A CLASS OF OPTIMAL ROW-COLUMN DESIGNS
WITH SOME EMPTY CELLS

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Abstract: Agrawal (1966) constructed a series of row column designs with row-
column incidence structure $J - I$. We show that there exist designs which strongly
dominate Agrawal’s designs for all $v \geq 4$ and are therefore to be preferred with
respect to any optimality criterion. These dominating designs are E-optimal within
the entire class of such row-column designs and are also highly A- and D-efficient.
Some methods for constructing such designs are also developed.

Key words and phrases: Structurally incomplete row-column design, row-column

1. Introduction

The study of optimality of structurally incomplete row-column designs where
treatments are allocated to some but not all of the combinations of rows and
columns has recently drawn considerable attention — see for example, Saharay
(1986), Shah and Sinha (1990), Stewart and Bradley (1991), Heiligers and Sinha
(1993). Shah and Sinha (1990) and Heiligers and Sinha (1993) investigated optimi-
ality aspects of the four types of structurally incomplete row-column designs,
constructed by Agrawal (1966), for which the row-column incidence matrix is
that of a balanced incomplete block design (BIBD). In most cases Agrawal de-
dsigns admit a completely symmetric (c.s.) C-matrix for each classification (i.e.
treatment, row or column). In view of the very strong optimality of BIBD’s one
would expect them to be optimal. Interestingly enough, Shah and Sinha (1990)
and Heiligers and Sinha (1993) came up with better designs for some values of $v$.

For the specific $v \times v$ row column set up with empty cells throughout the prin-
cipal diagonal, used for comparing $v$ treatments, Shah and Sinha (1990) obtained
better designs for $v = 7$ with respect to (w.r.t) the D-optimality criterion and
conjectured that Agrawal designs would be A-optimal for all $v$. In the present ar-
ticle we restrict our attention to this sort of structurally incomplete row-column
set up and examine the prospect of A- and E-optimality of Agrawal designs. It
is shown in Section 3 that there exist designs which strongly dominate Agrawal
designs for all $v \geq 4$ and are therefore to be preferred to the latter designs
w.r.t any meaningful optimality criterion. Finally, these dominating designs are
proven to be E-optimal for \( v \geq 4 \) within the entire class and A- and D-optimal for \( 4 \leq v \leq 14 \) within the binary and equireplicate class of connected row-column designs of the above structure. For larger values of \( v \) these designs are highly A- and D-efficient. The methods of constructing these designs are discussed in Section 4.

2. Preliminaries

Let \( d \) denote a design used for comparing \( v \) treatments applied to experimental units arranged in \( v \) rows and \( v \) columns such that in each row and in each column there is exactly one empty cell. Without loss of generality (w.l.o.g), the rows and the columns of the experimental set up can be rearranged so that the row-column incidence structure assumes the form \( J – I \). Throughout the paper we assume the usual fixed effects additive model with uncorrelated and homoscedastic errors. For a design \( d \), let \( L_d = ((l_{dhj})) \), \( M_d = ((m_{dhj})) \) and \( N_d = J – I \) stand, respectively, for treatment-row, treatment-column and row-column incidence matrices. The C-matrix for treatment effects of such a design \( d \) can be written as (cf. Saharay (1986))

\[
C_d = D_{r_d} - (L_dL_d')/v - (M_dM_d')/v - ((L_d + M_d)(L_d + M_d)')/v(v - 2) + (r_d r_d')/(v - 1)(v - 2),
\]

where \( D_{r_d} = \text{diag}(r_{d1}, \ldots, r_{dv}) \), \( r_d = (r_{d1}, \ldots, r_{dv}) \) and \( r_{di} \) is replication of treatment \( i \).

