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RELIABILITY ESTIMATION FROM LEFT-TRUNCATED AND RIGHT-CENSORED DATA USING SPLINES

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Abstract: Reliability data are often left truncated and right censored, because the data-collection process usually starts much later than the installation of the first product unit, and some units are still in service at the end of the data collection. The truncation introduces a sampling bias, making analyses of the lifetime data complicated. This study develops a nonparametric likelihood-based estimation procedure for left-truncated and right-censored data using B-splines. In terms of small-sample performance and large-sample efficiencies, the proposed spline-based estimators for the reliability function are shown to be more efficient than the existing nonparametric estimators. We further consider nonparametric two-sample tests for left-truncated and right-censored data. The new class of tests is useful for comparing the reliability of similar products. The test statistics are based on the cumulative weighted differences between the two estimated failure rates. Asymptotic distributions of the proposed statistics are derived and their finite-sample properties are evaluated using Monte Carlo simulations. The performance of the proposed test statistic is compared with that of the weighted Kaplan–Meier statistic. Lastly, a real-life example of high-voltage power transformers is used to illustrate the proposed method.

Key words and phrases: Asymptotic normality, B-splines, convergence rate, two-sample tests.

1. Introduction

Lifetime data collected from field operations contain important reliability information useful to asset management, such as preventive maintenance and remaining useful life predictions. Compared with reliability data collected from life tests, field failure data are usually subject to serious multiple censoring and truncation. In particular, the data are typically left truncated and right censored. The left truncation arises when the data collection starts later than the product launch/installation (Ye and Tang (2016)). Because of the high level of reliability, most products are still functioning when the data collection stops, leading to right censoring. This is common for assets used in infrastructure facilities, such as pipes in a water supply network (Carrión et al. (2010)) or power transformers

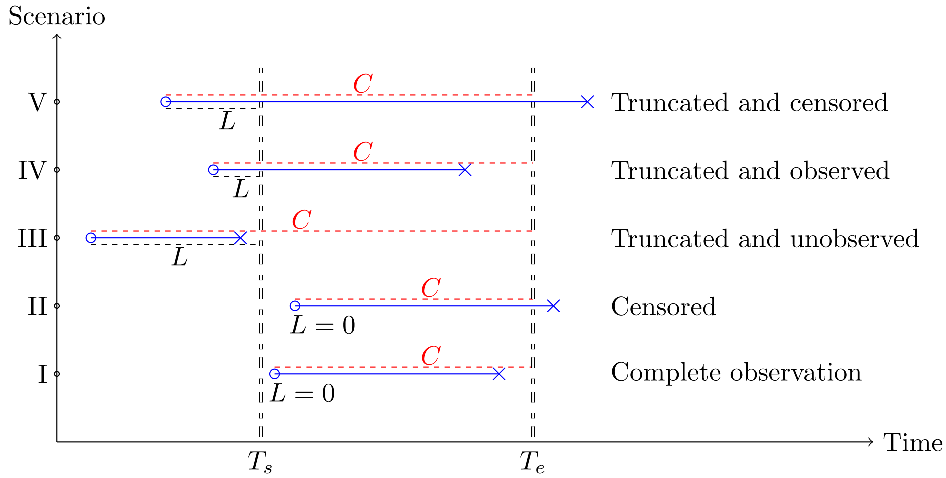


Figure 1. Schematic of the mechanism that generates left-truncated and right-censored data. The observation window in the calendar date is $[T_s, T_e]$. Here, "o" represents installation at date T_I, "$\times$" represents the failure event. The respective truncation and censoring times are $L = \max\{0, T_s - T_I\}$ and $C = \max\{0, T_e - T_I\}$.

used in a power grid (Hong, Meeker and Mccalley (2009)). An illustration of the data-generation mechanism is provided in Figure 1. The starting date is fixed for all product units. However, the installation dates (or sales dates) are generally random across the product population. The randomness in the left-truncation time results from the random installation dates. If a unit is installed before the date on which data begin to be collected, then the date are subject to left truncation. Furthermore, if the unit's lifetime is longer than the truncation time, it is a left-truncated observation (possibly subject to censoring); otherwise, the unit is truncated and unobserved, and the existence of the unit is unknown. The untruncated population corresponds to those units whose installation dates are later than the starting date of the data collection. The same data format is common in survival studies of clinical trials. For additional examples, see Tsai, Jewell and Wang (1987), Kevin et al. (2011), and Su and Wang (2012), among others.

Most of the literature on reliability data analyses has focused on right-censored data, because life tests are an important source of reliability data. However, the problem of left-truncation has begun attracting greater interest, owing to its prevalence in the increasingly important area of asset management. The recent literature on left-truncated and right-censored data features parametric models and related inferences. Hong, Meeker and Mccalley (2009) fitted a Weibull distribution to lifetime data of high-voltage power transformers from an energy company in the United States. The maximum likelihood (ML) estimates were obtained through direct maximization. An alternative method for ML estimation of the Weibull distribution is the EM algorithm developed by Balakrishnan and Mitra (2012). Parametric inferences for other distributions, such as the lognormal and gamma distributions, have been developed by Balakrishnan and Mitra (2011, 2013, 2014) and Emura and Shiu (2016), among others.

A problem with parametric inferences is that the estimation results, such as the reliability function and lifetime quantiles, may be sensitive to distributional assumptions. In addition, it may be difficult, if not impossible, to check the distributional assumption in the presence of heavy truncation (Kevin et al. (2011)). In view of these deficiencies, it is desirable to use nonparametric inference methods that impose fewer assumptions on the lifetimes. In this line, Turnbull (1976) proposed a nonparametric maximum likelihood estimation (NPMLE) procedure for arbitrarily censored and truncated data. He further developed a self-consistent algorithm to compute the NPMLE, which turns out to be a special case of the EM algorithm. Frydman (1994) corrected Turnbull's algorithm to make it applicable when the data are truncated and interval censored. The consistency and efficiency of the NPMLE have been established in a number of studies, such as Wang, Jewell and Tsai (1986) and Tsai, Jewell and Wang (1987). The EM algorithm converges quite slowly if the collected failure data are heavily truncated, and it can be sensitive to the initial values. With an appropriate adjustment of the definition of the risk set, Tsai, Jewell and Wang (1987) showed that the NPMLE of the survivor function and the cumulative failure rate can be obtained directly using the analogues of the Kaplan–Meier and Nelson–Aalen estimators.

In this study, we propose a nonparametric inference for left-truncated and right-censored data, using splines. A spline is a piecewise polynomial function that possesses a high degree of smoothness where the polynomial pieces connect. These connection points are known as knots. Once the knots are given, it is easy to compute the splines recursively for any desired degree of the polynomial (Schumaker (2007, Chap. IV)). The main advantages of spline interpolation are its stability and calculation simplicity. When applied to a nonparametric estimation, the number of parameters in the spline is usually much smaller than those of traditional nonparametric methods. This makes the estimation easier and reduces the computation time. Therefore, spline-based nonparametric estimations have received considerable attention in recent years. In Rosenberg (1995), nonnegative B-splines, also called M-splines, are applied to estimate the hazard function of censored survival data, where the nonnegativity is guaranteed by the nonnegativity coefficients. Specifically, the M-splines can be considered to be normalized versions of the B-splines, with a unit integral within the domain (Ramsay (1988)). Monotonic B-splines are also widely applied in the literature (e.g., Lu, Zhang and Huang (2009); Xie et al. (2018)), where the monotonicity is guaranteed by the nondecreasing order of the coefficients. On the other hand, I-splines, with bases that are integrals of the B-splines (Ramsay (1988)), are used to approximate the cumulative distribution function (CDF) in Wu and Ying (2012). I-splines naturally yield monotonicity with nonnegative coefficients, whereas B-splines require a nondecreasing order of coefficients to ensure monotonicity. Therefore, I-splines are often used to approximate monotone functions, which may simplify the numerical computation (Wu and Ying (2012); Hong et al. (2015); Lu, Zhang and Huang (2007)).

