Statistica Sinica **6**(1996), 389-402

# ECONOMICAL QUALITY CONTROL PROCEDURES BASED ON SYMMETRIC RANDOM WALK MODEL

M. S. Srivastava and Yanhong Wu

*University of Toronto and University of Alberta*

*Abstract.* Assuming the quality characteristic process follows a symmetric random walk model, the on-line control procedure is studied without the assumption of normality. Simple approximation for the long run average cost rate and the optimal control parameters are provided. In particular, the robustness of the on-line control procedure under the mixed normal random walk model is demonstrated.

Key words and phrases: Average cost rate, Brownian motion, control limit, inspection interval.

## 1. Introduction

The classical statistical process control (SPC) charts are usually applied when the quality characteristic process is assumed to be relatively stable. Thus the main goal is to monitor the process and detect sudden disruptions (e.g. shift in mean ) caused by malfunction of machine etc.. The Shewhart, CUSUM, EWMA and the so-called Shiryayev-Roberts procedures have been effectively studied for this purpose. However, the variation of the quality characteristic may change or increase. For example, a wearing or deteriorating system may cause increase of variation, and the random change of environment may increase the variation locally or globally. Thus, on-line control procedures are necessary to keep the quality of the products uniform by adjusting the process directly or indirectly when the deviation from the target value is too large. A variety of models have been considered by many authors, e.g. Barnard (1959), Bather (1963), Box and Jenkins (1963), Taguchi (1985), Taguchi et al. (1989), Box and Kramer (1992) and Srivastava and Wu (1994, 1995). The normal random walk plays the central role (see Adams and Woodall (1989) and Srivastava and Wu (1991) for more careful studies). In this paper, however, we proceed in a different direction by extending the results of Srivastava and Wu (1991) to the more general symmetric random walk without assuming normality. The study serves two purposes. One is to provide a more flexible model than the simple normal random walk model without losing the simplicity of the approximation for the average cost rate and of the formulae for the optimal control parameters. The other is to study the robustness of the on-line control procedures under the normal random walk by assuming that the underlying density of the random walk is a mixture of normal densities. The latter seems more interesting from practical point of view.

The organization of the paper is as follows. In Section 2, we first introduce the cost structure and some standard notation. A general formula for the long-run average cost rate is developed. In Section 3, we give an accurate approximation for the average cost rate under the symmetric random walk model, and simple formulae for the control parameters are also provided. The robustness of the on-line control procedure under the mixed normal random walk model is investigated in Section 4 in a more analytical approach; some numerical illustrations are given. Two examples are used for illustration in Section 5. Some concluding remarks are given in Section 6. Technical details are provided in the appendix.

## 2. Average Cost Rate

Denote by $\{Z_t\}$ the quality characteristic process. The time scale is always taken as the length between two consecutive products. Suppose the process is inspected periodically with inspection interval $m$. The measurement error is assumed to be negligible. Every time the observed quality characteristic is found to be out of the control region $(-d, d)$, an adjustment without lag time will be made which forces the process back to the target value, so a new cycle starts. As in Taguchi (1985), the following cost structure will be considered throughout our discussion.

(a) *Cost of Inspection* $C_I$: We assume that the cost of each inspection (observation) is $C_I$. When the inspection interval has length $m$, the inspection cost per item is $C_I/m$.

(b) *Cost of Adjustment* $C_A$: We assume that each adjustment costs $C_A$. If the average cycle length is denoted by $w$, the adjustment cost per item is thus $C_A/w$.

(c) *Loss due to deviations from the target value*: The target value will be assumed to be zero. The loss due to deviation from the target value is of the form

$$a' Z_t^2 I_{[|Z_t|<\Delta]} + C I_{[|Z_t|\geq\Delta]},$$

where $\Delta$ is the tolerance limit, $C$ is loss for a defective and $a' = C/\Delta^2$.

Although the loss function due to deviation is not quadratic, this assumption is made in the following evaluation because the optimal control limit is usually much smaller than the tolerance limit, so the possibility of producing defectives is negligible. This, however, deserves extra study when the cost of an undetected defective is considerably higher than the normal cost $C$.

Denote

$$N = \min\{n \geq 1 : \ |Z_{nm}| \geq d\},$$

and thus $Nm$ is the cycle length. Under the assumptions above, the long run average cost rate can be obtained by using standard renewal theory as

$$L(d,m) = \frac{C_I}{m} + \frac{C_A}{mE(N)} + \frac{a'}{mE(N)} E\Big(\sum_{i=1}^{mN} Z_i^2\Big). \tag{2.1}$$

In order to obtain the optimal control parameters $m$ and $d$, the problem boils down to evaluating $E(N)$ and $E(\sum_{i=1}^{mN} Z_i^2)$. This will depend on specific model assumptions for $Z_i$. Taguchi (1985) provides a solution under certain assumptions (see Srivastava and Wu (1995) for a more careful analysis). Adams and Woodall (1989) and Srivastava and Wu (1991) improve upon under the standard normal random walk. In the next section, we shall extend the results of Srivastava and Wu (1991) to the general symmetric random walk model without assuming normality for the underlying distribution.