Let \( \Omega \) denote the class of all connected \( v \times v \) (\( v \geq 4 \)) row-column designs described above and \( V \) denote the set of \( v \) treatments. Any design \( d \in \Omega \) is said to be binary if both \( L_d \) and \( M_d \) are \((0,1)\)-matrices. Let \( 0 = \lambda_{d0} < \lambda_{d1} \leq \lambda_{d2} \leq \cdots \leq \lambda_{dv-1} \) denote the eigenvalues of the \( C_d \) matrix. A design \( d^* \) is said to be A-, D- and E-optimal in a relevant class \( \Omega_0 \) if it minimizes \( \sum_{i=1}^{v-1} \lambda_{di}^{-1}, \prod_{i=1}^{v-1} \lambda_{di}^{-1} \) and \( \lambda_{d1}^{-1} \) respectively among the designs in \( \Omega_0 \). The A- and D-efficiency of \( d \in \Omega_0 \) is defined to be \( \sum \lambda_{di}^{-1}/\sum \lambda_{di}^{-1} \) and \( \left\{ \prod \lambda_{di}^{-1} / \prod \lambda_{di}^{-1} \right\}^{1/(v-1)} \) respectively. A design \( d_1 \in \Omega \) is said to strongly dominate another design \( d_2 \) if \( C_{d_1} - C_{d_2} \) is n.n.d.. We refer to Shah and Sinha (1989) for a detailed discussion on optimality criteria.

Let \( P \) be a permutation matrix of order \( v \) which can be written in the form \( P = [e_{i1} : e_{i2} : \ldots : e_{iv}] \) where \( e_i \) is a \( v \times 1 \) vector with 1 at the \( i \)th position and 0's at other positions. Then there exists a permutation \( \pi(P) \) of \( \{1, \ldots, v\} \) taking \( j \) to \( i_j, 1 \leq j \leq v \). Whenever there is exactly one cycle in \( \pi(P) \), we call \( P \) a cyclic permutation matrix. Using the properties of \( P \) and \( \pi(P) \) (see e.g. Birkhoff and MacLane (1977) and Hohn (1957)) the following theorems, helpful in the sequel, are immediate.

Theorem 2.1. The eigenvalues of a cyclic permutation matrix \( P \) of order \( v \) are given by

\[
\mu_j = \cos 2\pi j/v + i \sin 2\pi j/v, \quad 0 \leq j \leq v - 1.
\]
Theorem 2.2. Let $P$ be a permutation matrix of order $v$ and $\pi(P)$ be a product of $k$ disjoint cycles of $s$ distinct lengths, $n_t$ being the multiplicity of cycle length $l_t, 1 \leq t \leq s, \sum_{i=1}^{s} n_i = k$ and $\sum_{i=1}^{s} n_i l_i = v$. Then the eigenvalues of $P$ are given by

$$\mu_{tj} = \cos 2\pi j/l_t + i \sin 2\pi j/l_t, 1 \leq t \leq s, 0 \leq j \leq l_t - 1,$$

with multiplicity $n_t$.

Lemma 2.3. Let $P$ be a permutation matrix of order $v$. Then $2I - (P + P')$ is n.n.d.

Proof. The result follows immediately using the Cauchy Schwartz inequality for all $x \in \mathbb{R}^v$,

$$x'(P + P')x = x'Px + x'P'x \leq \|x\| \|Px\| + \|x\| \|P'x\| = 2x'x.$$

3. Designs Dominating Agrawal Designs

Agrawal (1966) constructed designs for $v \geq 4$, (henceforth denoted by $d_A$) based on his method 3 with $N_{d_A} = L_{d_A} = M_{d_A} = J - I$, which yields

$$C_{d_A} = (v(v - 3)/(v - 2))(I - J/v). \quad (3.1)$$

A simple way to construct $d_A$ is to start with a Latin Square design of order $v$ with diagonal elements all different and then to delete the diagonal.

