argument and Part 1 allows us take δ even smaller so that the diameter is still small with respect to the subgraph and fpp metrics.

LEMMA — *There exists a constant* $c_{\mathrm{fpp}} > 0$ *with*

$$\sup_{x,y \in \mathcal{M}(\mathsf{C}_n)} |d_{\mathsf{C}_n}(x,y) - c_{\mathrm{fpp}} d_{\mathcal{M}(\mathsf{C}_n)}(x,y)| = o_p(n^{1/4}).$$

PROOF STRATEGY

1. **Study first-passage percolation (FPP) on random simple triangulations. Their duals correspond to 3-connected cubic planar graphs.**
2. Relate FPP-distances to subspace distances on the 3-connected core $\mathscr{M}(\mathsf{C}_n)$. Hardest part of the proof. Requires new technique!
3. Approximate C_n by its 3-connected core $\mathscr{M}(\mathsf{C}_n)$.

PROOF STRATEGY: PART 1

Theorem — *Let $\mathcal{T}_n$ denote the uniform simple triangulation with $n+1$ vertices. Suppose that ι has finite exponential moments and that there exists a constant $\kappa > 0$ such that $\mathbb{P}(\iota \geq \kappa) = 1$. Then there exists a constant $c^{\mathcal{T}}_{\text{fpp}} > 0$ such that*

$$n^{-1/4} \sup_{x,y \in \mathrm{V}(\mathcal{T}_n)} \left| d^{\mathcal{T}_n}_{\text{fpp}}(x,y) - c^{\mathcal{T}}_{\text{fpp}} d_{\mathcal{T}_n}(x,y) \right| \xrightarrow{p} 0$$

as n tends to infinity.

Theorem — *Let $\mathscr{C}_n$ denote a uniformly selected 3-connected cubic planar graph with n labelled vertices. Suppose that ι has finite exponential moments and that there exists a constant $\kappa > 0$ such that $\mathbb{P}(\iota \geq \kappa) = 1$. Then there exists a constant $c^{\mathscr{C}}_{\text{fpp}} > 0$ such that*