## 3. Solution Under Symmetric Random Model

The basic assumption is that the underlying density function $f(x)$ of $Z_1$ for the random walk $\{Z_n\}$ is symmetric with variance $\sigma^2$ and fourth moment $c\sigma^4$. In this case, it is shown in the appendix that the average cost rate can be explicitly obtained as

$$L(d,m) = \frac{C_I}{m} + \frac{C_A}{mE(Y_N^2)} + a'\sigma^2\Big[\frac{mE(Y_N^4)}{6E(Y_N^2)} + \frac{1}{2} + \frac{1}{6}m(3-c)\Big], \tag{3.1}$$

where $Y_n = Z_{nm}/\sigma\sqrt{m}$ is the normalized random walk, and $N$ can be rewritten as

$$N = \min\Big\{n \geq 1 : \ |Y_n| \geq \frac{d}{\sigma\sqrt{m}} = (\frac{u}{m})^{1/2}\Big\},$$

where $u = (d/\sigma)^2$. By denoting

$$R_N = |Y_N| - (u/m)^{1/2}$$

as the overshoot quantity and

$$\tilde{\rho}_r = E(R_N^r), \quad \text{for} \quad r = 1, 2, \ldots,$$

we have

$$E(Y_N^2) = E(|Y_N| - (\frac{u}{m})^{1/2} + (\frac{u}{m})^{1/2})^2 = \frac{u}{m} + 2(\frac{u}{m})^{1/2}\tilde{\rho}_1 + \tilde{\rho}_2,$$

$$E(Y_N^4) = (\frac{u}{m})^2 + 4(\frac{u}{m})^{3/2}\tilde{\rho}_1 + 6\frac{u}{m}\tilde{\rho}_2 + 4(\frac{u}{m})^{1/2}\tilde{\rho}_3 + \tilde{\rho}_4.$$

Using a similar argument given in Srivastava and Wu (1991), it can be shown that under proper conditions for $f(x)$(e.g. strong non-arithmetic), one can show that $\tilde{\rho}_r$ converges to its limit, say $\rho_r$ exponentially fast as $u/m \to \infty$. Specific formulae to calculate $\rho_r$'s are given in Appendix (3). In the normal case, the simulation considered by Srivastava and Wu (1991) shows that the approximation to $\rho_r$ is very accurate as long as $u/m > 1$. This is roughly equivalent to the assumption that $C_A > C_I$ as we shall verify later from the approximation for the optimal control parameters. Estimation methods for $\sigma^2$, $c$ and $\rho_r$ will be suggested in the concluding remarks when $f(x)$ is unknown.

By substituting these approximations into (3.1), we get

$$\begin{aligned} L(d,m) \approx \frac{C_I}{m} + \frac{a'\sigma^2}{6} \ \frac{u^2 + 4\rho_1 u\sqrt{um} + 6\rho_2 um + 4\rho_3 m\sqrt{um} + m^2\rho_4}{u + 2\rho_1\sqrt{um} + \rho_2 m} \\ + \frac{C_A}{u + 2\rho_1\sqrt{um} + \rho_2 m} + a'\sigma^2\Big[\frac{1}{2} + \frac{1}{6}m(3-c)\Big]. \end{aligned}$$

Some simple Taylor expansions and calculations show that the optimal control parameters $d$ and $m$ are approximately equal to

$$d^* \approx \sigma\Big[(\frac{6C_A}{a'\sigma^2})^{1/4} - \rho_1 m^{*1/2}\Big]; \tag{3.2}$$

$$m^* \approx \max\Big(1, (\frac{C_I}{a'\sigma^2[2(\rho_2 - \rho_1^2)/3 + (3-c)/6]})^{1/2}\Big),$$

where we require that both $d^*$ and $m^*$ to be positive. A sketch for the derivation will be given in Appendix (2).

It is clear from the above formulae that in order to have the method work, the condition $C_A > C_I$ is necessary. The formula for the optimal inspection interval $m^*$ looks complicated; however, a more detailed analysis given in Appendix (2) shows that $\rho_2 - \rho_1^2 = \frac{c}{12}$, so $m^*$ can be rewritten as

$$m^* \approx \max\Big(1, (\frac{6C_I}{a'\sigma^2(c/3 + (3-c))})^{1/2}\Big). \tag{3.3}$$

In order for (3.3) to be meaningful, we assume that $c/3 + 3 - c > 0$, i.e.

$$c < 9/2,$$

which is reasonable if only the local effects are considered. When the condition is not satisfied, i.e. the vibration is large, the control limit has to be taken as small as possible in order to control the deviation.