Let $d^* \in \Omega$ be a design with

$$L_{d^*} = J - I = L_{d_A} \text{ and } M_{d^*} = J - P, \quad (3.2)$$

where $P$ is a cyclic permutation matrix of order $v$ different from the identity matrix. From (2.1) and (3.1),

$$C_{d^*} = C_{d_A} + (1/v(v - 2))(2I - (P + P')). \quad (3.3)$$

Using Lemma 2.3, it follows that $C_{d^*} - C_{d_A}$ is n.n.d. and nonzero. This suggests that $d^*$ is to be preferred to $d_A$ with respect to any meaningful optimality criterion. In the following, we establish that $d^*$ is E-optimal and highly A- and D-efficient. The various bounds that have been used in this context are listed below:

$$\lambda_{d^*} \leq (v/(v - 1)) \min_{h \in V} C_{dh}. \quad (3.4)$$

Lemma 3.1. For given positive integers $p$ and $t$, the minimum of $\sum_{i=1}^{p} x_i^2$ subject to $\sum_{i=1}^{p} x_i = t$, where the $x_i$'s are nonnegative integers, is obtained when $t - p[t/p]$ of the $x_i$'s are equal to $[t/p] + 1$, and $p - t + p[t/p]$ of the $x_i$'s are equal to $[t/p]$. ($[t/p]$ is the greatest integer $\leq t/p$).
Lemma 3.2. For given positive integer \( p \geq 3 \), the minimum value of \( \sum_{i=1}^{p} x_i^2 \) subject to \( \sum_{i=1}^{p} x_i = p-1 \), where the \( x_i \)'s are nonnegative integers and \( x_i \geq 2 \) for at least one \( i \), is equal to \( p + 1 \). It is attained when two of the \( x_i \)'s are 0, \( p - 3 \) of the \( x_i \)'s are 1 and one \( x_i \) is 2.

Proof. For any feasible \( x = (x_1, \ldots, x_p) \)

\[
\sum_{i=1}^{p} x_i^2 \geq 2 + (p-1) \geq p + 1
\]

and \( \sum_{i=1}^{p} x_i^2 = p + 1 \) for the particular \( x_i \)'s described in the lemma.

Theorem 3.3. Suppose that for given \( v, d^* \) defined by (3.2) exists. Then \( d^* \) is \( \text{E-optimal} \) in \( \Omega \).

Proof. Let us partition the collection of designs of \( \Omega \) in the following way:

\[
\Omega_1 = \{d : d \in \Omega, d \text{ is equireplicate and binary}\},
\Omega_2 = \{d : d \in \Omega, d \text{ is equireplicate and nonbinary}\},
\Omega_3 = \{d : d \in \Omega, d \text{ is nonequireplicate}\}.
\]

We organize our proof in three steps.

Step 1. Let \( d \) be a design in \( \Omega_1 \). Since there are exactly \( v - 1 \) feasible cells in each row and in each column and \( r_{di} = \bar{r} = v - 1 \) for all \( i \), note that

\[
L_d = J - P_{1d} \quad \text{and} \quad M_d = J - P_{2d}, \tag{3.5}
\]

where \( P_{1d} \) and \( P_{2d} \) are permutation matrices of order \( v \). Using (2.1) and Lemma 2.3 it can be easily verified that

\[
C_d = (v(v-3)/(v-2))(I - J/v) + (1/v(v-2))(2I - (P_{1d}P'_{2d} + P_{2d}P'_{1d}))
= C_{d_A} + A_d \geq C_{d_A},
\]

where \( A_d = (1/v(v-2))(2I - (P_{1d}P'_{2d} + P_{2d}P'_{1d})) \) and \( P_{1d}P'_{2d} = Q_d \) (say) is again a permutation matrix. Let the notations used in the context of Theorem 2.2 hold for \( Q_d \). Then the eigenvalues of \( A_d \) are

\[
\theta_{ij} = (2/v(v-2))(1 - \cos 2\pi j/l_i), \quad 0 \leq j \leq l_i - 1, \quad 1 \leq i \leq s,
\]

with multiplicity \( n_i \). Note that \( \theta_{i0} = 0 \), \( 1 \leq i \leq s \) and \( \sum_{i=1}^{s} n_i = k \). Since \( C_{d_A} \) and \( A_d \) commute, these two matrices can be simultaneously diagonalised and hence the nonzero eigenvalues of \( C_d \) are obtained by adding \( v(v-3)/(v-2) \) to
the \(\theta_{ij}\)'s, \((i,j) \neq (1,0)\) (assuming w.l.o.g that the eigenvectors corresponding to the eigenvalue 0 of \(C_{dA}\) and the eigenvalue \(\theta_{10} (= 0)\) of \(A_d\) are the same). Thus,

\[
\lambda_{d1} = \begin{cases} 
v(v-3)/(v-2) = \lambda_{dA1}, & \text{for } k \geq 2, \\
v(v-3)/(v-2) + (2/v(v-2))(1 - \cos 2\pi/v), & \text{for } k = 1. 
\end{cases}
\]

(3.6)

Note that \(d^*\) corresponds to \(k = 1\). Thus \(d^*\) is E-best in \(\Omega_1\).