Motivated by the promising performance, spline basis functions are adopted for left-truncated and right-censored data. Although splines come in many different forms, they are closely related (Ramsay (1988); Lu, Zhang and Huang (2007)). Using B-splines to approximate the failure rate is the same as using M-splines for the failure rate, which, in turn, is the same as using I-splines to approximate the cumulative failure rate. We use B-splines with nonnegative constraints on the spline coefficients to approximate the failure rate, and use I-splines to approximate the cumulative failure rate. We do not approximate the cumulative distribution and reliability function, because the approximation induces a normalization constraint on the spline function, which complicates the ML estimation. We show that the convergence rate of the estimated failure rate is faster than $O(n^{1/3})$. Based on the inferential results, we develop spline-based two-sample tests in order to compare two left-truncated and right-censored data sets. The results can be used to compare the reliability of similar products.

The rest of the paper is organized as follows. Section 2 formulates the spline-based likelihood estimation problem for left-truncated and right-censored data. The asymptotic properties of the spline estimators are presented in Section 3. Based on the asymptotic results, a nonparametric two-sample test is proposed to compare lifetime data from two products in Section 4. Section 5 conducts simulation studies to evaluate the finite-sample performance of the spline estimators. Section 6 applies the proposed spline methods to the power transformer example in Hong, Meeker and Mccalley (2009). Technical lemmas and the proofs of the theorems are provided in the Appendix.

2. B-Spline Approximation of the Failure Rate

Consider the lifetime T of a product unit with reliability $R(t)$, failure rate $\lambda_0(t)$, and cumulative failure rate $\Lambda_0(t)$, for $t \geq 0$. The lifetime T is subject to left truncation with truncation time L, for $L \geq 0$. A unit is observed only when $T > L$. The unit is further subject to right censoring, with random censoring time C and $C > L$. If the observation window of the product is a fixed interval, then $C - L$ is equal to the length of the interval if $L > 0$. See Figure 1 for an illustration. In terms of a calendar date, we let T_I be the installation date of a random unit, and let $[T_s, T_e]$ be the obvervation interval. Then, in terms of product age, the left-truncation time is $L = \max\{0, T_s - T_I\}$ and the right-censoring time is $C = \max\{0, T_e - T_I\}$. Naturally, the truncation and censoring times are bounded, because T_I cannot be earlier than the product launch date (Shen and Yan (2008); Shen (2014); Balakrishnan and Mitra (2012)). Therefore, we let $[\underline{L}, \bar{L}]$ and $[\underline{C}, \bar{C}]$ be the respective supports of L and C, and assume $\bar{L}, \bar{C} < \infty$. Furthermore, suppose T and (L, C) are independent. Because of the left-truncation and right-censoring, the lifetime information is only available within the interval $[\underline{L}, \bar{C}]$. As a result, the failure rate of T is identifiable in $[\underline{L}, \bar{C}]$.

When $T \geq L$, the unit enters our observation, and the observed lifetime is denoted as $Y = \min(T, C)$, $Y \geq L$. Let $\delta = I(T \leq C)$ be the censoring indicator. That is, $\delta = 1$ if the lifetime is observed, and 0 if censored. The observation from the unit is thus $X = (L, Y, \delta)$. Let $X_i = (L_i, Y_i, \delta_i)$, for $i = 1, \ldots, n$, be n independent and identically distributed (i.i.d.) copies of X, and let $\mathbf{D} = \{X_1, X_2, \ldots, X_n\}$. We are interested in estimating the failure rate $\lambda(t)$ using $\mathbf{D}$. Here, it suffices to consider the conditional log-likelihood (Wang (1987)),

$$\mathcal{L}(\lambda|\mathbf{D}) = \sum_{i=1}^{n} \left\{ \delta_i \ln \lambda(Y_i) - \int_{L_i}^{Y_i} \lambda(s) ds \right\}.$$

In order to implement the spline approximation, we first identify a finite closed interval $[a, b]$. The guideline for choosing a and b is that they should include all observed L_i and Y_i. In an application, we can let $a = \min\{L_i, i = 1, \ldots, n\}$ and $b = \max\{Y_i, i = 1, \ldots, n\}$. Given $[a, b]$, let $\mathcal{T} = \{t_j\}_1^{m_n+2l}$, with

$$a = t_1 = \cdots = t_l < t_{l+1} < \cdots < t_{l+m_n} < t_{l+m_n+1} = \cdots = t_{m_n+2l} = b,$$

be a sequence of knots that partition $[a, b]$ into $m_n + 1$ subintervals $J_j = [t_{l+j}, t_{l+j+1})$, for $j = 0, 1, \ldots, m_n$. To ensure the large-sample property, as discussed in the next section, the number m_n of inner knots is usually chosen as $O(n^{\nu})$, for some $\nu \in (0, 1/2)$. A common choice is $m_n = \lceil n^{1/3} \rceil$; see, for example, Lu, Zhang and Huang (2007, 2009) and Hua and Zhang (2012). With fixed m_n, the inner knots $\{t_j\}_{l+1}^{l+m_n}$ can be either equally spaced (Lu, Zhang and Huang (2007)), placed at the corresponding quantiles of the distinct observation times $\{Y_i\}_1^n$ (Hua and Zhang (2012)), or placed at the Chebyshev points. Based on our simulation experience, as well as the simulation experiments reported in the literature, such as Zhao et al. (2013), the estimation results are not sensitive to the selection of m_n and the placement of the knots. For ease of implementation, we recommend $m_n = \lceil n^{1/3} \rceil$ and equally spaced inner knots.