## 4. Robustness Under Mixture Normal Random Walk Model

In this section, we study the robustness of the control procedure under the normal random walk in which case $\rho_1 \approx 0.583$, and $c = 3$ . It turns out that we only need to study the behavior of $\rho_1$ and $c$. This is clear from (3.2) and (3.3) for the approximate optimal control parameters and the approximation for $L(d, m)$ given in the Appendix (2) as well as the relationship $\rho_2 = \rho_1^2 + c/12$.

We assume that the underlying density has the form

$$f(x) = \frac{1-p}{\sigma_1}\phi\Big(\frac{x}{\sigma_1}\Big) + \frac{p}{\sigma_2}\phi\Big(\frac{x}{\sigma_2}\Big),$$

where $\phi(x)$ is the standard normal density with $\sigma_1 < \sigma_2$ and

$$(1-p)\sigma_1^2 + p\sigma_2^2 = 1,$$

e.g. $p = (1-\sigma_1^2)/(\sigma_2^2 - \sigma_1^2)$ such that the variance of $f(x)$ is $\sigma^2 = 1$. This model has the following simple physical interpretation: Imagine that the deviations are caused by a shock from two different sources with probability $1-p$ and $p$ respectively. One of the sources can be thought of as unexpected. Suppose the deviations are additive, with normal increments having variances $\sigma_1^2$ and $\sigma_2^2$ respectively. Then, unconditionally, the model has the mixture normal underlying density.

The fourth moment of $f(x)$ is

$$c = (1-p)3\sigma_1^4 + 3p\sigma_2^4 = 3(\sigma_1^2 + \sigma_2^2 - \sigma_1^2\sigma_2^2),$$

where from the condition $c < 9/2$, $\sigma_1^2 + \sigma_2^2 - \sigma_1^2\sigma_2^2 < 3/2$. To study the robustness, we fix the value of $p$ and let $\sigma_1^2$ and $\sigma_2^2$ approach one at the same rate. For this, we write $\sigma_1^2 = 1 - \epsilon$ and thus $\sigma_2^2 = 1 + \epsilon(1-p)/p$. It is easy to see that $c = 3(1+\epsilon^2(1-p)/p)$. Thus, in order that $c < 9/2$ we require that $\epsilon < \sqrt{p/(2(1-p))}$. As $c/3 + 3 - c = 1 - 2\epsilon^2(1-p)/p$, we see that the local effect of nonnormality decreases the value of $m^*$ by a factor $\epsilon^2(1-p)/p$ if $C_I$ is assumed to be large.

To study the behavior of $\rho_1$, we note that the characteristic function of $f(x)$ is

$$g(\lambda) = (1-p)e^{-\lambda^2\sigma_1^2/2} + pe^{-\lambda^2\sigma_2^2/2}.$$

Thus, from (A.3) given in Appendix (3), we have

$$\begin{aligned}
\rho_1 = & -\frac{1}{\pi}\int_0^\infty \frac{1}{\lambda^2}\ln\frac{1-(1-p)e^{-\lambda^2\sigma_1^2/2} + pe^{-\lambda^2\sigma_2^2/2}}{\lambda^2/2}d\lambda \\
= & -\frac{1}{\pi}\int_0^\infty \frac{1}{\lambda^2}\ln\frac{1-e^{-\lambda^2/2}}{\lambda^2/2}d\lambda \\
& -\frac{1}{\pi}\int_0^\infty \frac{1}{\lambda^2}\ln\Big[1 + \frac{e^{-\lambda^2/2} - (1-p)e^{-\lambda^2\sigma_1^2/2} - pe^{-\lambda^2\sigma_2^2/2}}{1-e^{-\lambda^2/2}}\Big]d\lambda.
\end{aligned}$$

As $\epsilon \to 0$, a simple Taylor expansion gives

$$\rho_1 \approx 0.583 + \frac{\epsilon^2}{4\pi}\frac{1-p}{p}\int_0^\infty \frac{\lambda^2/2}{e^{\lambda^2/2}-1}d\lambda + o(\epsilon^2).$$

By noting that

$$\rho_1 m^{*1/2} \approx 0.583(6C_I/a')^{1/4}\Big(1+\big(\frac{1}{0.583\pi}\int_0^\infty \frac{\lambda^2/2}{e^{\lambda^2/2}-1}d\lambda+1\big)\frac{(1-p)}{2p}\epsilon^2+o(\epsilon^2)\Big),$$

where $\int_0^\infty \frac{\lambda^2/2}{e^{\lambda^2/2}-1}d\lambda \approx 1.637$, we see that the nonnormality decreases the optimal control limit but to a relatively insignificant order. Figure 1 gives the plot of the exact change $\rho_1 - 0.583$ for $p = 0.9, 0.8, 0.7, 0.6, 0.5$ and $\epsilon = 0.1, \ldots, 0.9$ (calculated by Mathematica Language). We see that for smaller values of $\epsilon$ and larger values of $p$, the change is not significant.

