Optimal Sequential Tests for Monitoring Changes in the Distribution of Finite Observation Sequences

Dong Han, Fugee Tsung, Jinguo Xian, Miaomiao Yu

Shanghai Jiao Tong University

Hong Kong University of Science and Technology

East China Normal University

Proofs of Theorems 1, 2 and 3.

Proof of Theorem 1. Let $T^* = T^*(c) = T^*_M(c, N)$ and

$$\xi_n = \sum_{k=1}^{n} (Y_{k-1} - cv_k), \tag{A. 1}$$

where $c > 0$. We will divide three steps to complete the proof of Theorem 1.

Step I. Show that

$$\mathbf{E}_0(\xi_T) \geq \mathbf{E}_0(\xi_{T^*}) \tag{A. 2}$$

for all $T \in \mathfrak{T}_N$ and the strict inequality of (A.2) holds for all $T \in \mathfrak{T}_N$ with $T \neq T^*$.

To prove (A.2), by Lemma 3.2 in Chow, Robbins and Siegmund (1971), we only need to prove the following two inequalities:

$$\mathbf{E}_\infty(\xi_{T^*}|\mathfrak{F}_n) \leq \xi_n \quad \text{on} \quad \{T^* > n\} \tag{A. 3}$$

and

$$\mathbf{E}_\infty(\xi_T|\mathfrak{F}_n) \geq \xi_n \quad \text{on} \quad \{T^* = n, T > n\} \tag{A. 4}$$

for each $n \geq 1$.

Let $B_{m,n+1}(N) = \{Y_k < l_k(c), n+1 \leq k \leq m\}$ for $n+1 \leq m \leq N$. By the similar method of proving Theorem 1 in Han, Tsung and Xian (2017), we can verify that

$$l_n(c) = cv_{n+1} + \mathbf{E}_0\Big(\sum_{m=n+1}^{N} B_{m,n+1}(N)[cv_{m+1} - Y_m]|\mathfrak{F}_n\Big) \tag{A. 5}$$

and

$$\begin{aligned} & \mathbf{E}_0\Big(\sum_{m=n+1}^{N} I(T > m)[cv_{m+1} - Y_m]|\mathfrak{F}_n\Big) \\ \leq \quad & (l_n(c) - cv_{n+1})I(T > n) \end{aligned} \tag{A. 6}$$

for $0 \leq n \leq N$ and $T \in \mathfrak{T}_N$, and therefore, by (A.1), (A.5) and (A.6),

$$\begin{aligned} & I(T^* > n)\mathbf{E}_0(\xi_{T^*} - \xi_n)|\mathfrak{F}_n) \\ = \quad & I(T^* > n)\sum_{m=n}^{N} \mathbf{E}_0(I(T^* > m)[\xi_{m+1} - \xi_m])|\mathfrak{F}_n) \\ = \quad & I(T^* > n)[Y_n - cv_{n+1} + \sum_{m=n+1}^{N} \mathbf{E}_0(I(B_{m,n+1})(Y_m - cv_{m+1})]|\mathfrak{F}_n) \\ = \quad & I(T^* > n)(Y_n - l_n(c)) < 0 \end{aligned} \tag{A. 7}$$

and

$$
\begin{aligned}
& I(T^* = n)I(T > n)\mathbf{E}_0(\xi_T - \xi_n)|\mathfrak{F}_n) \qquad \text{(A. 8)} \\
= \;& I(T^* = n)\sum_{m=n}^{N}\mathbf{E}_0(I(T > m)[\xi_{m+1} - \xi_m])|\mathfrak{F}_n) \\
= \;& I(T^* = n)[I(T > n)(Y_n - cv_{n+1}) \\
+ \;& \sum_{m=n+1}^{N}\mathbf{E}_0\Big(I(T > m)[Y_m - cv_{m+1}]|\mathfrak{F}_n\Big)] \\
\geq \;& I(T^* = n)[I(T > n)(Y_n - cv_{n+1}) + (cv_{n+1} - l_n(c))I(T > n)] \\
= \;& I(T^* = n)I(T > n)[Y_n - l_n(c)] \geq 0,
\end{aligned}
$$

for $1 \leq n \leq N$, where the last inequality in (A.7) comes from the definition of T^*. The two inequalities (A.7) and (A.8) mean respectively that (A.3) and (A.4) hold for $1 \leq n \leq N$. Hence, the inequality in (A.2) holds for all $T \in \mathfrak{T}_N$. Furthermore, from (A.7) and (A.8), it follows that the strict inequality in (A.2) holds for all $T \in \mathfrak{T}_N$ with $T \neq T^*$.