From the knot sequence, we can construct $q_n = m_n + l$ spline bases, denoted as B_k, for $1 \le k \le q_n$, using a recursive formula (Schumaker (2007, Chap. IV)). The class of polynomial splines of order l with the knot sequence $\mathcal{T}$ is the linear space spanned by these bases (Schumaker (2007, Thm. 4.18)). To satisfy the nonnegativity constraint of the failure rate approximation, we single out the following subclass of $\psi_{l,\mathcal{T}}$:

$$\psi_{l,\mathcal{T}} = \left\{ \sum_{k=1}^{q_n} \alpha_k B_k : \alpha_k \ge 0 \right\}.$$

According to Theorem 5.9 of Schumaker (2007), $\psi_{l,\mathcal{T}}$ is a class of nonnegative polynomial splines on $[a, b]$. The nonnegativity of the B-splines is guaranteed by the nonnegative coefficients. For each $h(\cdot) \in \psi_{l,\mathcal{T}}$, h is a polynomial of order l in the interval J_j for $0 \le j \le m_n$, and h is $l-2$ times continuously differentiable on $[a, b]$. Define $I_k(t) = \int_a^t B_k(s)ds$. Using the spline approximation, the log-likelihood function can be written as

$$\mathcal{L}(\boldsymbol{\alpha}|\mathbf{D}) = \sum_{i=1}^{n} \left\{ \delta_i \ln \left[\sum_{k=1}^{q_n} \alpha_k B_k(y_i) \right] + \sum_{k=1}^{q_n} \alpha_k I_k(L_i) - \sum_{k=1}^{q_n} \alpha_k I_k(y_i) \right\}. \quad (2.1)$$

Let $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_{q_n})$ be the spline coefficients that maximize (2.1), subject to the nonnegativity constraints $\alpha_k \ge 0$, for $k = 1, \ldots, q_n$. The spline log-likelihood function (2.1) is concave with respect to the unknown coefficients. Therefore, the spline estimation problem is equivalent to a nonlinear convex programming problem, subject to linear inequality constraints. The optimization can be easily solved by most software packages for scientific/statistical computation. Based on $\hat{\boldsymbol{\alpha}}$, the spline-based likelihood estimator for the failure rate is $\hat{\lambda}_n(t) = \sum_{k=1}^{q_n} \hat{\alpha}_k B_k(t)$.

3. Statistical Properties

In this section, we study the statistical properties of the spline-based likelihood estimator $\hat{\boldsymbol{\alpha}}$, with the L_2-metric d given by

$$d(\lambda_1, \lambda_2) = \|\lambda_1 - \lambda_2\|_2 = \left\{ \int |\lambda_1(t) - \lambda_2(t)|^2 dF^*(t) \right\}^{1/2},$$

where $F^*(t) = P(L \leq T \leq C, T \leq t)$ and λ_1, λ_2 are nonnegative functions. To ensure asymptotic convergence, we first require $m_n = O(n^\nu)$, for some $\nu \in (0, 1/2)$ (Stone (1994)). Below, we list the technical assumptions for the theoretical results of the proposed spline-based NPMLE.

- Condition 1: The maximum spacing of the knots satisfies

$$\Delta = \max_{l+1 \leq j \leq m_n+l+1} |t_j - t_{j-1}| = O(n^{-\nu}).$$

  Moreover, there exists a constant $M > 0$, such that $\Delta/\delta \leq M$ uniformly in n, where $\delta = \min_{l+1 \leq j \leq m_n+l+1} |t_j - t_{j-1}|$.

- Condition 2: The interval $[a, b]$ satisfies $P(\{Y \in [a, b]\}) = 1$.

- Condition 3: There exists a constant $C_0 > 0$, such that $\lambda_0(t) \geq C_0$, for $t \in [a, b]$. In addition, the true failure rate λ_0 is differentiable up to order r, and all derivatives are uniformly bounded by a constant M in $[a, b]$, where $r \geq 1$.

Remark 1. Condition 1 is a weak restriction on the knot sequence, and is satisfied when equally spaced knots are used. This condition is also adopted by Stone (1994). Condition 2 requires that $[\underline{L}, \bar{C}] \subset [a, b]$. Condition 3 is needed in the proof of the asymptotic normality in Theorem 3. It usually holds in practice. Product lifetime is a nonnegative and continuous random variable. Continuous parametric lifetime distributions, such as Weibull, lognormal, and inverse Gaussian distributions, have been widely used to model such lifetime data; see Balakrishnan and Mitra (2011, 2012, 2013), among others. The parametric distributions all have smooth hazard rate functions. As an extension from parametric to nonparametric estimations, the smoothness assumption in Condition 3 is natural and reasonable. This assumption is also used in Wang (2005) and Zhao and Zhang (2017).

Theorem 1 (Consistency)**.** *Suppose that Conditions* 1–3 *hold. Then, the estimated failure rate* $\hat{\lambda}_n$ *converges to the true failure rate* λ_0*, in probability; that is,*

$$\|\hat{\lambda}_n - \lambda_0\|_2 \to_p 0, \quad \textit{as } n \to \infty.$$

Theorem 2 (Rate of convergence)**.** *Suppose that Conditions* 1–3 *hold. If* ν *is chosen to be* $1/(2r+1)$*, then*

$$n^{r/(2r+1)}\|\hat{\lambda}_n - \lambda_0\|_2 = O_p(1).$$

Remark 2. Theorem 2 shows that the spline likelihood estimators have a convergence rate that is slower than $n^{-1/2}$, but faster than $n^{-1/3}$.

To discuss the asymptotic distributions of functions of $\hat{\lambda}_n$, define

$$\mathcal{H}_r = \left\{ h(\cdot) : |h^{(r-1)}(s) - h^{(r-1)}(t)| \le c_0|s-t| \text{ for all } a \le s, t \le b \right\},$$

where $h^{(r-1)}$ is the $(r-1)$th derivative of h, and $c_0 > 0$ is a constant. Let $\mathcal{U}_\lambda$ denote a neighborhood of the failure rate λ_0. We also define a sequence of maps G_n, mapping $\mathcal{U}_\lambda$ in the parameter space for λ into $\mathcal{L}^\infty(\mathcal{H}_r)$, as

$$G_n(\lambda)[h] = n^{-1}\sum_{i=1}^n \left\{ \delta_i \frac{h(Y_i)}{\lambda(Y_i)} - \int_{L_i}^{Y_i} h(t)dt \right\} = \mathbb{P}_n\phi(\lambda; X)[h].$$

The limit map $G : \mathcal{U}_\lambda \mapsto \mathcal{L}^\infty(\mathcal{H}_r)$ is

$$G(\lambda)[h] = P\phi(\lambda; X)[h] = P\left\{ \delta\frac{h(Y)}{\lambda(Y)} - \int_L^Y h(t)dt \right\},$$

where $X = (L, Y, \delta)$, $\mathbb{P}_n$ and P denote the empirical measure and probability measure, respectively, with $\mathbb{P}_n g = n^{-1}\sum_{i=1}^n g(X_i)$ and $Pg = \int g dP$.

Theorem 3 (Asymptotic normality)**.** *Suppose Conditions* 1–3 *hold. Then, for* $h \in \mathcal{H}_r$*,*

$$\sqrt{n}\int_a^b \frac{h(t)}{\lambda_0^2(t)}\{\hat{\lambda}_n(t) - \lambda_0(t)\}dF^*(t) = \sqrt{n}(G_n - G)(\lambda_0)[h] + o_p(1). \tag{3.1}$$

Remark 3. Theorem 3 does not require that $\hat{\lambda}_n$ be $\sqrt{n}$-consistent. Because we assume λ_0 is differentiable, it is easy to see that F^* is differentiable, with its derivative denoted as $f^*(t)$. Consider the situation $\underline{L} = 0$ and $f^*(t) > 0$, for all $t \in [0, \bar{C}]$. For any fixed time $\tau \in [0, \bar{C}]$, choose $h(t) = I_{(0,\tau]}(t)\lambda_0^2(t)/f^*(t)$ to see that the estimated cumulative hazard $\hat{\Lambda}_n(\tau)$ is $\sqrt{n}$-consistent for $\Lambda_0(\tau)$. Furthermore, a routine evaluation of the right-hand side of (3.1) shows that the asymptotic variance of $\hat{\Lambda}_n(\tau)$ is the same as that for the NPMLE of Λ_0 given in Wang, Jewell and Tsai (1986). This means that the proposed method leads to an efficient estimation of the cumulative failure rate. Moreover, the asymptotic normality can be used to construct new tests for the problem of multi-sample nonparametric comparisons of the reliability of left-truncated and right-censored data, as shown in the next section.