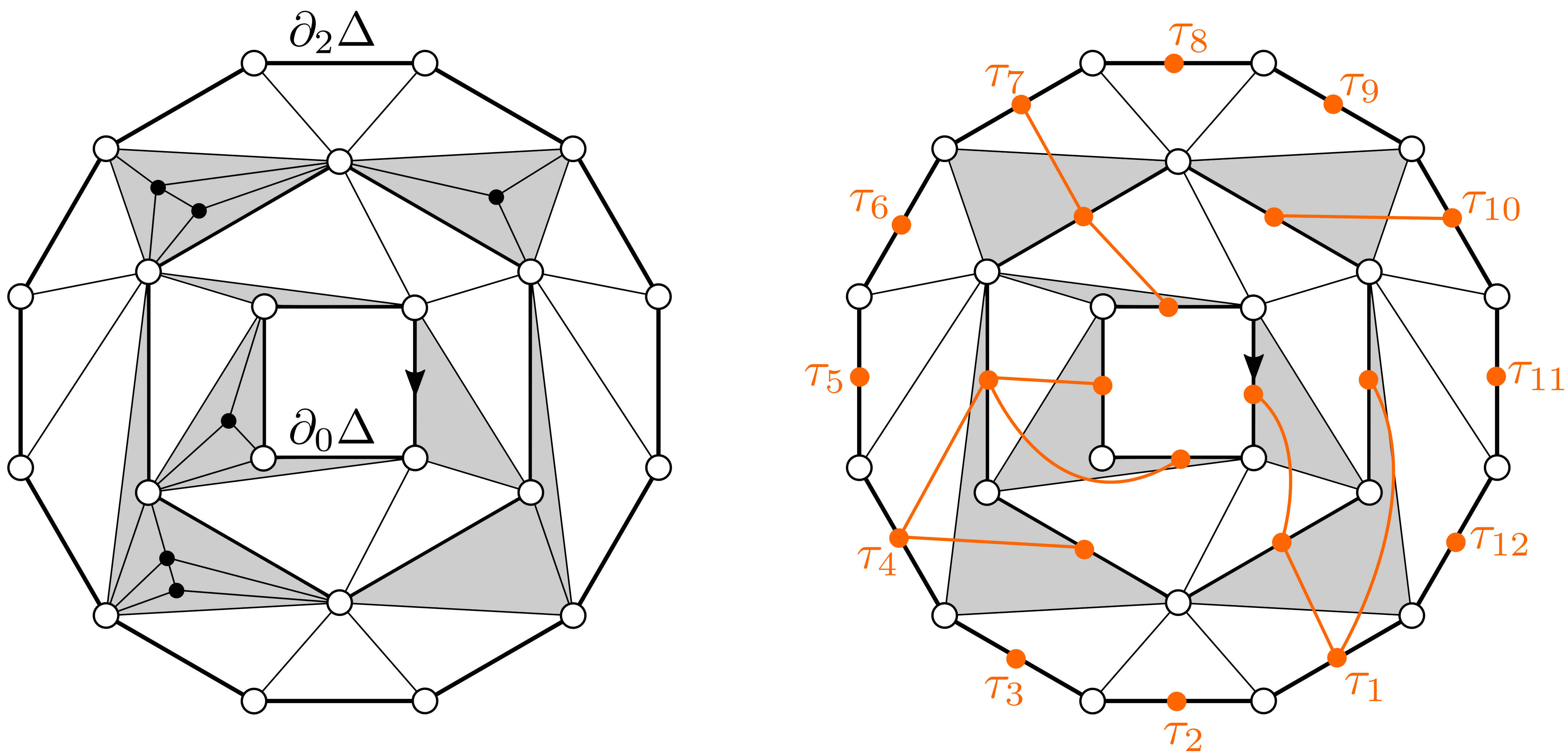
$$n^{-1/4} \sup_{x,y \in \mathrm{V}(\mathscr{C}(n))} \left| d^{\mathscr{C}_n}_{\text{fpp}}(x,y) - c^{\mathscr{C}}_{\text{fpp}} d_{\mathscr{C}_n}(x,y) \right| \xrightarrow{p} 0$$

as $n \in 2\mathbb{N}$ tends to infinity.

Methods: Angel and Schramm (2003); Krikun (2005)
Curien and Le Gall (2017)