**Step 2.** Suppose \(d\) is a design in \(\Omega_2\). Relying on (2.1), we obtain

\[
C_{dh} = \bar{r} - \sum_{j=1}^{v} l_{d}^2_{hj}/v - \sum_{j=1}^{v} m_{d}^2_{hj}/v - \sum_{j=1}^{v} (l_{d}hj + m_{d}hj)^2/v(v-2) + \bar{r}^2/(v-1)(v-2).
\]

(3.7)

Since \(d\) is nonbinary, using Lemmas 3.1 and 3.2, we get for some treatment \(h_0 \in V\),

\[
\sum_{j=1}^{v} l_{d}^2_{h0j} + \sum_{j=1}^{v} m_{d}^2_{h0j} \geq (v-1) + (v+1) = 2v
\]

and

\[
\sum_{j=1}^{v} (l_{d}h0j + m_{d}h0j)^2 \geq 4v - 6.
\]

Applying these two bounds in (3.7), note that

\[
C_{dh0} \leq (v^3 - 4v^2 + v + 6)/v(v-2) \leq (v^3 - 4v^2 + 3v)/v(v-2) = C_{dA_{h0h0}}
\]

for all \(v \geq 4\). Thus, (3.4) and the fact that \(C_{dA}\) is c.s. yield

\[
\lambda_{d1} \leq (v/(v-1))C_{dA_{h0h0}} = \lambda_{dA1} < \lambda_{d^*1}.
\]

(3.8)

Thus \(d^*\) is E-better than any design in \(\Omega_2\).

**Step 3.** Finally, for a design \(d \in \Omega_3\) we see that there exists a treatment \(h_0\), such that \(r_{d}h0 < \bar{r} = v - 1\) and, using (3.4),

\[
\lambda_{d1} \leq (v/(v-1))C_{dh0h0}.
\]

(3.9)

We now obtain an upper bound for \(C_{dh0h0}\) referring to Saharay (1986), p.51. Let \(g\) be a function defined by

\[
g(r) = \max_{\{d: d \in \Omega, r_{dh}=r\}} C_{dh}
\]

(3.10)

Saharay (1986) derived that

\[
g(r) = \begin{cases} 
g_1(r), & \text{if } 1 \leq r < v/2, \\
g_2(r), & \text{if } v/2 \leq r \leq v - 1,
\end{cases}
\]

(3.11)
where
\[ g_1(r) = r^2/(v-1)(v-2) + r(v^2 - 4v + 2)/v(v-2), \]
\[ g_2(r) = r^2/(v-1)(v-2) + r(v^2 - 4v - 2)/v(v-2) + 2/(v-2). \]

It can be easily checked that \( g_1 \) and \( g_2 \) are nondecreasing functions in \( r, 1 \leq r \leq v - 1 \)
and
\[ g_1(v/2) = g_2(v/2), \quad \text{if } v \text{ is even,} \]
\[ g_1((v-1)/2) < g_2((v+1)/2), \quad \text{if } v \text{ is odd.} \tag{3.12} \]

Using (3.9), (3.10), (3.11) and the properties of \( g_1 \) and \( g_2 \) we conclude the proof as follows:
\[ \lambda_{d_1} \leq (v/(v-1))g(r_{d_{h_0}}) \leq (v/(v-1))g_2(v-2) \leq v(v-3)/(v-2) = \lambda_{d_{A1}} < \lambda_{d^*1}. \]

4. Construction of \( d^* \)

In this section, we develop systematic methods of constructing \( d^* \). For convenience, the rows, columns and treatments are numbered as 0, 1, \ldots, \( v-1 \).