An important fact is that as $p$ increases to 1, the effect of unnormality diminishes to zero. The following examples serve to demonstrate the effect of nonnormality.

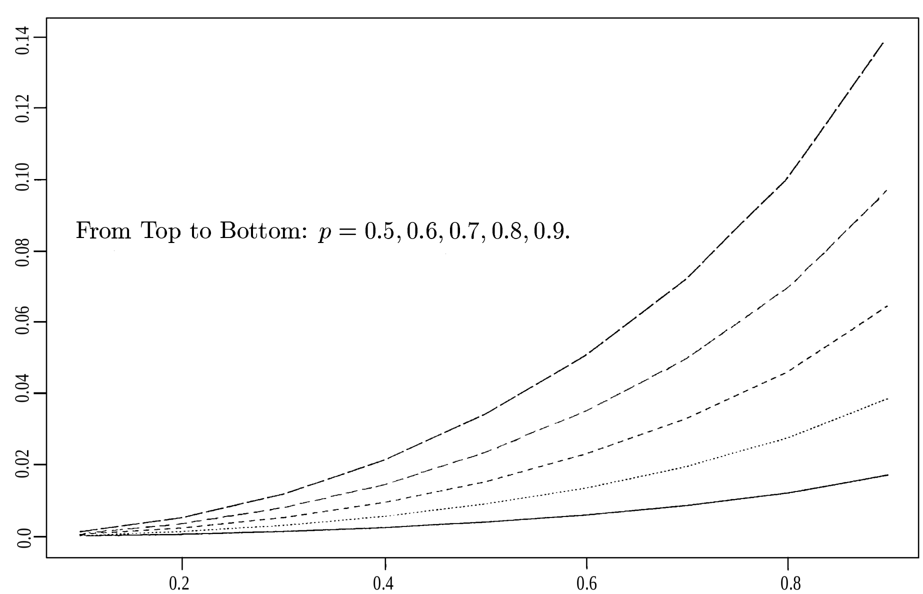


Figure 1. The difference of $\rho_1 - 0.583$ when $\epsilon$ changes from 0 to 1 for $p = 0.5, 0.6, 0.7, 0.8, 0.9$.

## 5. Illustrations

We use two typical examples to demonstrate the effects of the derived approximations.

**Example 1.** (Taguchi et al. (1989, p.71)). A manufacturer produces integrated circuits for a computer. The measurements are taken by an operator and the current system has the following parameters. Tolerance limit $\Delta = 10$; loss due to a defective piece $C = \$6.00$; measurement cost $C_I = \$1.50$ and adjustment cost $C_A = \$12.00$. The current control limit is $d_0 = 3.00$ and the observed average adjustment interval is $u_0 = 180$ units with every piece inspected.

In order to apply the derived approximations, we have to determine the variance $\sigma^2$ as well as $c$. The variance can be estimated by using the formula given by Taguchi et al. (1989) or by a simple formula from standard Brownian motion

$$\sigma^2 \approx d_0^2/u_0 \approx 0.05,$$

as the effect of discrete time is negligible. On the other hand, $a' = C/\Delta^2 \approx 0.06$.

Under the normal random walk model, the optimal control parameters are obtained as

$$d_0^* \approx 0.05^{1/2}\Big[(\frac{6 \times 12}{0.06 \times 0.05})^{1/4} - 0.583(\frac{6 \times 1.5}{0.06 \times 0.05})^{1/4}\Big] \approx 1.82;$$

and

$$m_0^* \approx \Big(\frac{6 \times 1.5}{0.06 \times 0.05}\Big)^{1/2} \approx 55.$$

However, if the true model based on the empirical data shows that there is a strong non-normality with $p = 0.6$ and $\epsilon = 0.5$ , say, this gives $\rho_1 = 0.607$ and $c = 3.5$. Then the optimal control parameters become

$$m^* \approx \Big(\frac{6 \times 1.5 \times 3}{0.06 \times 0.05 \times 2}\Big)^{1/2} \approx 67.4;$$

and

$$d^* \approx 0.05^{1/2}\Big[(\frac{6 \times 12}{0.06 \times 0.05})^{1/4} - 0.607 \times 67.4^{1/2}\Big] \approx 1.67.$$

We see that the effect on both control parameters is relatively large.

