Space-Time Estimation and Prediction under fixed-domain Asymptotics with Compactly Supported Covariance Functions

Online Supplement

Tarik Faouzi[1], Emilio Porcu[2], and Moreno Bevilacqua[3].

[1] Department of Statistics, Universidad del BioBio, 4081112 Concepcion, Chile
E-mail:tfaouzi@ubiobio.cl

[2] School of Computer Science and Statistics, Trinity College Dublin.
and Millennium Nucleus Center for the Discovery of Structures in Complex Data, Chile
E-mail: georgepolya01@gmail.com, porcue@tcd.ie

[3] Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Viña del Mar, Chile.
Millennium Nucleus Center for the Discovery of Structures in Complex Data, Chile
E-mail: moreno.bevilacqua@uai.cl

A. Background on Equivalence of Gaussian Measures

Throughout, when cross referencing to equations or theorems in blue color we refer to the corresponding numbering in the main manuscript.

Equivalence and orthogonality of probability measures are useful tools when assessing the asymptotic properties of both prediction and estimation for stochastic processes. Denote with P_i, $i = 0, 1$, two probability measures defined on the same measurable space $\{\Omega, \mathcal{F}\}$. P_0 and P_1 are called equivalent (denoted $P_0 \equiv P_1$) if $P_1(A) = 1$ for any $A \in \mathcal{F}$ implies $P_0(A) = 1$ and vice versa. On the other hand, P_0 and P_1 are orthogonal (denoted $P_0 \perp P_1$) if there exists an event A such that $P_1(A) = 1$ but $P_0(A) = 0$. For a stochastic process $Z = \{Z(\boldsymbol{s}, t), (\boldsymbol{s}, t) \in \mathbb{R}^d \times \mathbb{R}\}$, to define previous concepts, we restrict the event A to the σ-algebra generated by $\{Z(\boldsymbol{s}, t), (\boldsymbol{s}, t) \in D \times \mathcal{T}\}$ where $D \times \mathcal{T} \subset \mathbb{R}^d \times \mathbb{R}$. We emphasize this restriction by saying that the two measures are equivalent on the paths of Z.

Gaussian measures are completely characterized by their mean and covariance function. We write $P(C)$ for a Gaussian measure with zero mean and covariance function C. It is well known that two Gaussian measures are either equivalent or orthogonal on the paths of Z (Ibragimov and Rozanov, 1978).

Let d be a positive integer. Let $P(C_i)$, $i = 0, 1$ be two zero mean Gaussian measures associated with a process Z defined over a bounded set $\mathcal{D}\times\mathcal{T}$ of $\mathbb{R}^d\times\mathbb{R}$, with covariance function C_i such that $C_i=\sigma_i^2K_i$, for $K_i\in\Phi_{d,T}$ and associated spectral density $\widehat{C}_i(\boldsymbol{z},\tau)=\sigma_i^2 f_i(\|\boldsymbol{z}\|,|\tau|)$, with f_i as in (2.1). Using results in Skorokhod and Yadrenko (1973), Ibragimov and Rozanov (1978) and Stein (2004), Ip and Li (2017) have shown that, if for some $a>0$, $\widehat{C}_0(\boldsymbol{z},\tau)\,\|(\boldsymbol{z},\tau)\|^a$ is bounded away from 0 and ∞ as $(\|\boldsymbol{z},\tau\|)\to\infty$, where $(\|\boldsymbol{z},\tau\|)=\sum_{i=1}^d\ z_i^2+\tau^2$ and for some finite and positive c,

$$\int_{\mathcal{A}} z^{d-1}\left\{\frac{\sigma_1^2 f_1(z,\tau)-\sigma_0^2 f_0(z,\tau)}{\sigma_0^2 f_0(z,\tau)}\right\}^2 \mathrm{d}z\mathrm{d}\tau<\infty, \qquad \text{(A.1)}$$

where $\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3\cup\mathcal{A}_4\cup\mathcal{A}_5$ with $\mathcal{A}_1=\{z>c,\tau>c\};\mathcal{A}_2=\{z>c,0\le\tau<c\};\mathcal{A}_3=\{0\le z<c,\tau>c\};\mathcal{A}_4=\{z>c_1,0\le\tau<c\};\mathcal{A}_5=\{0\le z<c,\tau>c_2\}$, and c_1, c_2 be two constants satisfying $0<c_1,c_2<c$ such that $c_1^2+c_2^2<c^2$ (Ip and Li, 2017). Then, for any bounded subset $D\times\mathcal{T}\subset\mathbb{R}^d\times\mathbb{R}$, $P(C_0)\equiv P(C_1)$ on the paths of $Z(\boldsymbol{s},t),(\boldsymbol{s},t)\in D\times\mathcal{T}$. For the reminder of the paper, we denote with $P(\sigma^2K_{\mathcal{DM}}(\boldsymbol{\theta}))$, $P(\sigma^2K_{\mathcal{DGW}}(\boldsymbol{\chi})$, zero mean Gaussian measure induced by $\sigma^2K_{\mathcal{DM}}(\cdot,\cdot;\boldsymbol{\theta})$ and $\sigma^2K_{\mathcal{DGW}}(\cdot,\cdot;\boldsymbol{\chi})$ covariance functions, respectively.

B. Equivalence of Gaussian measures under $\mathcal{DGW}$ and $\mathcal{DM}$ classes

The following result is due to Ip and Li (2017). It characterizes the equivalence of two gaussian measures under $\mathcal{DM}$ covariance models, with a common smoothness parameter ν.

Theorem 1. *For a given $\nu > 0$, let $P(\sigma_i^2 K_{\mathcal{DM}}(\boldsymbol{\theta}_i))$, $i = 0, 1$, be two zero mean Gaussian measures, with $\boldsymbol{\theta}_i = (\nu, \zeta_i, \upsilon_i)^\top$. For any bounded infinite set $D \times \mathcal{T} \subset \mathbb{R}^d \times \mathbb{R}$, $d = 1, 2$,*

1. *for $\epsilon = 1$, $P(\sigma_0^2 K_{\mathcal{DM}}(\boldsymbol{\theta}_0)) \equiv P(\sigma_1^2 K_{\mathcal{DM}}(\boldsymbol{\theta}_1))$ if and only if $\sigma_0^2 \zeta_0^{2\nu-d} \upsilon_0^{2\nu-1} = \sigma_1^2 \zeta_1^{2\nu-d} \upsilon_1^{2\nu-1}$;*

2. *for $\epsilon = 0$, $P(\sigma_0^2 K_{\mathcal{DM}}(\boldsymbol{\theta}_0)) \equiv P(\sigma_1^2 K_{\mathcal{DM}}(\boldsymbol{\theta}_1))$ if and only if $\sigma_0^2 \zeta_0^{-d} \upsilon_0^{2\nu-1} = \sigma_1^2 \zeta_1^{-d} \upsilon_1^{2\nu-1}$ and $\upsilon_0/\zeta_0 = \upsilon_1/\zeta_1$;*

3. *for $0 < \epsilon_i < 1$, $P(\sigma_0^2 K_{\mathcal{DM}}(\boldsymbol{\theta}_0)) \equiv P(\sigma_1^2 K_{\mathcal{DM}}(\boldsymbol{\theta}_1))$ if $\sigma_0^2 \ell_0 \epsilon_0^{-2\nu} = \sigma_1^2 \ell_1 \epsilon_1^{-2\nu}$,*

where $\ell_i = \frac{\zeta_i^{2\nu-d} \upsilon_i^{2\nu-1} \Gamma(\nu)^2}{\Gamma(\nu-d/2)\Gamma(\nu-1/2)}$.

We now provide a characterization of the compatibility of two $\mathcal{DGW}$ functions having common spatial smoothness at the origin. In the following, we assume that $\lim\limits_{z,\tau\to\infty} z/\tau = k < \infty$, with k a positive constant.