4. Nonparametric Tests

As a result of technological advances and the availability of multiple suppliers, a fleet of assets usually consists of different brands or different generations of the same brand (Ye, Hong and Xie (2013)). These differences naturally stratify the field failure data into several categories. The transformer failure data analyzed in Hong, Meeker and Mccalley (2009) are a typical example. It is important that we know whether the categories might vary in terms of product reliability. This knowledge can be used to select a more reliable product. If there is no difference in reliability, the field data can be combined to achieve a more accurate estimation of the product lifetime distribution. In the literature on left-truncated and right-censored data, several extensions of nonparametric tests have been developed for two-sample comparisons. Examples include the Wilcoxon test, weighted Kaplan–Meier (WKM) statistic (Shen (2007)), and weighted log-rank statistic (Shen (2014)). These tests are based on estimates of the failure rates, cumulative failure rates, or survival functions. Similarly, we use the spline-based smooth estimator of the failure rate developed above, and propose a flexible class of nonparametric test statistics based on the integrated weighted differences between the two estimated failure rates. We examine the performance of the estimators using the weighted Kaplan–Meier statistic (Shen (2007)) in Section 5.

Consider two homogeneous groups. In group k, for $k = 1, 2$, the ith observed lifetime is $X_i^{(k)} = (L_i^{(k)}, Y_i^{(k)}, \delta_i^{(k)})$. The observed data of group k are $\mathbf{D}_k = \{X_i^{(k)}, i = 1, 2, \ldots, n_k\}$. Let $n = n_1 + n_2$. Assume that the failure rate and the cumulative failure rate functions of units from group k are λ_k and Λ_k, respectively. The goal is to test $H_0 : \lambda_1 = \lambda_2 = \lambda_0$, where λ_0 denotes the unknown common failure rate function when H_0 is true. The test statistics proposed here capitalize on the spline-based estimator developed in Section 3. Let $\hat{\lambda}_n^{(k)}(t)$ and $\hat{\lambda}_n(t)$ be the B-spline ML estimators of $\lambda_k(t)$ and $\lambda_0(t)$, respectively, based on $\mathbf{D}_k$ and the pooled data $\mathbf{D} = \mathbf{D}_1 \cup \mathbf{D}_2$. Motivated by a method commonly used in survival analyses (e.g., Pepe and Fleming (1989); Balakrishnan and Zhao (2009)), we propose the following test statistic

$$U_n = \sqrt{n} \int_a^b W_n(t)\{\hat{\lambda}_n^{(1)}(t) - \hat{\lambda}_n^{(2)}(t)\} dF_n^*(t), \tag{4.1}$$

where W_n is a bounded weight process (Zhao and Zhang (2017); Balakrishnan and Zhao (2009); Andersen et al. (1993, Chap. V)), and $F_n^*(t) = (\sum_{i=1}^n \delta_i I(Y_i \leq t))/(\sum_{i=1}^n \delta_i)$. The presence of the weight process $W_n(t)$ makes the above statistic flexible. A simple and natural choice for the weight is $W_n^{(1)}(t) = 1$. Another natural choice is $W_n^{(2)}(t) = Z_n(t) = 1/n \sum_{i=1}^n I(L_i < t \leq Y_i)$, in which the case weights are proportional to the number of subjects under observation. In addition, one may choose the weight process as

$$W_n^{(3)}(t) = \frac{Z_{n_1}(t) Z_{n_2}(t)}{Z_n(t)}, \quad W_n^{(4)}(t) = 1 - Z_n(t),$$

where $Z_{n_k}(t)$ is defined as $Z_n(t)$, with the summation over the subjects in sample k only. Weight processes similar to $W_n^{(3)}$ and $W_n^{(4)}$ have been used for recurrent event data (e.g., Andersen et al. (1993, Chap. V)). Now, we state the asymptotic distribution of U_n.

Theorem 4. *Suppose $\lambda_1 = \lambda_2 = \lambda_0$ and Conditions 1–3 hold for λ_0 and the spline estimators $\hat{\lambda}_n^{(1)}, \hat{\lambda}_n^{(2)}, \hat{\lambda}_n$. Furthermore suppose W_n are bounded weight processes, and that there exists a bounded function $W(t)$, such that $W \in \mathcal{H}_r$, and*

$$\left[\int_a^b \{W_n(t) - W(t)\}^2 dt \right]^{1/2} = o_p\left(n^{-1/(2(1+2r))} \right).$$

In addition, suppose that $n_1/n \to p$ as $n \to \infty$, with $0 < p < 1$. Then, U_n has an asymptotic normal distribution $N(0, \sigma_w^2)$, where

$$\sigma_w^2 = \frac{1}{p(1-p)} E\{\phi^2(\lambda_0; X)[h_w]\},$$

which can be estimated consistently by

$$\hat{\sigma}_w^2 = \frac{n}{n_1 n_2} \sum_{i=1}^n \{\phi^2(\hat{\lambda}_n; X_i)[\hat{h}_w]\},$$

with $h_w(t) = W(t)\{\lambda_0(t)\}^2$ and $\hat{h}_w(t) = W_n(t)\{\hat{\lambda}_n(t)\}^2$.

Remark 4. For the asymptotic normality of the proposed test statistics, we do not need the bounded Lipschitz condition for the selection of the weight processes, which is required by Balakrishnan and Zhao (2009).

5. Simulation Studies

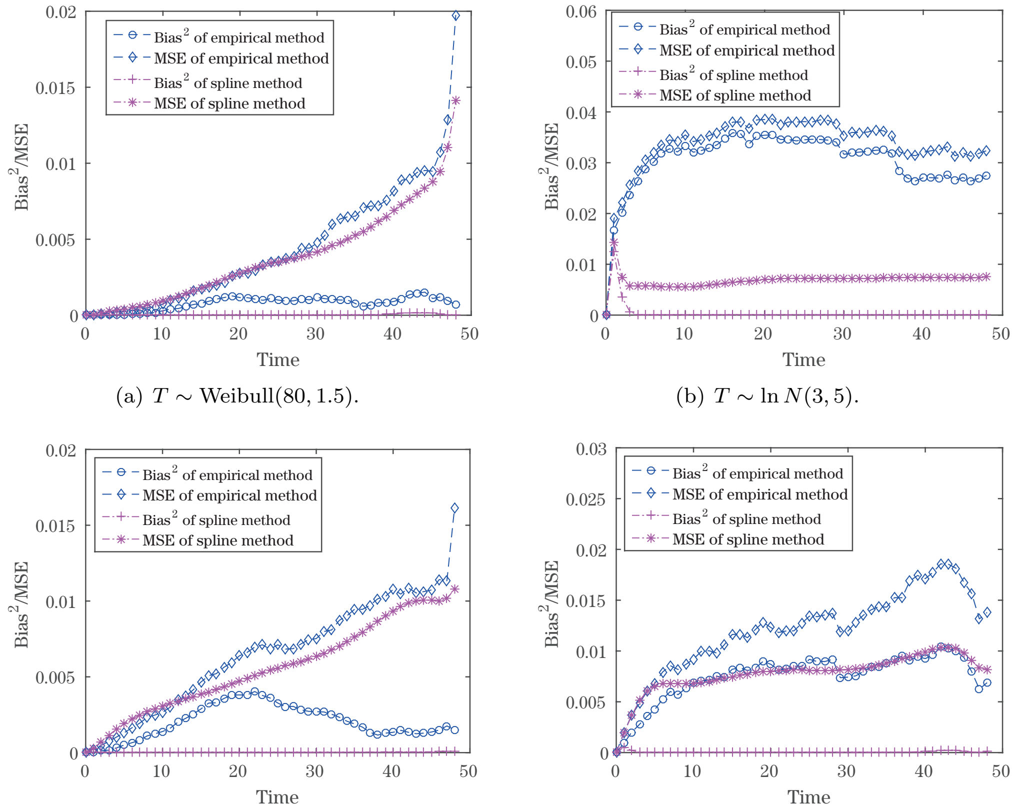
To verify the performance of the proposed spline-based estimators under finite samples, a Monte Carlo simulation is conducted. In the simulation study, we choose cubic B-splines with order $l = 4$, which are popular in the literature