The second example is slightly more complicated.

**Example 2.** (Taguchi et al. (1989, p.73)). Lot Type Production: An injection molding process produces 12 pieces at a time (12 pieces per “shot” ). The average value of the width of one piece is checked once every half hour at a cost of \$2.00. The checking process involves the immersion of the piece in ice water and then measurement of its width. The tolerance limit is $\Delta = 15$. The loss for each defective piece is \$0.16, but if the piece measured in a shot of 12 is out of specification, all 12 pieces of the shot are discarded. The current control limit for the width is $d_0 = 5$, and the average adjustment interval is 8 hours.

Suppose the adjustment cost is \$15.00. Assume that there is an hourly production rate of 120 shots, 2000 working hours per year and that one shot is the basic unit of production.

Clearly, the number of shots produced in a cycle is equal to $120 \times 8 = 960 = u_0$. Thus, the estimated variance is approximately

$$\sigma^2 \approx d_0^2/u_0 = 5^2/960 \approx 0.026,$$

and $a' = 0.16 \times 12/15^2 \approx 0.0085$. Also, we know $C_A = 15$ and $C_I = 2$.

Under the normal random walk model, the control parameters are

$$m_0^* \approx 233, \quad \text{and} \quad d_0^* \approx 2.64,$$

and the cost per shot is about

$$L(d_0^*, m_0^*) \approx 0.071.$$

However, if the mixed normal walk model is true with the same parameters as in the first example, then the optimal control parameters become

$$m^* \approx 285, \quad \text{and} \quad d^* \approx 2.42,$$

and the cost per shot is about

$$L(d^*, m^*) \approx 0.064.$$

Again, the effect is significant. Further, the yearly savings of total costs by using the optimal control parameters under the mixed normal random walk model instead of the normal random walk model are about

$$(0.071 - 0.064) \times 120 \times 200 \approx 1751.00.$$

## 6. Concluding Remarks

In this paper, we extended the results of Srivastava and Wu (1991) to the general symmetric random walk model and provided simple formulae for the inspection interval and control limit. In particular, the robustness of the optimal control procedure under the random walk model is investigated. Several remarks are made as follows:

(1) In the formulae for the optimal control parameters $m^*$ and $d^*$ as well as the average cost rate, the parameters $\sigma^2$, $c$ and $\rho_r$'s are assumed to be known. When $f(x)$ is unknown, they can be estimated adaptively. For example, we can

record the consecutive differences of the observations $Z_{mn} - Z_{(n-1)m}$ for an initial value of $m$ and estimate $\sigma^2$ and $c$ adaptively. These estimators are used for determining the optimal $m$ and $d$. The procedure continues until appropriate parameters are chosen. The value of the $\rho_r$'s can be estimated by recording the consecutive overshoot beyond the boundaries at each adjustment or calculated by using the empirically estimated characteristic function for $g(\lambda)$ based on the results given in the Appendix. The theoretical study is a much more difficult project than simply developing algorithms.

(2) When the measurement error exists or where the IMA(1,1) model holds, similar studies seem intractable as the de-noising procedure is more complicated (see Srivastava and Wu (1994) for a discussion under the normal random walk model).

(3) The main reason we study the symmetric random walk model is to assure that the symmetric control limits are still reasonable. Under the general random walk model, the symmetric control limits become questionable, at least from a theoretical point of view, as the skewness of the underlying density function will have significant effects on the selection of optimal lower and upper control limits as well as the inspection interval. How to evaluate the average cost rate when the control limits are unsymmetric is a challenging problem. A related natural question is: how large will the increment of the average cost rate be if we still use the symmetric control limit?

For other related problems, we refer interested readers to Srivastava and Wu (1994).

## Appendix

### (1) Derivation of (3.1)

We first give some elementary results associated with a randomly stopped random walk, and their proofs appear in Chow, Robbins and Teicher (1965).

**Lemma 1.** *Let $Y_n$ be a random walk with $E(Y_1) = 0$, $E(Y_1^2) = 1$, $E(Y_1^3) = 0$ and $E(Y_1^4) = c$. Then, the following processes are martingales*:

(a) $U_n = Y_n^2 - n$,

(b) $V_n = \sum_{k=1}^n (Y_{k-1}(Y_k - Y_{k-1}))$,

(c) $W_n = \sum_{j=2}^n (Y_{j-1}^2) - nY_n^2 + n(n+1)/2$,

(d) $B_n = Y_n^4 - 6nY_n^2 + 3n^2 + n(3-c)$,

(e) $C_n = \sum_{k=1}^n ((Y_k - Y_{k-1})^2) - n$.

