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BAYESIAN CONSISTENCY WITH THE SUPREMUM METRIC

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Abstract: We present conditions for Bayesian consistency in the supremum metric. The key to the technique is a triangle inequality that allows us to explicitly use weak convergence, a consequence of the standard Kullback–Leibler support condition for the prior. A further condition is to ensure that smoothed versions of densities are not too far from the original density, thus dealing with densities that could track the data too closely. Our main result is that we demonstrate supremum consistency using conditions comparable with those currently used to secure $\mathbb{L}_1$-consistency.

Key words and phrases: Fourier integral theorem, Prokhorov metric, sinc kernel, weak convergence.

1. Introduction

Bayesian consistency remains an open topic, and has seen much progress since the seminal papers of Barron, Schervish and Wasserman (1999) and Ghosal, Ghosh and Ramamoorthi (1999). A dominating sufficient, but not necessary condition is the Kullback–Leibler support condition for the prior,

$$\Pi\big(D(f_0, f) < \varepsilon\big) > 0, \tag{1.1}$$

for all $\varepsilon > 0$. Here, $D(f_0, f) = \int f_0 \log(f_0/f)$ denotes the Kullback–Leibler divergence between f_0 and f, and f_0 represents the true density function from which the identically distributed $(X_i)_{i=1:n}$ are observed. Furthermore, we write $\Pi(df)$ to denote the prior distribution on a space of probability density functions, say, $\mathbb{P}$.

It is well known that condition (1.1) is not sufficient for strong consistency. Strong consistency holds if

$$\Pi_n(A_\varepsilon) := \Pi(A_\varepsilon \mid X_{1:n}) \to 0 \quad \text{a.s.} \quad P_0^\infty, \tag{1.2}$$

for all $\varepsilon > 0$, where $A_\varepsilon = \{f : d_H(f_0, f) > \varepsilon\}$ and d_H is the Hellinger distance between f_0 and f. Note the the Hellinger distance is equivalent to the $\mathbb{L}_1$-distance. Barron, Schervish and Wasserman (1999) provide a counterexample that shows that a posterior is not strongly consistent, given only the Kullback–

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