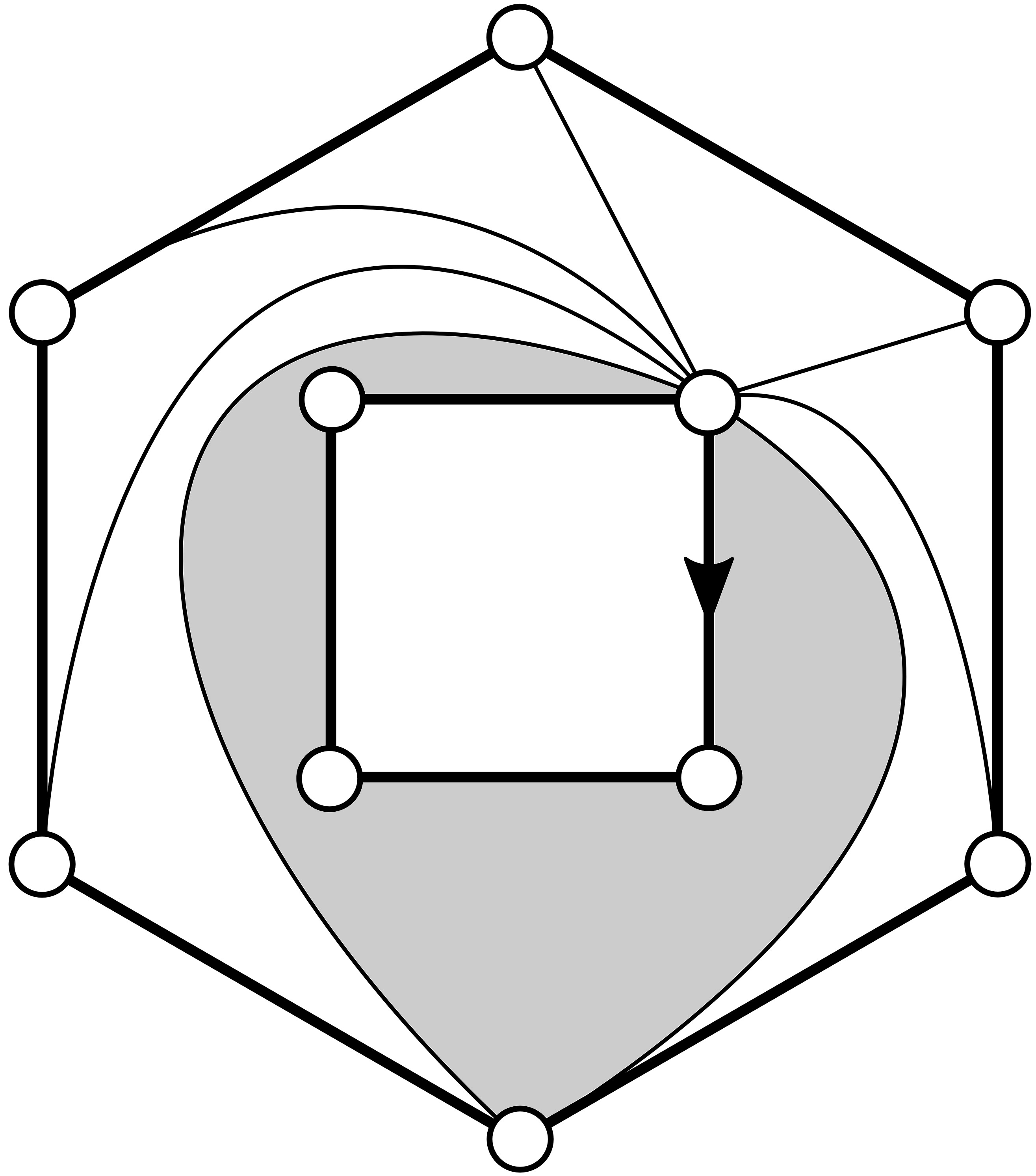
Scaling limit of simple triangulations:
Addario-Berry and Albenque (2017)