**Case(i): \( v \) even**

We construct \( d^* \) by assigning treatment symbol \( x \mod v \) to the \((i, j)\)th cell as indicated below:

<table>
<thead>
<tr>
<th>cell</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 \leq i &lt; v - 2 )</td>
<td>( 0 \leq j &lt; i )</td>
</tr>
<tr>
<td>( i = v - 2 )</td>
<td>( 0 \leq j \leq v/2 - 2 )</td>
</tr>
<tr>
<td>( i = v - 1 )</td>
<td>( v/2 - 1 \leq j \leq v - 3 )</td>
</tr>
<tr>
<td></td>
<td>( j = v - 1 )</td>
</tr>
<tr>
<td></td>
<td>( v/2 - 1 \leq j \leq v - 2 )</td>
</tr>
</tbody>
</table>

**Case(ii): \( v \) odd**

In this case, we first make an array \( A_{v \times v} \) by assigning treatment symbols to the first \((v - 2)\) rows following the rule described just above. The remaining cells of \( A \) are filled as shown below:

<table>
<thead>
<tr>
<th>cell</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = v - 2 )</td>
<td>( 0 \leq j \leq v - 3, j \neq (v-5)/2 )</td>
</tr>
<tr>
<td></td>
<td>( j = (v-5)/2 )</td>
</tr>
<tr>
<td></td>
<td>( j = v - 1 )</td>
</tr>
<tr>
<td>( i = v - 1 )</td>
<td>( 0 \leq j \leq v - 3, j \neq (v-5)/2 )</td>
</tr>
<tr>
<td></td>
<td>( j = (v-5)/2 )</td>
</tr>
<tr>
<td></td>
<td>( j = v - 2 )</td>
</tr>
</tbody>
</table>
Finally, $d^*$ is obtained from $A$ by interchanging treatment symbols $v - 5$ and $v - 3$ in the column numbers $0, 2, 4, \ldots, 2(p - 1)$ and $4p$ when $v = 4p + 1$, and in the column numbers $2p, 2p + 2, \ldots, 4p$ when $v = 4p + 3$.

In both cases, it can be easily verified that in $d^*$, treatment $i$ does not occur in the $i$th row as well as in the $(i + 1)$th (mod $v$) column, $0 \leq i \leq v - 1$, and hence (3.2) is ensured.

**Example 1.**

\[
\begin{array}{cccccccc}
  & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \\
1 & 3 & 4 & 5 & 6 & 7 & 0 & 1 \\
2 & 4 & 5 & 6 & 7 & 0 & 1 & 2 \\
3 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\
4 & 6 & 7 & 0 & 1 & 2 & 3 & 4 \\
5 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
6 & 1 & 2 & 3 & 4 & 5 & 6 & 0 \\
\end{array}
\]

**Remark.** It is interesting to note that for $v = 7, d^*$ is A- and D-better than the nonequireplicate design proposed by Shah and Sinha (1990).

**5. Concluding Remarks**

It is clear that $d^*$ is not completely symmetric and has maximum $\text{tr}(C_d)$ in the binary and equireplicate class $\Omega_1$. A computer search for A- and D-optimal designs for $v \leq 20$ within $\Omega_1$ indicates that $d^*$ remains A- and D-optimal for $4 \leq v \leq 14$, whereas from $v = 15$ onwards the structures of A- and D-optimal designs depend on $v$. For example, for $v = 15, d_0$ with $L_{d_0} = J - I, M_{d_0} = J - P_0, \pi^{(P_0)} = \pi_6 \pi_9$ is A- and D-optimal, whereas for $v = 16, d^*$ is A-optimal, $d_0$ with $L_{d_0} = J - I, M_{d_0} = J - P_0, \pi^{(P_0)} = \pi_5 \pi_{11}$ is D-optimal. However, the A- and D-efficiencies of $d^*$ in $\Omega_1$ turn out to be greater than 0.99 for $v \geq 15$. The determination of exact A- and D-optimal designs in $\Omega$ is still an open problem.

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**References**


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