Step II. Show that there is positive number c_γ such that

$$\mathcal{J}_{M,N}(T^*(c_\gamma)) = c_\gamma\Big(1 - \frac{\mathbf{E}_0(v_1)}{\gamma}\Big) - \frac{\mathbf{E}_0[l_1(c_\gamma) - Y_1]^+}{\gamma}.$$

As $\mathbf{E}_0(v_1) < \gamma < \sum_{k=1}^{N+1}\mathbf{E}_0(v_k)$, it follows that there is at least a $k \geq 2$ such that $\mathbf{E}_0(v_k) > 0$. Let $k^* = \max\{2 \leq k \leq N+1 : \ \mathbf{E}_0(v_k) > 0\}$, we have

$$\mathbf{E}_0(\sum_{k=1}^{T^*} v_k) = \sum_{k=1}^{k^*}\mathbf{E}_0(v_k I(T^* \geq k)) = \mathbf{E}_0(v_1) + \sum_{k=2}^{k^*}\mathbf{E}_0(v_k I(T^* \geq k)).$$

By the definition of $\{l_k(c), 1 \leq k \leq N+1\}$ and T^*, we know that

$$\begin{aligned}\lim_{c\to 0}\sum_{k=2}^{k^*}\mathbf{E}_0(v_k I(T^*\geq k)) &= 0\\ \lim_{c\to \infty}\sum_{k=2}^{k^*}\mathbf{E}_0(v_k I(T^*\geq k)) &= \sum_{k=2}^{k^*}\mathbf{E}_0(v_k)\end{aligned}$$

As $\sum_{k=2}^{k^*}\mathbf{E}_0(v_k I(T^*\geq k))$ is continuous and increasing on c, it follows that there is a positive number c_γ such that

$$\mathbf{E}_0(\sum_{k=1}^{T^*(c_\gamma)} v_k) = \sum_{k=1}^{k^*}\mathbf{E}_0(v_k I(T^*\geq k)) = \gamma. \tag{A. 9}$$

It follows from (A.5) that

$$\begin{aligned}&\sum_{m=1}^{N}\mathbf{E}_0([Y_m - cv_{m+1}]I(T^*\geq m+1))\\ =\ &\mathbf{E}_0\Big(\mathbf{E}_0(\sum_{m=1}^{N} B_{m,1}(N)[Y_m - cv_{m+1}]|\mathfrak{F}_0)\Big)\\ =\ &\mathbf{E}_0(cv_1 - l_0(c)). \qquad\qquad (A.\ 10)\end{aligned}$$

Thus, by (A.1), (A.9), and (A.10) we have

$$\begin{aligned}&\mathcal{J}_{M,N}(T^*(c_\gamma)) = \frac{\mathbf{E}_0(\sum_{m=1}^{T^*(c_\gamma)} Y_{m-1})}{\mathbf{E}_0(\sum_{m=1}^{T^*(c_\gamma)} v_k)}\\ =\ &\frac{\sum_{m=1}^{N}\mathbf{E}_0([Y_m - c_\gamma v_{m+1} + c_\gamma v_{m+1}]I(T^*(c_\gamma)\geq m+1))}{\gamma}\\ =\ &\frac{c_\gamma\sum_{m=1}^{N+1}\mathbf{E}_0(v_m I(T^*(c_\gamma)\geq m)) - \mathbf{E}_0(l_0(c_\gamma))}{\gamma}\\ =\ &\frac{c_\gamma\mathbf{E}_0(\sum_{m=1}^{T^*(c_\gamma)} v_k)}{\gamma} - \frac{\mathbf{E}_0(l_0(c_\gamma))}{\gamma} = c_\gamma - \frac{\mathbf{E}_0(l_0(c_\gamma))}{\gamma}\\ =\ &c_\gamma(1 - \frac{\mathbf{E}_0(v_1)}{\gamma}) - \frac{\mathbf{E}_0[l_1(c_\gamma) - Y_1]^+}{\gamma}.\end{aligned}$$

The last equality follows from the definition of $l_0(c)$ in (2.7). It proves (iii) of Theorem 1.

Step III. Show (i) and (ii) of Theorem 1. Let

$$\tilde{c}_\gamma = \mathcal{J}_{M,N}(T^*(c_\gamma)) = c_\gamma - \frac{\mathbf{E}_0(l_0(c_\gamma))}{\mathbf{E}_0(\sum_{m=1}^{T^*(c_\gamma)} v_m)}.$$