Now we evaluate $E(\sum_{n=1}^{mN} Z_n^2)$. Write $Z_n = a_1 + \cdots + a_n$ with $Ea_1 = 0$, $Ea_1^2 = \sigma^2$, $Ea_1^3 = 0$ and $Ea_1^4 = c\sigma^4$. As the techniques are similar to the ones used in Srivastava and Wu (1991), only the main steps will be given here. First, we write

$$\begin{aligned}E\Big(\sum_{n=1}^{mN} Z_n^2\Big) &= E\Big[E(\sum_{n=1}^{N}\sum_{i=1}^{m} Z^2_{m(n-1)+i} \mid Z_m,\ldots,Z_{mN})\Big]\\ &= E\Big[\sum_{n=1}^{N}\sum_{i=1}^{m} E(Z^2_{m(n-1)+i} \mid Z_{m(n-1)},Z_{mn})\Big]\\ &= E\Big[\sum_{n=1}^{N}\sum_{i=1}^{m} E(Z^2_{m(n-1)+i} \mid Z_{m(n-1)},Z_{mn}-Z_{m(n-1)})\Big],\end{aligned}$$

since $Z_{m(n-1)+i}$ is a random walk and its conditional distribution given $Z_m,\ldots,$ $Z_{mN}$ depends only on $Z_{m(n-1)}$ and $Z_{mn}$. Note that conditioning on $Z_{m(n-1)}, Z_{mn}-$ $Z_{m(n-1)}$, $a_{m(n-1)+1},\ldots,a_{m(n-1)+m}$ are identically distributed variables, so

$$E(a_{m(n-1)+i} \mid Z_{m(n-1)},Z_{mn}-Z_{m(n-1)}) = \frac{1}{m}(Z_{mn}-Z_{m(n-1)}).$$

Thus,

$$E(Z_{m(n-1)+i} \mid Z_{m(n-1)},Z_{mn}-Z_{m(n-1)}) = Z_{m(n-1)} + \frac{i}{m}(Z_{mn}-Z_{m(n-1)}).$$

Thus,

$$\begin{aligned}&E\sum_{n=1}^{N}\Big(\sum_{i=1}^{m} E[Z^2_{m(n-1)+i} \mid Z_{m(n-1)},Z_{mn}-Z_{m(n-1)}]\Big)\\ &= E\sum_{n=1}^{N}\Big(\sum_{i=1}^{m} E[(Z_{m(n-1)+i}-Z_{m(n-1)}-\frac{i}{m}(Z_{mn}-Z_{m(n-1)}))^2 \mid Z_{m(n-1)},Z_{mn}-Z_{m(n-1)}]\Big)\\ &\quad + E\Big\{\sum_{n=1}^{N}\sum_{i=1}^{m} E[\{Z_{m(n-1)}+\frac{i}{m}(Z_{mn}-Z_{m(n-1)})\}^2]\Big\}\\ &= A+B\,,\ \text{say.}\end{aligned}$$

To evaluate $A$, note that, for $n=1,2,\ldots$, the terms inside the first bracket are iid. So by Wald's identity, we have

$$\begin{aligned}A &= E(N)E\Big[\sum_{i=1}^{m} E[(Z_{m(n-1)+i}-Z_{m(n-1)}-\frac{i}{m}(Z_{mn}-Z_{m(n-1)}))^2 \,|\, Z_{mn}-Z_{m(n-1)}]\Big]\\ &= E(N)\sum_{i=1}^{m} E\Big[(Z_{m(n-1)+i}-Z_{m(n-1)}-\frac{i}{m}(Z_{mn}-Z_{m(n-1)}))^2\Big].\end{aligned}$$

A simple calculation yields $A=\sigma^2(m^2-1)E(N)/6$. To evaluate $B$, we get, by expansion,

$$\begin{aligned}B &= m^2\sigma^2 E\Big(\sum_{n=1}^{N} Y^2_{n-1}\Big) + \sigma^2\frac{(m+1)(2m+1)}{6}E\Big(\sum_{n=1}^{N}(Y_n-Y_{n-1})^2\Big)\\ &\quad + \sigma^2 m(m+1)E\Big(\sum_{n=1}^{N} Y_{n-1}(Y_n-Y_{n-1})\Big).\end{aligned}$$

A simple application of Lemma 1 finally yields

$$E\Big(\sum_{n=1}^{mN} Z_n^2\Big) = \frac{m^2\sigma^2}{6}E(Y_N^4) + \frac{m\sigma^2}{2}E(Y_N^2) + \frac{m^2\sigma^2}{6}E(N)(3-c).$$

In the normal case, $c = 3$, we get the results of Srivastava and Wu (1991).