Theorem 2. *Let $\boldsymbol{\chi}_i = (\mu, \kappa, \beta_i, \delta, \lambda, \gamma_i)^\top$, $i = 0, 1$, with $\gamma_i > 0$ and $\beta_i > 0$.Let $\eta_i := (d+1)/2 + \kappa_i$. For a given $\kappa \geq 0$, let $P(\sigma_i^2 K_{\mathcal{DGW}}(\boldsymbol{\chi}_i))$, $i = 0, 1$, be two* *zero mean Gaussian measures, and let* $\mu_i > \eta_i+1+\alpha_i$, $\lambda_i > \max((d+3)/2, 2\kappa_i+3)$ *and* $\delta_i > (d+1)/2$. *For any bounded infinite set* $D \times \mathcal{T} \subset \mathbb{R}^d \times \mathbb{R}$, $d = 1, 2$, $P(\sigma_0^2 K_{\mathcal{DGW}}(\boldsymbol{\chi}_0)) \equiv P(\sigma_1^2 K_{\mathcal{DGW}}(\boldsymbol{\chi}_1))$ *on the paths of* $Z(\boldsymbol{s}, t), (\boldsymbol{s}, t) \in D \times \mathcal{T}$ *if and only if*

$$\frac{\sigma_0^2 \beta_0^{-(2\kappa+1)}}{\gamma_0^{\delta}} = \frac{\sigma_1^2 \beta_1^{-(2\kappa+1)}}{\gamma_1^{\delta}}. \tag{B.1}$$

Proofs are deferred to Section C. Given Theorems 1 and 2, it becomes natural to ask whether the $P(\sigma^2 K_{\mathcal{DM}}(\cdot,\cdot;\boldsymbol{\theta}))$ and $P(\sigma^2 K_{\mathcal{DGW}}(\cdot,\cdot;\boldsymbol{\chi}))$ might be equivalent on the paths of a Gaussian field Z defined over the product space $\mathcal{D} \times \mathcal{T}$ being a bounded set of $\mathbb{R}^d \times \mathbb{R}$. The following result provides an answer when $d = 1$ or $d = 2$.

Theorem 3. *For given* $\mu > \eta+1+\alpha$, $\lambda > \max((d+3)/2, 2\kappa+3)$, $\delta = 1+2\kappa$, $\epsilon \in (0, 1]$ *and* $\boldsymbol{\chi}_2 = (\mu, \kappa, \beta, \delta, \lambda, \gamma)^\top$, *let* $P(\sigma_2^2 K_{\mathcal{DGW}}(\boldsymbol{\chi}_2))$ *and* $P(\sigma^2 K_{\mathcal{DM}}(\boldsymbol{\theta}))$ *be two zero mean Gaussian measures. Let* $\ell(\boldsymbol{\theta})$ *be as defined at (2.5). If*

$$\sigma_2^2 \varrho_{\lambda,\eta} c_3^{\varsigma} L^{\varsigma} \beta^{-(1+2\kappa)} = \sigma^2 \ell(\boldsymbol{\theta}) \epsilon^{-2\nu}, \qquad \textit{with} \qquad \nu = \eta,$$

then, for any bounded infinite set $D \times \mathcal{T} \subset \mathbb{R}^d \times \mathbb{R}$, $d = 1, 2$, $P(\sigma_2^2 K_{\mathcal{DGW}}(\boldsymbol{\chi}_2)) \equiv P(\sigma^2 K_{\mathcal{DM}}(\boldsymbol{\theta}))$ *on the paths of* $Z(\boldsymbol{s}, t), (\boldsymbol{s}, t) \in D \times \mathcal{T}$.

C. Proofs

C.1 Proof of Theorem 2

We first consider the case $\kappa > 0$. Let us start with the sufficient part of the assertion. By Theorem 1 (Point *3*), there exist constants c_i and C_i, $i = 0, 1$ such that

$$c_i \leq z^{2\eta}\tau^{\delta+1} f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}_i) \leq C_i, \qquad (z,\tau) \in (0,\infty)^2, \quad \text{with} \quad \boldsymbol{\chi}_i = (\mu,\kappa,\beta_i,\delta,\lambda,\gamma_i)^\top.$$

We proceed by direct construction, and, using Theorem 1 (Points *2* and *3*), we find that, as $z \to \infty$ and $\tau \to \infty$ with z/τ converging to a constant k,

$$\begin{aligned}
&\left|\frac{\sigma_1^2 f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}_1) - \sigma_0^2 f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}_0)}{\sigma_0^2 f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}_0)}\right| \\
&\leq c_0^{-1} z^{2\eta}\tau^{\delta+1} \times \Bigg| \sigma_1^2 \beta_1^d L^{\varsigma} c_3^{\varsigma} (z\beta_1)^{-2\eta} \times \\
&\Bigg(\left[\varrho_{\lambda,\eta}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right] + \left[\varrho_{\lambda,\eta+1}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right]\mathcal{O}(z^{-2})\Bigg) \\
&+ \left[\varrho_{\lambda,0}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right]\mathcal{O}(z^{-(\mu+\eta)}) - \sigma_0^2 \beta_0^d L^{\varsigma} c_3^{\varsigma} (z\beta_0)^{-2\eta} \Bigg(\left[\varrho_{\lambda,\eta}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right] \\
&+ \left[\varrho_{\lambda,\eta+1}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right]\mathcal{O}(z^{-2})\Bigg) + \left[\varrho_{\lambda,0}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right]\mathcal{O}(z^{-(\mu+\eta)})\Bigg|.
\end{aligned}$$

For some positive and finite c, condition (A.1) can be written as

$$\int_{\mathcal{A}} z^{d-1} \left(\frac{\sigma_1^2 f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}_1) - \sigma_0^2 f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}_0)}{\sigma_0^2 f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}_0)} \right)^2 \mathrm{d}z\mathrm{d}\tau < \infty, \qquad \text{(C.1)}$$

where $\mathcal{A}$ has been defined around Equation (A.1).

It is easy to verify that (C.1) is satisfied if

$$\frac{\sigma_0^2\beta_0^{-(2\kappa+1)}}{\gamma_0^\delta} = \frac{\sigma_1^2\beta_1^{-(2\kappa+1)}}{\gamma_1^\delta},$$

for $\mu > \eta + 1 + \alpha$, $\delta > \frac{d+1}{2}$, $\lambda > 2\kappa + 3$ and $d = 1, 2$. Following the steps in the proof of Theorem 1 of Zhang (2004), we obtain the necessity part. □

C.2 Proof of Theorem 3

We need to find conditions such that for some positive and finite c,

$$\int_{\mathcal{A}} z^{d-1}\left(\frac{\sigma_2^2 f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}_2) - \sigma^2 f_{\mathcal{DM}}(z,\tau;\boldsymbol{\theta})}{\sigma^2 f_{\mathcal{DM}}(z,\tau;\boldsymbol{\theta})}\right)^2 \mathrm{d}z\mathrm{d}\tau < \infty, \tag{C.2}$$

where $\mathcal{A}$ depends on c as specified through (A.1). It is known that $f_{\mathcal{DM}}(z,\tau;\boldsymbol{\theta})z^{2\nu}$ is bounded away from 0 and ∞ as $z, \tau \to \infty$, with z/τ converging to a constant k (Ip and Li, 2017). Using Theorem 1 (Point *2*) and Theorem 2 (Point *2*) when $\epsilon \in (0, 1]$, we have, as $z, \tau \to \infty$,

$$\left|\frac{\sigma_2^2 f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}_2) - \sigma^2 f_{\mathcal{DM}}(z,\tau;\boldsymbol{\theta})}{\sigma^2 f_{\mathcal{DM}}(z,\tau;\boldsymbol{\theta})}\right| = \left|\frac{\sigma_2^2 f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}_2)}{\sigma^2 f_{\mathcal{DM}}(z,\tau;\boldsymbol{\theta})} - 1\right|$$

$$= \Bigg|\ell(\boldsymbol{\theta})^{-1}(\epsilon r\tau)^{2\nu}\Big\{\beta^d L^\varsigma c_3^\varsigma (z\beta)^{-2\eta}\times$$