If $\mathcal{J}_{M,N}(T) \geq c_\gamma$, then $\mathcal{J}_{M,N}(T) \geq \tilde{c}_\gamma = \mathcal{J}_{M,N}(T^*(c_\gamma))$. If $\mathcal{J}_{M,N}(T) < c_\gamma$, then, by (A.1), (A.2), and $\mathbf{E}_0(\sum_{m=1}^{T} v_m) \geq \gamma$, we have

$$\begin{aligned}
[\mathcal{J}_{M,N}(T) - c_\gamma]\gamma &\geq [\frac{\mathbf{E}_0(\sum_{m=1}^{T} Y_{m-1})}{\mathbf{E}_0(\sum_{m=1}^{T} v_m)} - c_\gamma]\mathbf{E}_0(\sum_{m=1}^{T} v_m) \\
&= [\mathbf{E}_0(\sum_{m=1}^{T} Y_{m-1}) - c_\gamma \mathbf{E}_0(\sum_{m=1}^{T} v_m)] \\
&\geq [\mathbf{E}_0(\sum_{m=1}^{T^*(c_\gamma)} Y_{m-1}) - c_\gamma \mathbf{E}_0(\sum_{m=1}^{T^*(c_\gamma)} v_m)] \\
&= [\frac{\mathbf{E}_0(\sum_{m=1}^{T^*(c_\gamma)} Y_{m-1})}{\mathbf{E}_0(\sum_{m=1}^{T^*(c_\gamma)} v_m)} - c_\gamma]\mathbf{E}_0(\sum_{m=1}^{T^*(c_\gamma)} v_m) \\
&= [\mathcal{J}_{M,N}(T^*(c_\gamma)) - c_\gamma]\gamma.
\end{aligned}$$

This means that $\mathcal{J}_{M,N}(T) \geq \mathcal{J}_{M,N}(T^*(c_\gamma))$ for all $T \in \mathfrak{T}_N$ with $\mathbf{E}_0(\sum_{m=1}^{T} v_m) \geq \gamma$. That is, (i) of Theorem 1 is true. The strict inequality in (ii) of Theorem 1 comes from the strict inequality in (A.2) when $T \neq T^*(c_\gamma)$ with $\mathbf{E}_0(\sum_{m=1}^{T} v_m) = \mathbf{E}_0(\sum_{m=1}^{T^*(c_\gamma)} v_m) = \gamma$. This completes the proof of Theorem 1.

Proof of Theorem 2. Since $Y_k = (Y_{k-1} + w_k(Y_{k-1}, A_{n,p_1}))\Lambda_k$ and

$$\Lambda_k = \frac{p_{\mathbf{1}k}(X_k|X_{k-1}, ..., X_{k-j})}{p_{\mathbf{0}k}(X_k|X_{k-1}, ..., X_{k-i})}$$

for $1 \le k \le N$, it follows that $(Y_k, X_k), 0 \le k \le N$, is a two-dimensional **p**-order Markov chain, where $p = \max\{i, j\}$. Let $1 \le p \le N$. By the definition of the optimal control limits, we have

$$
\begin{aligned}
l_k(c) &= cv_{k+1}(Y_k, A_{n,p_2}) \\
&\quad +\mathbf{E}_0\Big([l_{k+1}(c) - (Y_k + w_{k+1}(Y_k, A_{n,p_1}))\Lambda_{k+1}]^+ | Y_k, A_{n,0}\Big)
\end{aligned}
\tag{A. 11}
$$

for $0 \le k \le p-1$ and

$$
\begin{aligned}
l_k(c) &= cv_{k+1}(Y_k, A_{n,p_2}) \\
&+ \mathbf{E}_0\Big([l_{k+1}(c) - (Y_k + w_{k+1}(Y_k, A_{n,p_1}))\Lambda_{k+1}]^+ | Y_k, A_{n,p}\Big)
\end{aligned}
\tag{A. 12}
$$

for $p \le k \le N$. Let $p = 0$, we have similarly

$$
\begin{aligned}
l_k(c) &= l_k(c, Y_k) \\
&= cv_{k+1}(Y_k) + \mathbf{E}_0\Big([l_{k+1}(c) - (Y_k + w_{k+1}(Y_k))\Lambda_{k+1}]^+ | Y_k\Big)
\end{aligned}
\tag{A. 13}
$$

for $0 \le k \le N$.

Proof of Theorem 3. Let $p = 0$. As the observations $X_k, 0 \le k \le N$, are independent, it follows from the definition of $\{Y_k, 1 \le k \le N\}$ that $\{Y_k, 1 \le k \le N\}$ is a 1-order Markov chain. Thus, the optimal control limits, $l_k(c), 0 \le k \le N$, satisfy (A.13).