PROOF STRATEGY: PART 1

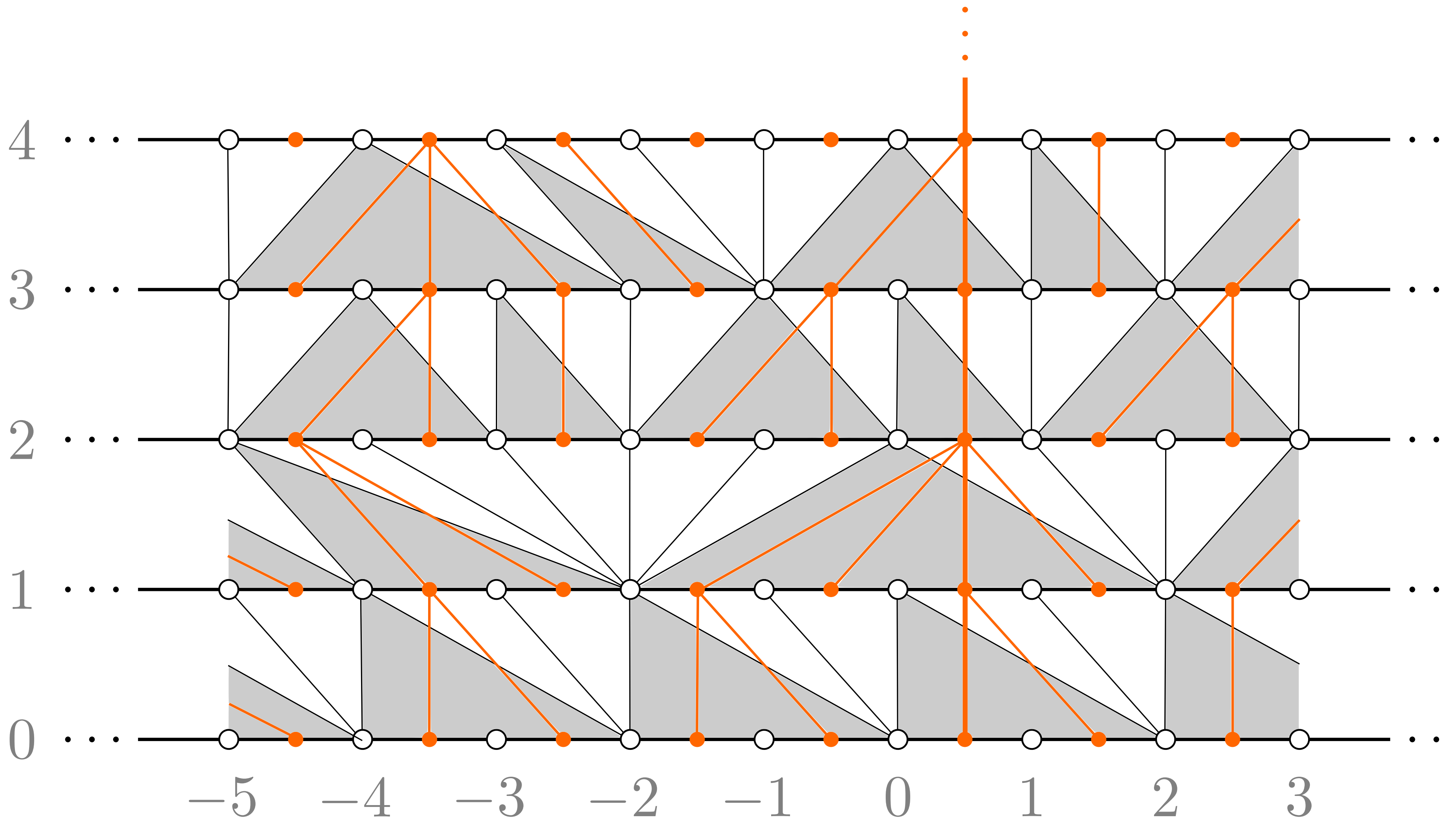


Skeleton Decomposition

PROOF STRATEGY: PART 1



PROOF STRATEGY: PART I



- UIPT of p-gon: $\mathscr{T}_\infty^{(p)} = \lim_{n\to\infty} \mathscr{T}_n^{(p)}$
- UHPT: $\mathscr{U} = \lim_{p\to\infty} \mathscr{T}_\infty^{(p)}$
- Use Liggett's version of Kingman's subadditive ergodic theorem

SCALING LIMIT OF CUBIC PLANAR GRAPHS

And the proof is complete!

Thm. (S., 2022+) Let C_n denote the uniform connected cubic planar graph with n labelled vertices. Let d_{C_n} denote the graph distance and μ_{C_n} the uniform measure on its vertex set. There is a constant $\gamma > 0$ such that in distribution

$$(C_n, \gamma n^{-1/4} d_{C_n}, \mu_{C_n}) \to (M, d_M, \mu_M)$$

as $n \in 2\mathbb{N}$ tends to infinity. Convergence is in distribution in the Gromov-Hausdorff-Prokhorov sense, with (M, d_M, μ_M) denoting the Brownian sphere.

Thanks for your attention.