$$\left(\left[\varrho_{\lambda,\eta}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right] + \left[\varrho_{\lambda,\eta+1}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right]\mathcal{O}(z^{-2})\right)$$

$$+\left[\varrho_{\lambda,0}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right]\mathcal{O}(z^{-(\mu+\eta)})\Big\}\times$$

$$\left(1 + \frac{\nu\zeta^2\upsilon^2}{\epsilon^2 r^2\tau^2} + \frac{\nu\upsilon^2}{\epsilon^2\tau^2} + \frac{\nu\zeta^2}{\epsilon^2 r^2} + \mathcal{O}(\tau^{-4}r^{-4})\right) - 1\Bigg|,$$

and, $d = 1, 2$ if $\nu = \eta$, $\delta = 1 + 2\kappa$, $\lambda > 2\kappa + 3$, $\mu \geq \eta + 1 + \alpha$ and

$$\sigma_2^2\varrho_{\lambda,\eta}c_3^\varsigma L^\varsigma\beta^{-(1+2\kappa)} = \sigma^2\ell(\boldsymbol{\theta})\epsilon^{-2\nu},$$

so that (C.2) holds. $\square$

C.3 Proof of Theorem 1

The proof of Points *1*, *2* and *3* are based results on results of Bevilacqua et al. (2019) and Lim and Teo (2009). In particular, to prove Point *1*, we note that

$$f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}) \;=\; \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}} e^{-\mathrm{i}\tau t}\int_{\mathbb{R}^d} e^{-\mathrm{i}<z\mathrm{e}_1,\boldsymbol{x}>} K_{\mathcal{DGW}}(\|\boldsymbol{x}\|,t;\boldsymbol{\chi})\mathrm{d}\boldsymbol{x}\mathrm{d}t,$$

where e_1 denotes a unit vector. Next, using the arguments in Bevilacqua et al. (2019), and by standard Fourier calculus, we get

$$f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}) \;=\; \frac{\beta^d L^{\varsigma}}{2\pi}\int_{\mathbb{R}} e^{-\mathrm{i}\tau t} h_{\delta,\gamma}(t)^{d+\lambda}{}_1F_2\Big(\eta;\eta+\frac{\mu}{2},\eta+\frac{\mu}{2}+\frac{1}{2};-\frac{(z\beta\; h_{\delta,\gamma}(t))^2}{4}\Big)\mathrm{d}t.$$

Using the definition of the function ${}_1F_2$ in concert with the fact that standard arguments allow for exchange of series and integrals, we obtain

$$f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}) \;=\; \beta^d L^{\varsigma}\sum_{j=0}^{\infty}\frac{(-1)^j(\eta)_j(z\beta/2)^{2j}}{j!(\eta+\mu/2)_j(\eta+\mu/2+1/2)_j}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mathrm{i}t\tau}\; h_{\delta,\gamma}(t)^{d+\lambda+2j}\mathrm{d}t.$$

We can now make use of dominated convergence (because $\delta(\lambda+d)>1$) to exchange series and integrals and replace the series within the integral by ${}_1F_2$, to obtain

$$f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}) = -\beta^d\tau^{1/2}\sqrt{2}\gamma^{3/2}\pi^{-3/2}L^{\varsigma}\times$$
$$\times\Im\left(\int_0^{\infty}\frac{\mathcal{K}_{1/2}(\gamma t\tau){}_1F_2\Big(\eta;\eta+\frac{\mu}{2},\eta+\frac{\mu}{2}+\frac{1}{2};-\frac{(z\beta(1+e^{i\pi\delta/2}t^{\delta})^{-1})^2}{4}\Big)}{(1+\exp(\mathrm{i}\frac{\pi\delta}{2})t^{\delta})^{\lambda+d}}t^{1/2}\mathrm{d}t\right).$$

Point *2* can be verified by showing that, as $z \to \infty$ and for $\eta = \frac{d+1}{2} + \kappa$,

$$
\begin{aligned}
f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}) &= -\beta^d \tau^{1/2}\sqrt{2}\gamma^{3/2}\pi^{-3/2} L^{\varsigma} \Im\Bigg(\int_0^\infty \frac{\mathcal{K}_{1/2}(\gamma t \tau)}{(1+\exp(i\frac{\pi\delta}{2})t^\delta)^{\lambda+d}} \\
&\times \Big(c_3^{\varsigma} (z\beta(1+e^{i\pi\delta/2}t^\delta)^{-1})^{-2\eta}\Big(1+\mathcal{O}((1+e^{i\pi\delta/2}t^\delta)^2 z^{-2})\Big) \\
&+ \mathcal{O}((1+e^{i\pi\delta/2}t^\delta)^{\mu+\eta} z^{-(\mu+\eta)})\Big) t^{1/2}\mathrm{d}t\Bigg).
\end{aligned}
$$

Note that we replaced $c_4^{\varsigma} z^{-(\mu+\eta)}\left\{\cos(z\beta - c_5^{\varsigma}) + \mathcal{O}\left(z^{-1}\right)\right\}$ by $\mathcal{O}\left(z^{-(\mu+\eta)}\right)$.

Now, by letting τ tend to infinity we obtain

$$
\begin{aligned}
&f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}) = \beta^d L^{\varsigma} c_3^{\varsigma}(z\beta)^{-2\eta}\Bigg(\left[\varrho_{\lambda,\eta}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right] \\
&+ \left[\varrho_{\lambda,\eta+1}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right]\mathcal{O}(z^{-2})\Bigg) + \left[\varrho_{\lambda,0}\tau^{-(1+\delta)} - \mathcal{O}\left(\tau^{-(1+2\delta)}\right)\right]\mathcal{O}(z^{-(\mu+\eta)}).
\end{aligned}
$$

Point *3* comes from Bevilacqua et al. (2019). The last point is a direct application of the arguments from Lim and Teo (2009)[Proposition 3.2] and Bevilacqua et al. (2019)[Theorem 2]. □

C.4 Proof of Theorem 3

The proof of Point *1.* follows the same arguments of the proof of Theorem 3 in Zhang (2004), so that we omit it.

For the proof of Point *2.*, we follow the arguments in Wang and Loh (2011) and Wang (2010), applied to the $\mathcal{DGW}$ case. As in Wang and Loh (2011), without loss of generality, we assume $D \times \mathcal{T} = [0,T]^{d+1}$, $0 < T < \infty$, is a bounded subset of $\mathbb{R}^{d+1}$, $d = 1,2$. Let $R_{\boldsymbol{\chi}} = R_{nm}(\boldsymbol{\chi})$ and $\hat{\sigma}^2_{nm} = \hat{\sigma}^2_{nm}(\boldsymbol{\chi})$

for notation convenience, and let σ^2, β and γ be three positive constants such that $\sigma_0^2\beta_0^{-(2\kappa+1)}/\gamma_0^\delta = \sigma^2\beta^{-(2\kappa+1)}/\gamma^\delta$. Then, we have

$$\begin{aligned}
&\sqrt{nm}\Big(\hat{\sigma}_n^2\beta^{-(1+2\kappa)}\gamma^{-\delta} - \sigma_0^2\beta_0^{-(1+2\kappa)}\gamma_0{-\delta}\Big) \\
&= \frac{\sigma_0^2\beta_0^{-(1+2\kappa)}\gamma^{-\delta}}{\sqrt{nm}}\left(\frac{1}{\sigma^2}\boldsymbol{Z}_{nm}^\top R_{\boldsymbol{\chi}}^{-1}\boldsymbol{Z}_{nm} - \frac{1}{\sigma_0^2}\boldsymbol{Z}_{nm}^\top R_{\boldsymbol{\chi}_0}^{-1}\boldsymbol{Z}_{nm}\right) \\
&\quad + \frac{\sigma_0^2\beta_0^{-(1+2\kappa)}\gamma_0^{-\delta}}{\sqrt{nm}}\left(\frac{1}{\sigma_0^2}\boldsymbol{Z}_{nm}^\top R_{\boldsymbol{\chi}_0}^{-1}\boldsymbol{Z}_{nm} - nm\right).
\end{aligned}$$