Let $y = cv_{N+1}(y)$. As $v_{N+1}(y)$ is non-increasing, it follows that there is a positive number $y_N(c)$ such that $y_N(c) = cv_{N+1}(y_N(c))$. Hence, $Y_N \geq l_N(c) = cv_{N+1}(Y_N)$ if and only if $Y_N \geq y_N(c)$. Therefore, we let $\widetilde{l_N}(c) = y_N(c)$. Take $k = N-1$ in (A.13) and let

$$\begin{aligned} y = f_0(y) &= cv_N(y) \\ &+ \mathbf{E}_0\Big([cv_{N+1}((y + w_N(y))\Lambda_N) - (y + w_N(y))\Lambda_N]^+ | Y_{N-1} = y\Big). \end{aligned}$$

Note that the two functions $(y + w_N(y))$ and $v_N(y)$ are non-decreasing and non-increasing on $y \geq 0$, respectively. Therefore, the function $f_0(y)$ is non-increasing on $y \geq 0$, and it follows that there is a positive number y_{N-1} such that $y_{N-1} = f_0(y_{N-1})$; that is,

$$\begin{aligned} y_{N-1} &= cv_N(y_{N-1}) \\ &+ \mathbf{E}_0\Big([cv_{N+1} - (y_{N-1} + w_N(y_{N-1}))\Lambda_N]^+ | Y_{N-1} = y_{N-1}\Big). \end{aligned}$$

This implies that $Y_{N-1} \geq l_{N-1}(c)$ if and only if $Y_{N-1} \geq y_{N-1}$. Therefore, we let $\widetilde{l_{N-1}}(c) = y_{N-1}$. Similarly, there are positive numbers $y_k, 1 \leq k \leq N-2$ such that $Y_k \geq l_k(c)$ if and only if $Y_k \geq y_k$ for $1 \leq k \leq N-2$, where

$$y_k = cv_{k+1}(y_k) + \mathbf{E}_0\Big([l_{k+1}(c) - (y_k + w_{k+1}(y_k))\Lambda_{k+1}]^+ | Y_k = y_k\Big)$$

for $1 \leq k \leq N-2$. Taking $\widetilde{l_k}(c) = y_k$ for $1 \leq k \leq N$, we know that the control limit $\{\widetilde{l_k}(c)\}$ is an equivalent control limit of the optimal sequential test $T_M^*(c, N)$ and it consists of a series of nonnegative non-random numbers. This proves (ii) of Theorem 3.

Let $1 \leq p \leq N$. As $\{(Y_k, X_k), 0 \leq k \leq N\}$ is a two-dimensional p-order Markov chain, it follows that (A.12) and (A.11) hold for $p \leq k \leq N$ and $0 \leq k \leq p-1$, respectively. When $k = N$ in (A.12), we take $\widetilde{l_N}(c) = y_N(c)$, where $y_N(c) = cv_{N+1}(y_N(c))$. For any fixed observation values $a_{k,p} = \{x_k, ..., x_{k-p+1}\}$ for $p \leq k \leq N-1$ and $a_{k,0} = \{x_k, ..., x_0\}$ for $0 \leq k \leq p-1$, let

$$\begin{aligned} y = f_p(y) &= cv_{k+1}(y, a_{k,p}) \\ &+ \mathbf{E}_0\Big([l_{k+1}(c) - (y + w_{k+1}(y, a_{k,p}))\Lambda_{k+1}]^+ | Y_k = y, A_{k,p} = a_{k,p}\Big) \end{aligned}$$

for $p \leq k \leq N-1$ and

$$\begin{aligned} y = g_p(y) &= cv_{k+1}(y, a_{k,0}) \\ &+ \mathbf{E}_0\Big([l_{k+1}(c) - (y + w_{k+1}(y, a_{k,0}))\Lambda_{k+1}]^+ | Y_k = y, A_{k,0} = a_{k,0}\Big) \end{aligned}$$

for $0 \leq k \leq p-1$. As the two functions $f_p(y)$ and $g_p(y)$ are non-increasing on $y \geq 0$, it follows that there are positive numbers $y_k = y_k(c, a_{k,p})$ for $p \leq k \leq N-1$ and $y_k = y_k(c, a_{k,0})$ for $1 \leq k \leq p-1$ such that $y_k = f_p(y_k)$ for $p \leq k \leq N-1$ and $y_k = g_p(y_k)$ for $1 \leq k \leq p-1$. Therefore, $Y_k \geq l_k(c)$ if and only if $Y_k \geq y_k$. Taking $\widetilde{l_k}(c) = y_k(c, X_k, ..., X_{k-p+1})$ for $p \leq k \leq N$ and $\widetilde{l_k}(c) = y_k(c, X_k, ..., X_0)$ for $1 \leq k \leq p-1$, we have $\widetilde{T_M^*}(c, N) = T_M^*(c, N)$. That is, $\{\widetilde{l_k}(c), 1 \leq k \leq N+1\}$ is an equivalent control limit of the optimal sequential test $T_M^*(c, N)$ that does not directly depend on the statistic, $Y_k, 1 \leq k \leq N$. This completes the proof of (i) of Theorem 3.