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NONPARAMETRIC MAXIMUM LIKELIHOOD ESTIMATION UNDER A LIKELIHOOD RATIO ORDER

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Abstract: Comparing two univariate distributions based on independent samples from them is a fundamental problem in statistics, with applications in a variety of scientific disciplines. In many situations, we might hypothesize that the two distributions are stochastically ordered, meaning that samples from one distribution tend to be larger than those from the other. One type of stochastic order is the likelihood ratio order, in which the ratio of the density functions of the two distributions is monotone nondecreasing. In this article, we derive and study the nonparametric maximum likelihood estimator of the individual distribution functions and the ratio of their densities under the likelihood ratio order. Our work applies to discrete distributions, continuous distributions, and mixed continuous-discrete distributions. We demonstrate convergence in distribution of the estimator in certain cases, and illustrate our results using numerical experiments and an analysis of a biomarker for predicting bacterial infection in children with systemic inflammatory response syndrome.

Key words and phrases: Biomarker evaluation, density ratio, monotonicity constraint, odds ratio, ordinal dominance curve, shape-constrained inference.

1. Introduction

Comparing the distributions of two independent samples is a fundamental problem in statistics. Suppose that $X_1, \dots, X_{n_1}$ and $Y_1, \dots, Y_{n_2}$ are independent real-valued samples with distribution functions F_0 and G_0, respectively. In many situations, we might hypothesize that F_0 and G_0 are *stochastically ordered*, meaning intuitively that samples from F_0 tend to be larger than those from G_0. A particular type of stochastic order that arises in many applications is the *likelihood ratio order*. Specifically, G_0 and F_0 satisfy a likelihood ratio order if the density ratio f_0/g_0 is monotone nondecreasing over the support $\mathscr{G}_0$ of G_0, where $f_0 := dF_0/d\eta$ and $g_0 := dG_0/d\eta$, for some dominating measure η. For this reason, the likelihood ratio order is also called a *density ratio order*.

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