Under the measure $P(\sigma_0^2 K_{\mathcal{DGW}}(\boldsymbol{\theta}_0))$, we have $\sigma_0^{-2}\boldsymbol{Z}_{nm}^\top R_{\boldsymbol{\chi}_0}^{-1}\boldsymbol{Z}_{nm} \sim \chi^2_{nm}$ (a centered chi-squared distribution with $n \times m$ degrees of freedom) and

$$\frac{\sigma_0^2\beta_0^{-(1+2\kappa)}\gamma_0^{-\delta}}{\sqrt{nm}}\left(\frac{1}{\sigma_0^2}\boldsymbol{Z}_{nm}^\top R_{\boldsymbol{\chi}_0}^{-1}\boldsymbol{Z}_{nm} - nm\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 2\sigma_0^2\beta_0^{-(1+2\kappa)}\gamma_0^{-\delta})$$

as $nm \to \infty$. To prove the result, it is sufficient to show that

$$\frac{1}{\sqrt{nm}}\left(\frac{1}{\sigma^2}\boldsymbol{Z}_{nm}^\top R_{\boldsymbol{\chi}}^{-1}\boldsymbol{Z}_{nm} - \frac{1}{\sigma_0^2}\boldsymbol{Z}_{nm}^\top R_{\boldsymbol{\chi}_0}^{-1}\boldsymbol{Z}_{nm}\right) \xrightarrow{p} 0, \text{ as } nm \to \infty, \tag{C.3}$$

where $\xrightarrow{p}$ denotes convergence under $P(\sigma_0^2 K_{\mathcal{DGW}}(\boldsymbol{\theta}_0))$.

Specifically, we need to show that for any $\vartheta > 0$,

$$\begin{aligned}
&P_{\sigma_0^2,\beta_0,\gamma_0}\left(\frac{1}{\sqrt{nm}}\left|\frac{1}{\sigma^2}\boldsymbol{Z}_{nm}^\top R_{\boldsymbol{\chi}}^{-1}\boldsymbol{Z}_{nm} - \frac{1}{\sigma_0^2}\boldsymbol{Z}_{nm}^\top R_{\boldsymbol{\chi}_0}^{-1}\boldsymbol{Z}_{nm}\right| > \vartheta\right) \\
&= P_{\sigma_0^2,\beta_0,\gamma_0}\left(\frac{1}{\sqrt{n}}\left|\sum_{k=1}^{nm}(\lambda_{k,nm}^{-1} - 1)Y_k^2\right| > \vartheta\right) \longrightarrow 0, \text{ as } n, m \to \infty,
\end{aligned} \tag{C.4}$$

where Y_k and $\lambda_{k,nm}$ are defined below.

Following Wang and Loh (2011), the quantity in (C.3) can be written as

$$\frac{1}{\sqrt{nm}}\sum_{k=0}^{nm}(\lambda_{k,nm}^{-1}-1)Y_k^2, \tag{C.5}$$

where $(Y_1,\ldots,Y_{nm})^\top \sim \mathcal{N}_{nm}(0,I_{nm})$ under $P(\sigma_0^2 K_{\mathcal{DGW}}(\boldsymbol{\theta}_0))$ and $\lambda_{k,nm}$, $k=1,\ldots,nm$, satisfy

$$\sigma^2\big[\sigma_0^{-1}R_{\boldsymbol{\chi}_0}^{-1/2}\big]^\top R_{\boldsymbol{\chi}}\big[\sigma_0^{-1}R_{\boldsymbol{\chi}_0}^{-1/2}\big] = \mathrm{diag}(\lambda_{k,nm})_{k\in\{1,\ldots,nm\}}.$$

Here, I_{nm} denotes the identity matrix of dimension $nm\times nm$. For the rest of the proof $|\cdot|$ denotes the Euclidean norm, and $|\mathbf{x}|_{\max}=\max\{|x_1|,\ldots,|x_{d+1}|\}$ with $\mathbf{x}=(x_1,\ldots,x_{d+1})^\top\in\mathbb{R}^{d+1}$.

Let $\xi_0:\mathbb{R}^d\to\mathbb{R}$ be defined as $\xi_0(\boldsymbol{\omega},v)=\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathrm{e}^{-i(\mathbf{x}^\top\boldsymbol{\omega}+vu)}c_0(\mathbf{x},u)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{u}$, where $c_0(\mathbf{x},u)=|(\mathbf{x},u)|^{\zeta^*-d-1}\mathbf{1}_{\{|(\mathbf{x},u)|\le 1\}}$ and $\zeta^*=\frac{\delta+d+1+2\kappa}{2p}$, with $p=\lfloor\delta+d+1+2\kappa\rfloor+1$. Here, $\lfloor x\rfloor$ is the largest integer less than or equal to x. Next, we show that ξ_0 is a positive function for $d\ge 1$.

Lemma 1. *The function $\xi_0:\mathbb{R}^{d+1}\to\mathbb{R}$ is a continuous, isotropic strictly positive function and $\xi_0(\boldsymbol{\omega},v)\asymp|(\boldsymbol{\omega},v)|^{-\zeta^*}$ as $|\boldsymbol{\omega}|,|v|\to\infty$.*

Proof of Lemma 1. Let U_d be the uniform probability measure on $S^d=\{\mathbf{u}\in$

$\mathbb{R}^{d+1} : |\mathbf{u}| = 1\}$. By isotropy, we have for all $(\boldsymbol{\omega}, v) \in \mathbb{R}^{d+1}$

$$
\begin{aligned}
\xi_0(\boldsymbol{\omega}, v) &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathrm{e}^{-i(\mathbf{x}^\top \boldsymbol{\omega} + vu)} c_0(\mathbf{x}, u) \mathrm{d}\mathbf{x}\mathrm{d}u \\
&= \int_{|(\mathbf{x},u)|\le 1} \int_{S^d} e^{-i(|(\boldsymbol{\omega},v)|\mathbf{y}^\top)(\mathbf{x},u)} |(\mathbf{x}, u)|^{-\zeta^*-d-1} U_d(\mathrm{d}\mathbf{y}) \mathrm{d}(\mathbf{x}, u) \\
&= (2\pi)^{\frac{d+1}{2}} |(\boldsymbol{\omega}, v)|^{\frac{1-d}{2}} \int_0^1 r^{\zeta^* - \frac{d+1}{2}} J_{\frac{d-1}{2}}(|(\boldsymbol{\omega}, v)| r) \mathrm{d}r \\
&= (2\pi)^{\frac{d+1}{2}} |(\boldsymbol{\omega}, v)|^{-\zeta^*} \int_0^{|(\boldsymbol{\omega},v)|} r^{\zeta^* - \frac{d+1}{2}} J_{\frac{d-1}{2}}(r) \mathrm{d}r \\
&= 2(\zeta^*)^{-1} \pi^{(d+1)/2} \Gamma\left(\frac{d+1}{2}\right)^{-1} {}_1F_2\Big(\zeta^*/2; \zeta^*/2 + 1, (d+1)/2; -(|(\boldsymbol{\omega}, v)|/2)^2\Big).
\end{aligned}
\tag{C.6}
$$

From Lemmas 2 and 3 of Bevilacqua et al. (2019) we have ξ_0 is a continuous, isotropic and strictly positive on $\mathbb{R}^{d+1}$.