(Lu, Zhang and Huang (2009); Hong et al. (2015); Xie et al. (2018)). In addition, m_n is set as $\lceil n^{1/3} \rceil$. The other simulation settings follow the work of Balakrishnan and Mitra (2012).

The starting date T_s of data collection is fixed as 1980, and the end date T_e is 2008. Let n be the number of observed units, and let p be the proportion of truncated observations; that is, $100p\%$ of the observed units are installed before 1980. Let $T_{I,i}$ be the installation time of unit i, for $i = 1, \ldots, n$, which are assigned as follows. The earliest installation date $\underline{T}_I$ is 1960. For the period 1960–1979, a proportion of 0.15 is attached to each of the first five years, and the remainder is distributed equally over the remaining years of this period. For the period 1980–1995, a proportion of 0.1 is attached to each of the first six years, and a proportion of 0.04 is attached to each of the remaining years of this period. Accordingly, the left-truncation time of unit i is $L_i = \max\{0, T_s - T_{I,i}\}$, and the right-censoring time of unit i is $C_i = \max\{0, T_e - T_{I,i}\}$, for $i = 1, \ldots, n$. For additional details, see Balakrishnan and Mitra (2012).

Four distributions are considered for the product lifetime T: Weibull, lognormal, a mixture of two Weibull distributions, and a mixture of lognormal and gamma distributions (Balakrishnan and Mitra (2012, 2011, 2013)). The generated data are fitted using the proposed spline method and Turnbull's NPMLE (Tsai, Jewell and Wang (1987)). Here, we consider two fixed proportions of truncated observations (i.e., $p = 40\%$ and $p = 80\%$) and two sample sizes (i.e., $n = 100$ and 200). Based on 50,000 Monte Carlo replications, the squared biases and the mean squared errors (MSEs) of the reliability estimators using the two methods are computed. The results are presented in Figures 2–5. From the plots, we can see that the squared biases and the MSEs of the spline-based reliability estimators are smaller than those of Turnbull's NPMLE for both proportions of truncated observations. Furthermore, a comparison of Figures 2 and 3 (or Figures 4 and 5) shows that when the sample size n doubles, the MSEs of the spline-based reliability estimators drop substantially, which supports the asymptotic consistency of these estimators (Theroem 1).

Next, we examine the finite-sample properties of the proposed two-sample test statistic U_n. Assume the lifetimes of the units in the two groups follow Weibull distributions, with different values of the scale parameter α and shape parameter β. To guarantee truncation and censoring, we generate the data using the simulation setting of Balakrishnan and Mitra (2012) for each group. The null hypothesis $H_0 : \lambda_1 = \lambda_2 = \lambda_0$ is equivalent to $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. If the null is true, then $T_n = U_n/\hat{\sigma}_w$ is approximately standard normal, where U_n in (4.1)



(a) $T \sim$ Weibull$(80, 1.5)$.

(b) $T \sim \ln N(3, 5)$.

(c) $T \sim 0.5$Weibull$(80, 1.5) + 0.5$Weibull$(80,\ 0.8)$.

(d) $T \sim 0.8 \ln N(4.5, 3.5) + 0.2$Gamma$(2.5, 30)$.

Figure 2. Comparisons of the spline estimator and the NPMLE (Turnbull (1976)) for estimating the reliability function when $n = 100$ and $p = 40\%$.

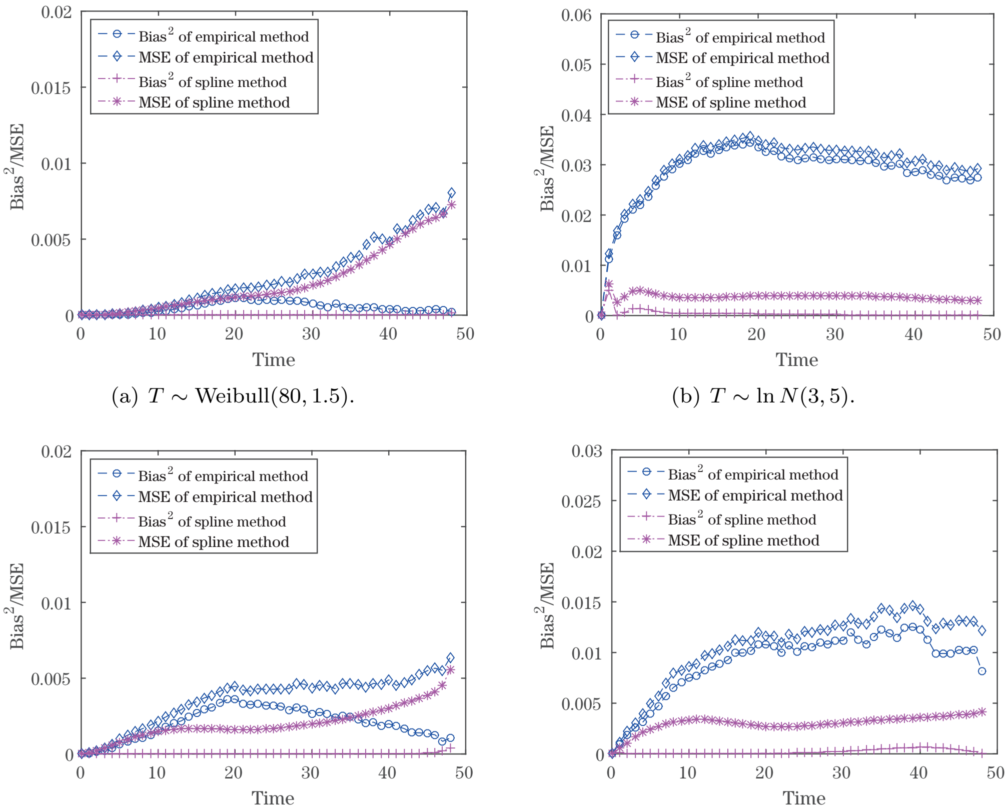
can be expressed as

$$U_n = \frac{\sqrt{n}}{\sum_{i=1}^{n} \delta_i} \sum_{i=1}^{n} \delta_i W_n(Y_i) \left\{ \hat{\lambda}_1(Y_i) - \hat{\lambda}_2(Y_i) \right\},$$

and $\hat{\sigma}_w$ is given in Theorem 4. Let T_H denote the WKM statistic developed by Shen (2007). Here, we evaluate the performance of T_n and compare it with that of T_H. We consider two scenarios:

- Case 1. Two groups, with the same shape parameter and different scale parameters.