**(2) Derivation of (3.2) and (3.3)**

The derivation is based on direct calculations, so we only give the main ideas. By deleting higher order terms of $(m/u)$ in (2) for $EY_N^4$, we have approximately

$$L(d,m) \approx \frac{a'\sigma^2}{2} + \frac{C_I}{m} + \frac{C_A}{u + 2\rho_1\sqrt{um} + \rho_2 m} + \frac{a'\sigma^2}{6}[u + 2\rho_1\sqrt{um} + \rho_2 m] \\ + ma'\sigma^2\Big[\frac{2}{3}(\rho_2 - \rho_1^2) + \frac{1}{6}(3-c)\Big] + \frac{a'\sigma^2}{2}.$$

By considering $u + 2\rho_1\sqrt{um} + \rho_2$ as a new variable, we know that the optimal $u^*$ and $m^*$ satisfy

$$u + 2\rho_1\sqrt{um} + \rho_2 m = \Big(\frac{6C_A}{a'\sigma^2}\Big)^{\frac{1}{2}}$$

and

$$m^* = \max\Big\{1, \big(\frac{C_I}{a'\sigma^2\{\frac{2}{3}(\rho_2 - \rho_1^2) + \frac{1}{6}(3-c)\}}\big)^{\frac{1}{2}}\Big\},$$

which gives

$$d^* \approx \sigma\Big[\big(\frac{6C_A}{a'\sigma^2}\big)^{\frac{1}{4}} - \rho_1(m^*)^{\frac{1}{2}}\Big].$$

**(3) Calculation of $\rho_i$'s**

Let $Y_n$ be a symmetric random walk where $Y_1$ has variance 1 and fourth moment $c$. Denote $f(x)$ as its underlying density function which is assumed to be strongly non-arithmetic (e.g. Siegmund (1985, Appendix 4)) and characteristic function

$$g(\lambda) = E(e^{i\lambda Y_1}) = \int_{-\infty}^{\infty} e^{i\lambda x} f(x)dx.$$

Let

$$\tau_+ = \min\{n > 0 : \ Y_n > 0\},$$

denote the ladder time and $Y_{\tau_+}$ the ladder height and

$$\tau_d = \min\{n > 0 : \ |Y_n| > d\}$$

the crossing time at the boundary $d$. Then from the strong renewal theorem, we can show that as $d \to \infty$,

$$\lim_{d\to\infty} P(|Y_{\tau_d}| - d > y) = \lim_{d\to\infty} P(Y_{\tau_d} - d > y | Y_{\tau_d} > d) = \frac{1}{E(Y_{\tau_+})}\int_0^{\infty} P(Y_{\tau_+} > x)dx,$$

and the convergence is exponentially fast in terms of $d$. From this, it is easy to see that

$$\rho_r = \frac{E(Y_{\tau_+}^{r+1})}{(r+1)E(Y_{\tau_+})}.$$

Thus, the problem reduces to evaluating the moments for the ladder height $Y_{\tau_+}$. From Siegmund (1985, (10.60)), it is known that the Laplace transform of $Y_{\tau_+}$ has the form

$$\begin{aligned} h(\alpha) = E[e^{-\alpha Y_{\tau_+}}] &= 1 - \exp\Big[\frac{1}{\pi}\int_0^\infty \frac{\alpha}{\alpha^2+\lambda^2}\ln(1-g(\lambda))d\lambda\Big] \\ &= 1 - \frac{\alpha}{\sqrt{2}}\exp\Big[\frac{1}{\pi}\int_0^\infty \frac{\alpha}{\alpha^2+\lambda^2}\ln\frac{1-g(\lambda)}{\lambda^2/2}d\lambda\Big] \qquad \text{(A.1)} \end{aligned}$$

as

$$\frac{1}{\pi}\int_0^\infty \frac{\alpha}{\alpha^2+\lambda^2}\ln(\lambda^2/2)d\lambda = \ln(\alpha/\sqrt{2}).$$

From (A.1), by letting $\alpha \to 0$, we obtain

$$(1-h(\alpha))/\alpha \to E(Y_{\tau_+}) = 1/\sqrt{2},$$

which is the average ladder height no matter what the underlying density function is. To obtain the values of the $\rho_i$'s, we first write

$$\begin{aligned} -\ln\frac{1-h(\alpha)}{\alpha E(Y_{\tau_+})} &= -\ln\Big(1-\rho_1\alpha+\frac{\rho_2}{2}\alpha^2-\frac{\rho_3}{6}\alpha^3+\frac{\rho_4}{24}\alpha^4+o(\alpha^4)\Big) \\ &= \rho_1\alpha-\nu_2\alpha^2+\nu_3\alpha^3-\nu_4\alpha^4+o(\alpha^4), \quad \text{say}, \end{aligned}$$