Moreover, from Luke (1969, p. 203 (4)) we have, as $|\boldsymbol{\omega}| \to \infty$,

$$
\begin{aligned}
{}_1F_2(\zeta/2; \zeta/2 + 1, d/2; -(|\boldsymbol{\omega}|/2)^2) &= \frac{2^\zeta \Gamma(d/2)}{\Gamma(d/2 - \zeta/2)} |\boldsymbol{\omega}|^{-\zeta} \\
&+ \frac{\Gamma(d/2)}{\pi^{1/2}\Gamma(\zeta/2)} |\boldsymbol{\omega}|^{-(1+d)/2} \exp\Big(4w_3|\boldsymbol{\omega}|^{-2} + \mathcal{O}(|\boldsymbol{\omega}|^{-4})\Big) \\
&\times \cos\Big(|\boldsymbol{\omega}| - \frac{\pi(d+1)}{2} - 2w_4|\boldsymbol{\omega}|^{-1} - 8w_5|\boldsymbol{\omega}|^{-3} + \mathcal{O}(|\boldsymbol{\omega}|^{-5})\Big),
\end{aligned}
$$

where $\{w_k\}_{k=3,4,5}$ are constants not depending on $\boldsymbol{\omega} \in \mathbb{R}^d$. Thus

$$
{}_1F_2(\zeta^*/2; \zeta^*/2 + 1, (d+1)/2; -(|(\boldsymbol{\omega}, v)|/2)^2) \asymp \frac{2^{\zeta^*}\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d+1}{2} - \zeta^*/2)} |(\boldsymbol{\omega}, v)|^{-\zeta^*}.
$$

Under the assumption that there exists a positive constant $\hbar$ such that $\lim_{|\boldsymbol{\omega}|,|v|\to\infty} \frac{|\boldsymbol{\omega}|}{|v|} = \hbar$, in concert with Equation (C.6), we have that $\xi_0(\boldsymbol{\omega}, v) \asymp |(\boldsymbol{\omega}, v)|^{-\zeta^*}$, as $|\boldsymbol{\omega}|, |v| \to \infty$. The proof is completed. $\square$

Let $\xi_1(\boldsymbol{\omega}, v) = \int_{\mathbb{R}^{d+1}} e^{-i(\boldsymbol{\omega}^\top \mathbf{x} + \mathbf{v}\mathbf{u})} c_1^*(\mathbf{x}, u) \mathrm{d}\mathbf{x}\mathrm{d}u = \xi_0(\boldsymbol{\omega}, v)^{2p}$, for all $(\boldsymbol{\omega}, v) \in \mathbb{R}^{d+1}$, where $c_1^* = c_0 * \ldots * c_0$ denote the $2p$-fold self convolution of the function c_0^*. We define

$$\eta^*(\boldsymbol{\omega}, v) = \frac{f_{\mathcal{DGW}}(|\boldsymbol{\omega}|, |v|; \boldsymbol{\theta}) - f_{\mathcal{DGW}}(|\boldsymbol{\omega}|, |v|; \boldsymbol{\theta}_0)}{\xi_1(\boldsymbol{\omega}, v)}, \qquad \forall (\boldsymbol{\omega}, v) \in \mathbb{R}^{d+1}.$$

From Theorem 1, Point 3 and Lemma 1, we have

$$\frac{f_{\mathcal{DGW}}(|\boldsymbol{\omega}|, |v|; \boldsymbol{\theta}_0)}{\xi_1(\boldsymbol{\omega}, v)} \asymp 1, \qquad \text{as } |\boldsymbol{\omega}|, |v| \to \infty.$$

Furthermore, this ratio is well defined and continuous on any arbitrary compact interval of $\mathbb{R}_+$ with $\xi_1 > 0$, so there exist two constants c_{ξ_1} and C_{ξ_1} not depending on $|\boldsymbol{\omega}|$, neither on $|v|$, such that

$$c_{\xi_1} \le \frac{f_{\mathcal{DGW}}(|\boldsymbol{\omega}|, |v|; \boldsymbol{\theta}_0)}{\xi_1(\boldsymbol{\omega}, v)} \le C_{\xi_1}, \qquad \text{as } |\boldsymbol{\omega}|, |v| \to \infty. \tag{C.7}$$

Thus, for an arbitrary constant $C_\eta > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^{d+1}} \eta^*(\boldsymbol{\omega}, v)^2 \mathrm{d}\boldsymbol{\omega}\mathrm{d}v = \frac{4\pi^{d/2}}{\Gamma(d/2)} \Bigg[& \int_0^{C_\eta} \int_0^{C_\eta} \left(\frac{f_{\mathcal{DGW}}(r, t; \boldsymbol{\theta}) - f_{\mathcal{DGW}}(r, t; \boldsymbol{\theta}_0)}{\xi_1(\boldsymbol{r}, v)} \right)^2 \mathrm{d}r\mathrm{d}t \\ & + \int_0^{C_\eta} \int_{C_\eta}^{\infty} \left(\frac{f_{\mathcal{DGW}}(r, t; \boldsymbol{\theta}) - f_{\mathcal{DGW}}(r, t; \boldsymbol{\theta}_0)}{\xi_1(\boldsymbol{r}, v)} \right)^2 \mathrm{d}r\mathrm{d}t \\ & + \int_{C_\eta}^{\infty} \int_0^{C_\eta} \left(\frac{f_{\mathcal{DGW}}(r, t; \boldsymbol{\theta}) - f_{\mathcal{DGW}}(r, t; \boldsymbol{\theta}_0)}{\xi_1(\boldsymbol{r}, v)} \right)^2 \mathrm{d}r\mathrm{d}t \\ & + \int_{C_\eta}^{\infty} \int_{C_\eta}^{\infty} \left(\frac{f_{\mathcal{DGW}}(r, t; \boldsymbol{\theta}) - f_{\mathcal{DGW}}(r, t; \boldsymbol{\theta}_0)}{\xi_1(\boldsymbol{r}, v)} \right)^2 \mathrm{d}r\mathrm{d}t \Bigg]. \end{aligned} \tag{C.8}$$

where $(\boldsymbol{r}, v) \in \mathbb{R}^d \times \mathbb{R}$, with $|\boldsymbol{r}| = r$, $t = |v|$ and $\frac{r}{t} \to \hbar$ when $r, t \to \infty$.

Since $d = 1, 2$, $\mu > \eta + 1 + \alpha$, $\lambda > 2\kappa + 3$, $\delta > \frac{d+1}{2}$ and $\sigma^2 \beta^{-(1+2\kappa)} \gamma^{-\delta} =$

$\sigma_0^2\beta_0^{-(1+2\kappa)}\gamma_0^{-\delta}$, all terms of Equation (C.8) are finite. Thus, η^* is square integrable. From the theory of Fourier transforms of $\mathrm{L}^2(\mathbb{R}^{d+1})$ functions, there exists a square integrable function $g : \mathbb{R}^{d+1} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^{d+1}} \left(\eta^*(\boldsymbol{\omega}, v) - \hat{g}_k(\boldsymbol{\omega}, v)\right)^2 \mathrm{d}\boldsymbol{\omega}\mathrm{d}v \to 0, \qquad \text{as } k \to \infty,$$

where

$$\hat{g}_k(\boldsymbol{\omega}, v) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}(\boldsymbol{\omega}^\top \mathbf{x} + vu)} g(\mathbf{x}, u) \mathbf{1}_{\{|(\mathbf{x},u)|_{\max} \le k\}} \mathrm{d}\mathbf{x}\mathrm{d}u, \qquad \forall (\boldsymbol{\omega}, v) \in \mathbb{R}^{d+1}, \quad k > 0. \tag{C.9}$$