- Case 2. Two groups, with the same scale parameter and different shape parameters.



(a) $T \sim$ Weibull$(80, 1.5)$. (b) $T \sim \ln N(3, 5)$.

(c) $T \sim 0.5$Weibull$(80, 1.5) + 0.5$Weibull$(80,\ 0.8)$. (d) $T \sim 0.8 \ln N(4.5, 3.5) + 0.2$Gamma$(2.5, 30)$.

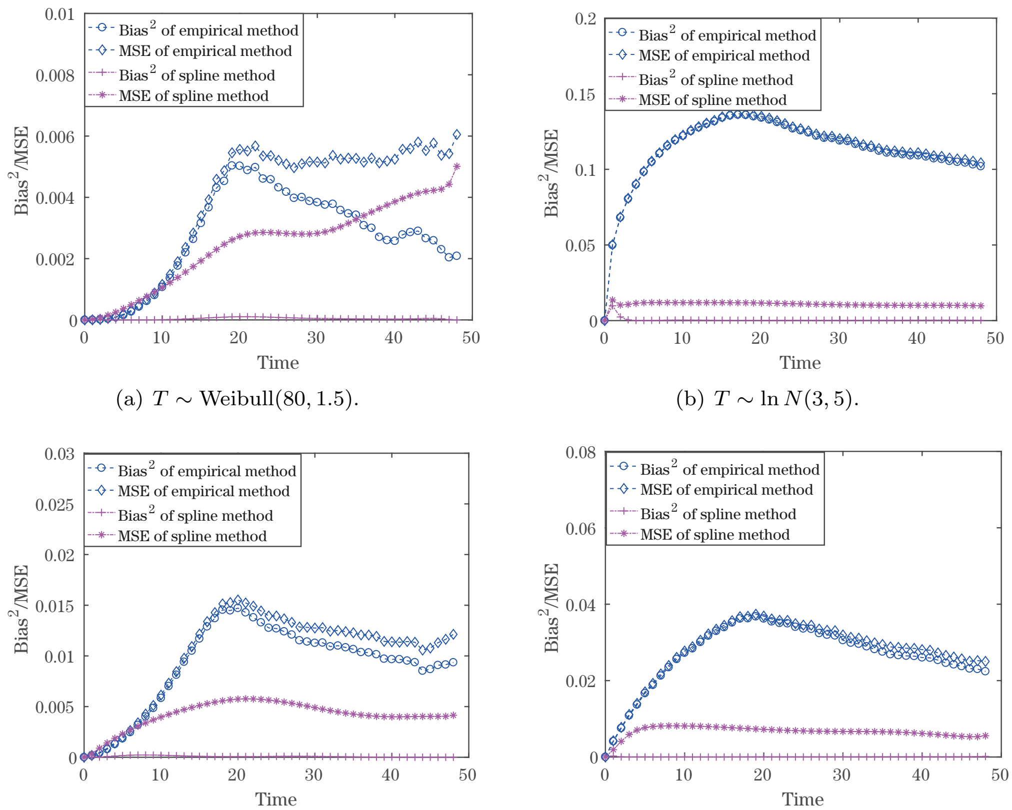
Figure 3. Comparisons of the spline method and the NPMLE (Turnbull (1976)) for estimating the reliability function when $n = 200$ and $p = 40\%$.

In Case 1, the two failure rates do not overlap, whereas the true failure rates intersect in Case 2. For each case, we consider two sample sizes, $n_1 = n_2 = 100$ and 200, respectively. As with Section 4, we choose the four weight processes

$$W_n^{(1)}(t) = 1, \quad W_n^{(2)}(t) = Z_n(t) = \frac{1}{n}\sum_{i=1}^{n} I(L_i < t \leq Y_i),$$

$$W_n^{(3)}(t) = \frac{Z_{n_1}(t)Z_{n_2}(t)}{Z_n(t)}, \quad W_n^{(4)}(t) = 1 - Z_n(t),$$

where $Z_{n_k}(t)$ is defined as $Z_n(t)$, with the summation over the subjects in group k only. All results reported here are based on 50,000 Monte Carlo replications. Tables 1 and 2 present the estimated sizes and powers, respectively, of the proposed test statistics T_n and the WKM statistic T_H (Shen (2007)), respectively,



(a) $T \sim$ Weibull$(80, 1.5)$.

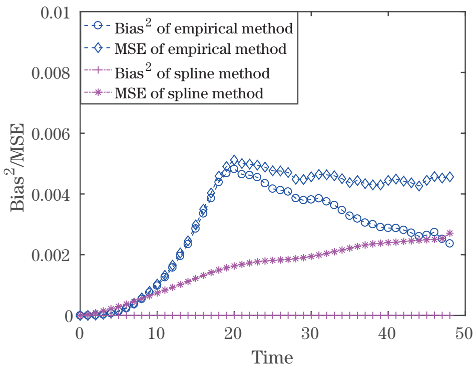
(b) $T \sim \ln N(3, 5)$.

(c) $T \sim 0.5$Weibull$(80, 1.5) + 0.5$Weibull$(80, 0.8)$.

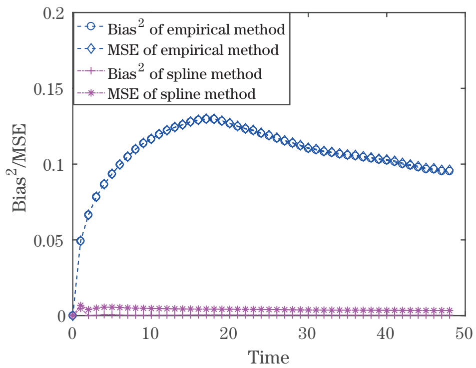
(d) $T \sim 0.8 \ln N(4.5, 3.5) + 0.2$Gamma$(2.5, 30)$.

Figure 4. Comparisons of the spline method and the NPMLE (Turnbull (1976)) for estimating the reliability function when $n = 100$ and $p = 80\%$.