where $\nu_2 = \frac{1}{2}(\rho_2-\rho_1^2)$, $\nu_3 = \frac{1}{6}(\rho_3-3\rho_1\rho_2+2\rho_1^3)$, and $\nu_4 = \frac{1}{24}(\rho_4-3\rho_2^2-4\rho_1\rho_3+12\rho_1^2\rho_2-6\rho_1^4)$. By rewriting (A.1) as

$$\rho_1-\nu_2\alpha+\nu_3\alpha^2-\nu_4\alpha^3+o(\alpha^3) = -\frac{1}{\pi}\int_0^\infty \frac{1}{\alpha^2+\lambda^2}\ln\frac{1-g(\lambda)}{\lambda^2/2}d\lambda, \qquad \text{(A.2)}$$

we get

$$\rho_1 = -\frac{1}{\pi}\int_0^\infty \frac{1}{\lambda^2}\ln\frac{1-g(\lambda)}{\lambda^2/2}d\lambda, \qquad \text{(A.3)}$$

as given by Theorem 10.55 of Siegmund (1985) in the symmetric case.

By using (A.3), we rewrite (A.2) as

$$\nu_2-\nu_3\alpha+\nu_4\alpha^2+o(\alpha^2) = -\frac{1}{\pi}\int_0^\infty \frac{\alpha}{\lambda^2(\alpha^2+\lambda^2)}\ln\frac{1-g(\lambda)}{\lambda^2/2}d\lambda, \qquad \text{(A.4)}$$

which gives

$$\begin{aligned}\nu_2 &= \lim_{\alpha\to 0} -\frac{1}{\pi}\int_0^\infty \frac{1}{1+\lambda^2}\frac{\ln\frac{1-g(\lambda\alpha)}{(\lambda\alpha)^2/2}}{(\lambda\alpha)^2}d\lambda \\ &= \frac{c}{12\pi}\int_0^\infty \frac{1}{1+\lambda^2}d\lambda = \frac{c}{24}. \end{aligned} \tag{A.5}$$

This means that the variance of the overshoot is about $c/12$, depending only on the fourth moment $c$ of $Y_1$. In the normal case, it becomes $1/4$.

Similarly, by using (A.4), we obtain $\nu_3$ and $\nu_4$ as

$$\begin{aligned}a\nu_3 &= \lim_{\alpha\to 0}\frac{1}{\pi}\int_0^\infty \frac{1}{\lambda^2}\frac{\lambda^2-\alpha^2}{(\lambda^2+\alpha^2)^2}\ln\frac{1-g(\lambda)}{\lambda^2/2}d\lambda \\ &= \lim_{\alpha\to 0}\frac{1}{\pi}\int_0^\infty \frac{\lambda^2-\alpha^2}{(\lambda^2+\alpha^2)^2}\Big(\frac{\ln\frac{1-g(\lambda)}{\lambda^2/2}}{\lambda^2}+\frac{c}{12}\Big)d\lambda \\ &= \frac{1}{\pi}\int_0^\infty \frac{\ln\frac{1-g(\lambda)}{\lambda^2/2}+\frac{c\lambda^2}{12}}{\lambda^4}d\lambda, \\ \nu_4 &= \lim_{\alpha\to 0}\frac{1}{\pi}\int_0^\infty \frac{\alpha(3\lambda^2-\alpha^2)}{(\lambda^2+\alpha^2)^3}\Big(\frac{\ln\frac{1-g(\lambda)}{\lambda^2/2}}{\lambda^2}+\frac{c}{12}\Big)d\lambda \\ &= \lim_{\alpha\to 0}\frac{1}{\pi}\int_0^\infty \frac{(3\lambda^2-1)}{(1+\lambda^2)^3}\frac{1}{\alpha^2}\Big(\frac{\ln\frac{1-g(\lambda\alpha)}{(\lambda\alpha)^2/2}}{(\lambda\alpha)^2}+\frac{c}{12}\Big)d\lambda \\ &= \frac{1}{\pi}\int_0^\infty \frac{(3\lambda^2-1)\lambda^2}{(1+\lambda^2)^3}d\lambda\Big(-\frac{g^{(6)}(0)}{360}-\frac{c^2}{288}\Big) \\ &= \frac{1}{2}\Big(-\frac{g^{(6)}(0)}{360}-\frac{c^2}{288}\Big). \end{aligned}$$

The values of $\rho_3$ and $\rho_4$ can thus be evaluated numerically.

## Acknowledgement

The first author's research was supported by NSERC of Canada and the second author's research was partially supported by the Central Research Fund of the University of Alberta. We are grateful to the referees and the Editor for many helpful suggestions and comments which greatly improved the presentation.

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Department of Statistics, University of Toronto, Toronto, Ontario, M5S 1A1, Canada.

Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, T6G 2G1, Canada.

(Received September 1993; accepted May 1995)