In order to illustrate the following Lemma, some notation is needed. According to Equation (2.44) of Wang (2010), define

$$e_{nm}(\mathbf{x}, u) = \frac{1}{C_e \epsilon_{nm}^d \epsilon_{nm}^\top} \tilde{c}_1\left(\frac{\mathbf{x}}{\epsilon_{nm}}, \frac{u}{\epsilon_{nm}}\right), \qquad \forall (\mathbf{x}, u) \in \mathbb{R}^d \times \mathbb{R}, \tag{C.10}$$

and

$$\tilde{\xi}_1(\boldsymbol{\omega}, v) = \int_{\mathbb{R}^{d+1}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{\omega}^\top \mathbf{x} + uv)} \tilde{c}_1(\mathbf{x}, v) \mathrm{d}\mathbf{x}\mathrm{d}u,$$

where $C_e = \int_{\mathbb{R}^{d+1}} \tilde{c}_1(\mathbf{x}, u)\mathrm{d}\mathbf{x}\mathrm{d}u$, $\tilde{c}_1 = \tilde{c}_0 * \ldots * \tilde{c}_0$ with

$$\tilde{c}_0(\mathbf{x}, u) = |(\mathbf{x}, u)|^{\frac{a+d+1}{2p_a} - d} \mathbf{1}_{\{|(\mathbf{x},u)| \le 1\}},$$

with $p_a = \lfloor a + d + 1 \rfloor + 1$. Here a is an arbitrary positive constant. Write

$$\hat{e}_{nm}(\boldsymbol{\omega}, v) = \int_{\mathbb{R}^{d+1}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{\omega}^\top \mathbf{x} + uv)} \mathrm{e}_{nm}(\mathbf{x}, u)\mathrm{d}\mathbf{x}\mathrm{d}u = \frac{\tilde{\xi}_1(\epsilon_{nm}\boldsymbol{\omega}, \epsilon_{nm}u)}{C_e}$$

for the Fourier transform of e_{nm}. Note that there exists a constant $C_{\hat{e}}$ not depending on $\boldsymbol{\omega}$, v, n and m such that

$$|\hat{e}_{nm}(\boldsymbol{\omega}, v)| \leq \frac{C_{\hat{e}}}{(1+\epsilon_n|(\boldsymbol{\omega}, v)|)^{a+d+1}}, \quad \forall(\boldsymbol{\omega}, v) \in \mathbb{R}^{d+1}. \tag{C.11}$$

Lemma 2. *Let* $(\epsilon_{nm})_{nm} : \epsilon_{nm} \in (0,1]$, $\forall n, m \in \mathbb{N}$, *and additionally,* $\epsilon_{nm} \to 0$, *when* $n, m \to \infty$. *Let* g *as in Equation* (C.9), e_{nm} *as in Equation* (C.10), *and* ι_0 *a constant satisfying* $0 < \iota_0 < \min\{2(\mu-\eta-\alpha), 2\delta-d-1, 3-d\}$. *Then, there exists a constant* C_{ι_0} *such that*

$$\int_{\mathbb{R}^d} \left|e_{nm} * g(\mathbf{x}, u) - g(\mathbf{x}, u)\right|^2 \mathrm{d}\mathbf{x} \leq C_{\iota_0}\epsilon_{nm}^{\iota_0}. \tag{C.12}$$

Proof. Lemma 2 can be proved by noting that

$$\begin{aligned}\int_{\mathbb{R}^{d+1}} \left|g(\mathbf{x}-\mathbf{y}, v-u) - g(\mathbf{x}, v)\right|^2 \mathrm{d}\mathbf{x} &= \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \left|(e^{-i(\boldsymbol{w}^\top \mathbf{y}+su)} - 1)\eta^*(\boldsymbol{w}, s)\right|^2 \mathrm{d}\boldsymbol{w}\mathrm{d}s \\ &\leq \frac{2^{2-\iota_0}|(\mathbf{y}, u)|^{\iota_0}}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} |(\boldsymbol{w}, u)|^{\iota_0}|\eta^*(\boldsymbol{w}, u)|^2 \mathrm{d}\boldsymbol{w}\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}&\left[\int_{\mathbb{R}^{d+1}} \left|e_n * g(\mathbf{x}, v) - g(\mathbf{x}, v)\right|^2 \mathrm{d}\mathbf{x}\mathrm{d}v\right]^{1/2} \\ &= \left[\int_{\mathbb{R}^{d+1}} \left|\int_{|(\mathbf{y},u)|\leq 2m_a\epsilon_n} (g(\mathbf{x}-\mathbf{y}, v-u) - g(\mathbf{x}, v))e_n(\mathbf{y}, u)\mathrm{d}\mathbf{y}\mathrm{d}u\right|^2 \mathrm{d}\mathbf{x}\mathrm{d}v\right]^{1/2} \\ &\leq \frac{2^{(2-\iota_0)/2}(2m_a\epsilon_{nm})^{\iota_0/2}}{(2\pi)^{d+1}} \left[\int_{\mathbb{R}^{d+1}} |(\boldsymbol{w}, v)|^{\iota_0}|\eta^*(\boldsymbol{w}, v)|^2 \mathrm{d}\boldsymbol{w}\mathrm{d}v\right]^{1/2}.\end{aligned}$$

A simple calculus shows that $\int_{\mathbb{R}^{d+1}} |(\boldsymbol{w}, v)|^{\iota_0}|\eta(\boldsymbol{w}, v)|^2 \mathrm{d}\boldsymbol{w}\mathrm{d}v$ is finite if $\max\{0, 1/\lambda - d - \alpha\} < \iota_0 < \min\{2(\mu-\kappa-d/2-5/2), 3-d\}$ and the conditions of Theorem 2 hold. Thus, the proof is completed. □

Let $b(\mathbf{x},\mathbf{y}) = E_{f_{\mathcal{DGW}(\boldsymbol{\theta})}}[Z(\mathbf{x})Z(\mathbf{y})] - E_{f_{\mathcal{DGW}(\boldsymbol{\theta}_0)}}[Z(\mathbf{x})Z(\mathbf{y})]$, $\forall \mathbf{x},\mathbf{y} \in D \times \mathcal{T} = [0,T]^{d+1}$.

From Wang and Loh (2011, (2.24)), and observing that $supp(c_1) \subseteq [-2p,2p]^{d+1}$, we obtain for $\mathbf{x}$, $\mathbf{y} \in D$,

$$
\begin{aligned}
b(\mathbf{x},\mathbf{y}) &= (2\pi)^{d+1} \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} g(\mathbf{s}-\mathbf{t}) c_1(\mathbf{x}-\mathbf{s}) c_1(\mathbf{y}-\mathbf{t}) \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t} \\
&= (2\pi)^{d} \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} e_{nm} * g(\mathbf{s}-\mathbf{t}) c_1(\mathbf{x}-\mathbf{s}) c_1(\mathbf{y}-\mathbf{t}) \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t} \\
&\quad + (2\pi)^{d} \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} h^*_{nm}(\mathbf{s},\mathbf{t}) c_1(\mathbf{x}-\mathbf{s}) c_1(\mathbf{y}-\mathbf{t}) \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t},
\end{aligned}
$$

where $h^*_{nm}(\mathbf{s},\mathbf{t}) = \left[g(\mathbf{s}-\mathbf{t}) - e_n * g(\mathbf{s}-\mathbf{t})\right] \mathbf{1}_{\{|\mathbf{s}+\mathbf{t}|_{\max} \le 4p+2T\}}$, $\forall \mathbf{s}$, $\mathbf{t} \in \mathbb{R}^{d+1}$.

Let $\eta^{**}_{nm} : \mathbb{R}^d \to \mathbb{C}$ denote the Fourier transform of $g - e_{nm} * g$. This implies that

$$
\int_{\mathbb{R}^{d+1}} \left|\eta^{**}_{nm}(\boldsymbol{w}) - \hat{g}^*_{nm,k}(\boldsymbol{w})\right|^2 \mathrm{d}\boldsymbol{w} \to 0, \quad \text{as } k \to \infty, \tag{C.13}
$$

where $\hat{g}^*_{nm,k}(\boldsymbol{w}) = \int_{\mathbb{R}^{d+1}} e^{-i\boldsymbol{w}^\top \mathbf{x}} \left[g(\mathbf{x}) - e_{nm} * g(\mathbf{x})\right] \mathbf{1}_{\{|\mathbf{x}|_{\max} \le k\}} \mathrm{d}\mathbf{x}$.