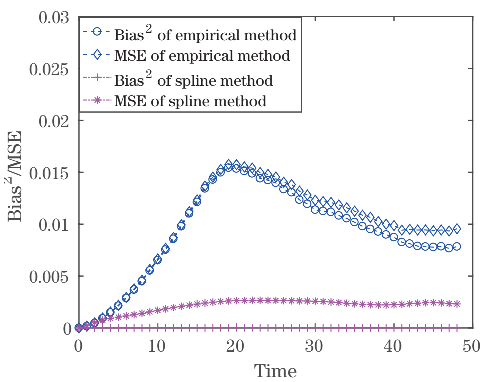
at a significance level of $\alpha = 0.05$ for the different cases and four weight processes. As expected, the powers of all test statistics increase with the sample size. Under H_0, when the proportion of truncated observations is serve (40%), the proposed test T_n outperforms T_H. For Case 1, Table 1 shows good power properties of the proposed test T_n for the four weight processes. The proposed test with weight $W_n^{(1)}(t)$ has the best power performance. On the other hand, the powers rely heavily on the choices of the weight processes in Case 2, as can be seen from Table 2. The simulation results suggest that the proposed test T_n with $W_n^{(4)}(t)$ has the best power performance. The differing performance of the four test statistics is due to the intersection between the two failure rate functions. The difference between the failure rate functions changes sign at the intersection point. If the weight W_n is approximately the same at the two sides of the inter-



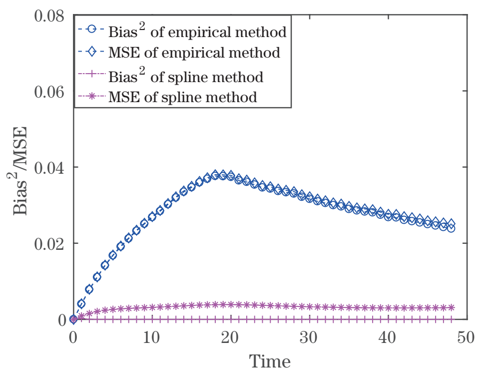
(a) $T \sim$ Weibull$(80, 1.5)$.



(b) $T \sim \ln N(3, 5)$.



(c) $T \sim 0.5$Weibull$(80, 1.5) + 0.5$Weibull$(80,\ 0.8)$.



(d) $T \sim 0.8 \ln N(4.5, 3.5) + 0.2$Gamma$(2.5, 30)$.

Figure 5. Comparisons of the spline method and the NPMLE (Turnbull (1976)) for estimating the reliability function when $n = 200$ and $p = 80\%$.

section point, the value of U_n will be small, leading to poor powers of the test. The weight $W_n^{(4)}(t)$ puts unequal weight on the two sides and, thus, exhibits the best performance. From the simulation, we recommend using $W_n^{(4)}(t)$ for the test.

6. A Real Example: Power Transformer Failure Data

The power transformer is one of the most important components in a power grid. Unexpected failures of transformers cause power a shortage and lead to large economic losses. Therefore, it is important to know the failure behaviors of a transformer in the field. Such information can be extracted from field failure data of the transformers. However, because of the long lifetime of a transformer

Table 1. Estimated sizes and powers of $T_n = U_n/\hat{\sigma}_w$ and T_H with a Weibull distribution (α, β), where the shape parameters $\beta_1 = \beta_2 = 1.5$, and the scale parameters $\alpha_1 = 30, \alpha_2 = 30, 40, 80$. Here, $W_n^{(k)}$ are the weight processes, $k = 1, 2, 3, 4$.

T_n					T_H				
α_1/α_2	$W_n^{(1)}$	$W_n^{(2)}$	$W_n^{(3)}$	$W_n^{(4)}$	α_1/α_2	$W_n^{(1)}$	$W_n^{(2)}$	$W_n^{(3)}$	$W_n^{(4)}$
$n_1 = n_2 = 100$									
30/30	0.058	0.052	0.052	0.056	30/30	0.065	0.062	0.062	0.064
30/40	0.697	0.583	0.585	0.650	30/40	0.705	0.657	0.657	0.683
30/80	1.000	0.981	0.981	0.997	30/80	1.000	1.000	1.000	1.000
$n_1 = n_2 = 200$									
30/30	0.052	0.049	0.049	0.053	30/30	0.060	0.058	0.058	0.061
30/40	0.769	0.718	0.720	0.750	30/40	0.827	0.654	0.670	0.716
30/80	1.000	1.000	1.000	1.000	30/80	1.000	1.000	1.000	1.000

Table 2. Estimated sizes and powers of $T_n = U_n/\hat{\sigma}_w$ and T_H with a Weibull distribution (α, β), where the scale parameters $\alpha_1 = \alpha_2 = 30$, and the shape parameters $\beta_1 = 1.5, \beta_2 = 1.5, 1.2, 0.8, 0.5$. Here, $W_n^{(k)}$ are the weight processes, $k = 1, 2, 3, 4$.

T_n					T_H				
β_1/β_2	$W_n^{(1)}$	$W_n^{(2)}$	$W_n^{(3)}$	$W_n^{(4)}$	β_1/β_2	$W_n^{(1)}$	$W_n^{(2)}$	$W_n^{(3)}$	$W_n^{(4)}$
$n_1 = n_2 = 100$									
1.5/1.5	0.058	0.052	0.052	0.056	1.5/1.5	0.065	0.062	0.062	0.064
1.5/1.2	0.264	0.073	0.078	0.401	1.5/1.2	0.093	0.082	0.085	0.191
1.5/0.8	0.643	0.189	0.192	0.981	1.5/0.8	0.202	0.119	0.122	0.411
1.5/0.5	0.499	0.325	0.377	0.998	1.5/0.5	0.407	0.421	0.473	0.813
$n_1 = n_2 = 200$									
1.5/1.5	0.052	0.049	0.049	0.053	1.5/1.5	0.060	0.058	0.058	0.061
1.5/1.2	0.399	0.080	0.085	0.627	1.5/1.2	0.265	0.111	0.092	0.380
1.5/0.8	0.870	0.201	0.214	0.999	1.5/0.8	0.394	0.157	0.168	0.553
1.5/0.5	0.668	0.477	0.565	1.000	1.5/0.5	0.637	0.558	0.563	0.895

and the recent development of data recording systems, transformer lifetime data are left truncated and right censored. Figure 6 displays the data set "MC_Old65," which are recorded in operating time (Hong, Meeker and Mccalley (2009)).

The data set "MC_Old65," of 80 transformers and the installation dates of these transformers are recorded. The starting year of observation T_s is 1980, and the end date for data collection date T_e is 2008. The earliest installation date $\underline{T}_I$ is 1950 and 69 transformers are installed before 1980. As a result, their lifetime observations are left-truncation observed. The proportion of truncated observations of tranformers is 86%. In the data set, 65 transformers continue to function after 2008 and the proportion of censored observed transformers is 81%.

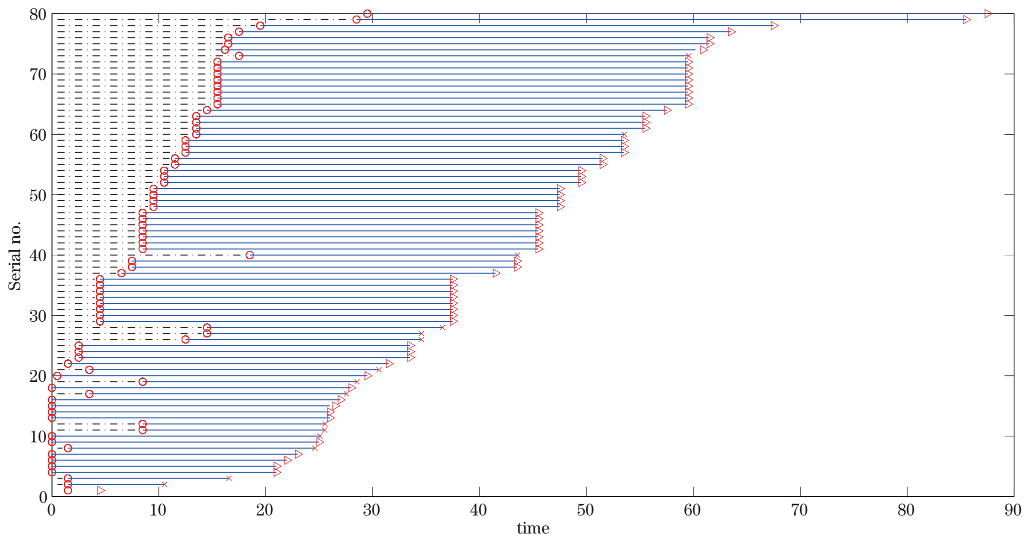


Figure 6. Service-time event plot of a subset of the transformer lifetime data: "o" represents the install time, "×" represents the failure time, and "▷" represents the censored time.