Thus, as in Wang (2010, (2.27)), we have

$$
\begin{aligned}
&(2\pi)^{-d} \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} h^*_{nm}(\mathbf{s},\mathbf{t}) c_1(\mathbf{x}-\mathbf{s}) c_1(\mathbf{y}-\mathbf{t}) \mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t} \\
&\quad = (2\pi)^{-d} \int_{\mathbb{R}^{2(d+1)}} e^{i(\boldsymbol{\omega}^\top \mathbf{x} - \mathbf{v}^\top \mathbf{y})} \eta^{**}_{nm}\left(\frac{\boldsymbol{\omega}+\mathbf{v}}{2}\right) \Delta\left(\frac{\boldsymbol{\omega}-\mathbf{v}}{2}\right) \xi_1(\boldsymbol{\omega}) \xi_1(\mathbf{v}) \mathrm{d}\boldsymbol{\omega}\mathrm{d}\mathbf{v},
\end{aligned} \tag{C.14}
$$

where $\delta(\mathbf{x}) = 2^{-(d+1)} \int_{\mathbb{R}^{d+1}} \mathrm{e}^{-\mathrm{i}\mathbf{t}^\top \mathbf{x}} \mathbf{1}_{\{|\mathbf{t}|_{\max} \le 4p+2T\}} \mathrm{d}\mathbf{t}$, $\mathbf{x} \in \mathbb{R}^{d+1}$.

We observe that Δ is continuous and

$$
\int_{\mathbb{R}^{d+1}} \Delta(\boldsymbol{\omega})^2 \mathrm{d}\boldsymbol{\omega} < \infty. \tag{C.15}
$$

Now we define

$$h^{**}_{nm}(\mathbf{s},\mathbf{t}) = \int_{|u|_{\max}\le 2p+2p_a+T} e_{nm}(\mathbf{s}-\mathbf{u})g(\mathbf{u}-\mathbf{t})\mathrm{d}\mathbf{u}, \quad \forall \mathbf{s},\, \mathbf{t}\in\mathbb{R}^{d+1},$$

so that

$$\begin{aligned}
&(2\pi)^{-(d+1)}\int_{\mathbb{R}^{d+1}}\int_{\mathbb{R}^{d+1}} e_{nm}*g(\mathbf{s}-\mathbf{t})c_1(\mathbf{x}-\mathbf{s})c_1(\mathbf{y}-\mathbf{t})\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\\
&= (2\pi)^{-(d+1)}\int_{\mathbb{R}^{2(d+1)}} h^{**}_{nm}(\mathbf{s},\mathbf{t})c_1(\mathbf{x}-\mathbf{s})c_1(\mathbf{y}-\mathbf{t})\mathrm{d}\mathbf{s}\mathrm{d}\mathbf{t}\\
&= (2\pi)^{-(d+1)}\int_{\mathbb{R}^{2(d+1)}} \mathrm{e}^{i(\boldsymbol{\omega}^\top\mathbf{u}-\mathbf{v}^\top\mathbf{u})}\xi_1(\boldsymbol{\omega})\xi_1(\mathbf{v})\\
&\quad\times\Big(\int_{|\mathbf{u}|_{\max}\le 2p+2p_a+T}\mathrm{e}^{-i(\boldsymbol{\omega}^\top\mathbf{u}-\mathbf{v}^\top\mathbf{u})}\hat{e}_{nm}(\boldsymbol{\omega})\eta(\mathbf{v})\mathrm{d}\mathbf{u}\Big)\mathrm{d}\mathbf{v}\mathrm{d}\boldsymbol{\omega}.
\end{aligned}\tag{C.16}$$

It follows, from equations (C.14) and (C.16), that for $\mathbf{x},\mathbf{y}\in D\times\mathcal{T} = [0,T]^{d+1}$,

$$\begin{aligned}
b(\mathbf{x},\mathbf{y}) &= \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^{2(d+1)}}\mathrm{e}^{i(\boldsymbol{\omega}^\top\mathbf{x}-\mathbf{v}^\top\mathbf{y})}\eta^{**}_{nm}\Big(\frac{\boldsymbol{\omega}+\mathbf{v}}{2}\Big)\theta\Big(\frac{\boldsymbol{\omega}-\mathbf{v}}{2}\Big)\xi_1(\boldsymbol{\omega})\xi_1(\mathbf{v})\mathrm{d}\boldsymbol{\omega}\mathrm{d}\mathbf{v}\\
&\quad+\frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^{2(d+1)}}\mathrm{e}^{i(\boldsymbol{\omega}^\top\mathbf{x}-\mathbf{v}^\top\mathbf{y})}\xi_1(\boldsymbol{\omega})\xi_1(\mathbf{v})\\
&\quad\times\Big(\int_{|\mathbf{u}|_{\max}\le 2p+2p_a+T}\mathrm{e}^{-i(\boldsymbol{\omega}^\top\mathbf{u}-\mathbf{v}^\top\mathbf{u})}\hat{e}_{nm}(\boldsymbol{\omega})\eta^*(\mathbf{v})\mathrm{d}\mathbf{u}\Big)\mathrm{d}\mathbf{v}\mathrm{d}\boldsymbol{\omega}.
\end{aligned}$$

Let $\{\psi_1,\ldots,\psi_{nm}\}$ be as in (2.15) of Wang (2010). Then using (2.16) and (2.60) in Wang (2010), we have

$$<\psi_k,\psi_k>_{\widehat{\psi}_{\mathbf{x}}} - <\psi_k,\psi_k>_{\widehat{\psi}_{\mathbf{x}_0}} = \lambda_{k,nm}-1 = \varpi^*_{k,nm}+\varpi^{**}_{k,nm},$$

where

$$\varpi^{*}_{k,nm} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^{2(d+1)}} \psi_k(\boldsymbol{\omega})\overline{\psi_k(\mathbf{v})}\eta^{**}_{nm}\Big(\frac{\boldsymbol{\omega}+\mathbf{v}}{2}\Big)\theta\Big(\frac{\boldsymbol{\omega}-\mathbf{v}}{2}\Big)\xi_1(\boldsymbol{\omega})\xi_1(\mathbf{v})\mathrm{d}\boldsymbol{\omega}\mathrm{d}\mathbf{v}, \text{ and}$$

$$\varpi^{**}_{k,nm} = \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^{2(d+1)}} \psi_k(\boldsymbol{\omega})\overline{\psi_k(\mathbf{v})}\xi_1(\boldsymbol{\omega})\xi_1(\mathbf{v}) \times \Big(\int_{|\mathbf{u}|_{\max}\le 2p+2p_a+T} e^{-i(\boldsymbol{\omega}^\top\mathbf{u}-\mathbf{v}^\top\mathbf{u})}\hat{e}_{nm}(\boldsymbol{\omega})\eta^{*}(\mathbf{v})\mathrm{d}\mathbf{u}\Big)\mathrm{d}\mathbf{v}\mathrm{d}\boldsymbol{\omega}.$$

Using Bessel's inequality, we have

$$\sum_{k=1}^{nm}|\varpi^{*}_{k,nm}|^2 \le 2^{-d-1}\pi^{-d}\sup_{\mathbf{s}\in\mathbb{R}^{d+1}}\frac{\xi_1(\mathbf{s})^2}{\widehat{\psi}_{\boldsymbol{\chi}_0}(\mathbf{s})}\int_{\mathbb{R}^d}|\eta^{**}_{nm}(\boldsymbol{\omega})|^2\mathrm{d}\boldsymbol{\omega}\int_{\mathbb{R}^d}|\theta(\mathbf{v})|^2\mathrm{d}\mathbf{v},$$

and

$$\sum_{k=1}^{nm}|\varpi^{**}_{k,nm}| \le 2^{-d-2}\pi^{-(d+1)}\sup_{\mathbf{s}\in\mathbb{R}^{d+1}}\frac{\xi_1(\mathbf{s})^2}{\widehat{\psi}_{\boldsymbol{\chi}_0}(\mathbf{s})}\int_{|\mathbf{u}|_{\max}\le 2p+2p_a+T}\mathrm{d}\mathbf{u} \times\Big(\int_{\mathbb{R}^{d+1}}|\hat{e}_{nm}(\boldsymbol{\omega})|^2\mathrm{d}\boldsymbol{\omega}+\int_{\mathbb{R}^{d+1}}\eta^{*}(\mathbf{v})^2\mathrm{d}\mathbf{v}\Big).$$

From Equations (C.7), (C.8), (C.11), (C.12), (C.13), (C.15), there exists constants C, C_1, C_2 not depending on n such that

$$\sum_{k=1}^{nm}|\varpi^{*}_{k,nm}|^2 \le C\epsilon_{nm}^{\iota_0}, \qquad \sum_{k=1}^{nm}|\varpi^{*}_{k,nm}| \le \sqrt{Cnm\epsilon_{nm}^{\iota_0}} \quad \text{and}$$

$$\sum_{k=1}^{nm}|\varpi^{**}_{k,nm}| \le \big(C_1/\epsilon_{nm}^{d+1}+C_2\Upsilon\big)$$

with $\Upsilon = \int_{\mathbb{R}^{d+1}}\eta^{*}(\mathbf{v})^2\mathrm{d}\mathbf{v}$ being finite.

So we conclude that

$$\sum_{k=1}^{nm}|\lambda_{k,nm}-1| \le \sqrt{Cnm\epsilon_{nm}^{\iota_0}}+\frac{C_1}{\epsilon_{nm}^{d}}+C_2\Upsilon. \tag{C.17}$$

We further observe that there exist constants $c^{**} > 0$ and C^{**} such that

$$c^{**} \leq \frac{\widehat{\psi}_{\boldsymbol{\chi}}}{\widehat{\psi}_{\boldsymbol{\chi}_0}} \leq C^{**}, \qquad \forall \boldsymbol{\omega} \in \mathbb{R}^{d+1}.$$

This implies that $c^* \leq \lambda_{k,nm} \leq C^* \ \forall k \in \{1, 2, \ldots, nm\}$.

Finally for any $\vartheta > 0$, using Markov's inequality, (C.17), and using (C.5) we obtain

$$\begin{aligned}
&P_{\sigma_0^2,\beta_0,\gamma_0}\left(\frac{1}{\sqrt{nm}}\left|\frac{1}{\sigma^2}\boldsymbol{Z}_{nm}^{\top}R_{\boldsymbol{\chi}}^{-1}\boldsymbol{Z}_{nm} - \frac{1}{\sigma_0^2}\boldsymbol{Z}_{nm}^{\top}R_{\boldsymbol{\chi}_0}^{-1}\boldsymbol{Z}_{nm}\right| > \vartheta\right) \\
&\quad = P_{\sigma_0^2,\beta_0,\gamma_0}\left(\frac{1}{\sqrt{nm}}\left|\sum_{k=1}^{nm}(\lambda_{k,nm}^{-1} - 1)Y_k^2\right| > \vartheta\right) \\
&\quad \leq P_{\sigma_0^2,\beta_0,\gamma_0}\left(\frac{1}{\sqrt{nm}}\sum_{k=1}^{nm}|\lambda_{k,nm}^{-1} - 1|Y_k^2 > \vartheta\right) \\
&\quad \leq \frac{1}{\vartheta\sqrt{nm}}\sum_{k=1}^{nm}|\lambda_{k,nm}^{-1} - 1| \\
&\quad \leq \frac{1}{\vartheta\sqrt{nm}}\max_{i\in[1,nm]}\left\{\lambda_{i,nm}^{-1}\right\}\sum_{k=1}^{nm}|\lambda_{k,nm} - 1| \\
&\quad \leq \frac{C^{1/2}\epsilon_{nm}^{\iota_0/2}}{c^*\vartheta} + \frac{1}{c^*\vartheta(nm)^{1/2}}\left(C_1/\epsilon_{nm}^{d+1} + C_2\Upsilon\right).
\end{aligned} \tag{C.18}$$

Choose ϵ_{nm} such that $\epsilon_{nm} \to 0$ and $(nm)^{1/2}\epsilon_{nm}^{d+1} \to \infty$ as $n, m \to \infty$. It follows that (C.18) tends to 0 as $n, m \to \infty$. □

C.5 Proof of Theorem 4

The spectral density $f_{\mathcal{DGW}}(z, \tau; \boldsymbol{\chi}_0)$ is bounded away from zero and infinity. Also, from Theorem 1, Point *3.* and if $\mu > \eta + 1 + \alpha$, then for all $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\sup_{\|(\boldsymbol{z},u)\|>C_\varepsilon}\left|\frac{\sigma_1^2 f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}_1)}{b\sigma_0^2 f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}_0)}-1\right|<\varepsilon$$

with $b=\frac{\sigma_1^2\beta_1^{-(2\kappa+1)}\gamma_1^{-\delta}}{\sigma_0^2\beta_0^{-(2\kappa+1)}\gamma_0^{-\delta}}$.

Using Theorem 1 of Stein (1993), we obtain (5.4). If $\sigma_1^2\beta_1^{-(2\kappa+1)}\gamma_1^{-\delta} = \sigma_0^2\beta_0^{-(2\kappa+1)}\gamma_0^{-\delta}$ and using Theorem 2 of Stein (1993), we obtain (5.6). Similarly, since $f_{\mathcal{DM}}(z,\tau;\boldsymbol{\theta}_0)$ is bounded away from zero and infinity see Ip and Li (2017), then for all $\epsilon > 0$, exists a constant $C_\epsilon > 0$ such that

$$\sup_{\|(\boldsymbol{z},u)\|>C_\epsilon}\left|\frac{f_{\mathcal{DGW}}(z,\tau;\boldsymbol{\chi}_1)}{k f_{\mathcal{DM}}(z,\tau;\boldsymbol{\theta}_0)}-1\right|<\epsilon$$

with $k=\frac{\sigma_2^2\varrho_{\lambda,\eta}c_3^{\varsigma}L^{\varsigma}\beta^{-(1+2\kappa)}}{\ell(\boldsymbol{\theta}_0)\epsilon^{-2\nu}}\mathbf{1}_{\{\epsilon\in(0,1]\}}$.

Using Theorem 1 of Stein (1993), we obtain (5.5). If

$$\frac{\sigma_1^2\varrho_{\lambda,\eta}c_3^{\varsigma}\beta^{-2\eta}(\lambda+d\alpha-2(\eta+1))}{\ell(\boldsymbol{\theta}_0)\epsilon^{-2\nu}}\mathbf{1}_{\{\epsilon\in(0,1]\}}=1$$

and using Theorem 2 of Stein (1993), we obtain (5.7).

References

Bevilacqua, M., T. Faouzi, R. Furrer, and E. Porcu (2019). Estimation and prediction using Generalized Wendland covariance functions under fixed domain asymptotics. *Annals of Statistics 47*(2), 828–856.

Ibragimov, I. A. and Y. A. Rozanov (1978). *Gaussian Random Processes.* New York: Springer.

Ip, R. H. and W. Li (2017). On some Matérn covariance functions for spatio-temporal random fields. *Statistica Sinica 27*, 805–822.

Lim, S. and L. Teo (2009). Gaussian fields and gaussian sheets with generalized cauchy covariance structure. *Stochastic Processes and Their Applications 119*(4), 1325–1356.

Luke, Y. (1969). *The Special Functions and their approximations. Vol. 1.* New York: Academic Press.

Skorokhod, A. and M. Yadrenko (1973). On absolute continuity of measures corresponding to homogeneous Gaussian fields. *Theory of Probability and Its Applications 18*, 27–40.

Stein, M. L. (1993). A simple condition for asymptotic optimality of linear predictions of random fields. *Statistic and Probability Letters 17*, 399–404.

Stein, M. L. (2004). Equivalence of gaussian measures for some nonstationary random fields. *Journal of Statistical Planning and Inference 123*, 1–11.

Wang, D. (2010). *Fixed-domain asymptotics and consistent estimation for Gaussian random field models in spatial statistics and computer experi-*

ments. Ph. D. thesis, Department of Statistics and Applied Probability, National University of Singapore.

Wang, D. and W.-L. Loh (2011). On fixed-domain asymptotics and covariance tapering in gaussian random field models. *Electronic Journal of Statistics 5*, 238–269.

Zhang, H. (2004). Inconsistent estimation and asymptotically equivalent interpolations in model-based geostatistics. *Journal of the American Statistical Association 99*, 250–261.