We use the proposed spline method with $m_n = 5$ equally spaced inner knots, Turnbull's NPMLE, and the Weibull distribution (Hong, Meeker and Mccalley (2009); Balakrishnan and Mitra (2012)) to fit the data. Figure 7 presents the estimated reliability functions based on the three methods. We also tried $m_n = 4$ and 6 (not shown); here, the estimated reliability function is almost the same as that with $m_n = 5$. In general, the spline estimate and the empirical estimate agree quite well. The empirical estimate becomes constant when t is greater than the largest failure time, which is 42.1 in this example. By contrast, the spline method estimates the reliability function up to the largest observation time, which corresponds to a censoring time of 58. The wider range shows the greater flexibility of the spline method. Moreover, it is clear that spline-based estimator is more smooth. The comparisons of the spline-based estimator and the Weibull estimator show that the spline-based method can be used to assess the goodness-of-fit of a parametric model. To improve the quality of the uncertainty in the spline estimates, the random weighted bootstrap procedure (Hong, Meeker and Mccalley (2009)) with 50,000 resamples is used to construct a pointwise 95% confidence band of the reliability function, as shown in Figure 8.

Hong, Meeker and Mccalley (2009) also collected the failure times of transformers from the same manufacturer as "MC_Old65" but different generations.

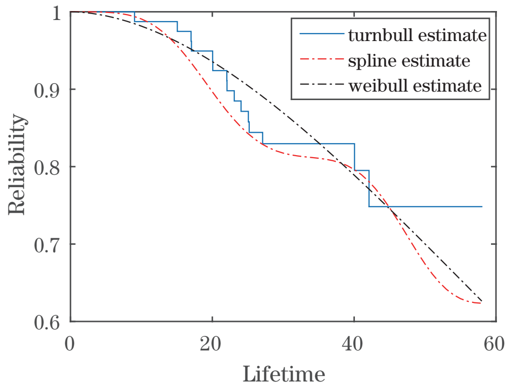


Figure 7. Comparisons of estimators for the reliability function based on "MC_Old65" data (Hong, Meeker and Mccalley (2009)): The stair line is the Trunbull estimate, the dash line is for the Weibull estimate, and the chain line is the spline estimate.

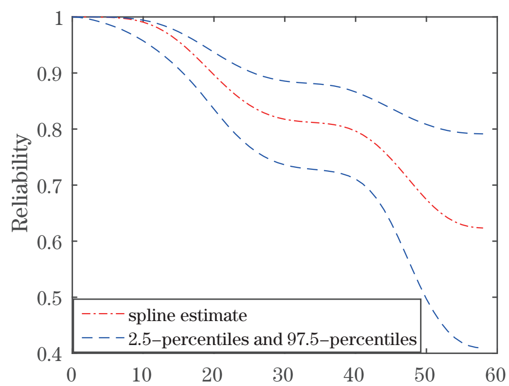


Figure 8. Spline estimates of the reliability function based on "MC_Old65" (Hong, Meeker and Mccalley (2009)), and the pointwise 95% two-sided confidence band based on 50,000 simulations.

We choose the data set "MC_Old55" as the second group. The difference between the two groups is the type of insulation. The problem of interest here is to compare the two groups and check whether the data from the two groups can be merged. The test statistics (4.1) developed in Section 4 are used for the comparison. We obtain $T_n = 7.643$, 10.392, 6.028, and 4.653, with $W_n(t) = W_n^{(k)}(t)$, $k = 1, 2, 3, 4$, defined in Section 5. All values correspond to p-values $\ll 0.0001$. The proposed tests suggest that the two groups are significantly different. Therefore, the effect of insulation type cannot be ignored, and the two data sets "MC_Old65" and "MC_Old55" cannot be combined.

Acknowledgement

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Appendix

Proof of Theorem 1 (Consistency)

The log-likelihood function for λ is

$$\mathcal{L}(\lambda|\mathbf{D}) = \sum_{i=1}^{n} \left\{\delta_i \ln \lambda(Y_i) - [\Lambda(Y_i) - \Lambda(L_i)]\right\}.$$

With the knot sequence $\mathcal{T} = \{t_j\}_1^{m_n+2l}$ specified in Section 2, there exists a spline $\lambda_n(t) \in \psi_{l,\mathcal{T}}$ with order $l \geq r+2$ such that $\|\lambda_n(t) - \lambda_0\|_\infty = \sup_{t\in[a,b]} |\lambda_n(t) - \lambda_0(t)| = O(n^{-\nu r})$, according to Corollary 6.21 of Schumaker (2007, p. 227). Choose a positive function $h_n \in \psi_{l,\mathcal{T}}$ such that $\|h_n\|_2^2 = O(n^{-2\nu r} + n^{-(1-\nu)/2})$. Therefore, for any $\alpha > 0$, $\|\lambda_n - \lambda_0 + \alpha h_n\|_2^2 = O(n^{-2\nu r} + n^{-(1-\nu)/2})$ for sufficiently large n.

Denote $n\mathbb{M}_n(\lambda) = \mathcal{L}(\lambda|\mathbf{D})$ and $H_n(\alpha) = \mathbb{M}_n(\lambda_n + \alpha h_n)$. The first and second derivatives of H_n are

$$H_n'(\alpha) = n^{-1} \sum_{i=1}^{n} \left\{ \frac{\delta_i h_n(Y_i)}{\lambda_n(Y_i) + \alpha h_n(Y_i)} - \int_{L_i}^{Y_i} h_n(x)dx \right\},$$

$$H_n''(\alpha) = -n^{-1} \sum_{i=1}^{n} \frac{\delta_i h_n^2(Y_i)}{[\lambda_n(Y_i) + \alpha h_n(Y_i)]^2} < 0.$$

Thus $H_n'(\alpha)$ is a non-increasing function. Therefore, to prove Theorem 1, it is sufficient to show that, for any $\alpha_0 > 0$, $H_n'(\alpha_0) < 0$ and $H_n'(-\alpha_0) > 0$ except on an event with probability converging to zero. Then $\hat{\lambda}_n$ must be between $\lambda_n - \alpha_0 h_n$ and $\lambda_n + \alpha_0 h_n$ with probability converging to one, so that $P(\|\hat{\lambda}_n - \lambda_n\|_2 \leq \alpha_0 \|h_n\|_2) \to 1$ as $n \to \infty$. We first show $H_n'(\alpha_0) < 0$. Express $H_n'(\alpha_0)$ as

$$H_n'(\alpha_0) = \underbrace{(\mathbb{P}_n - P)\left\{\frac{\delta h_n(Y)}{\lambda_n(Y) + \alpha_0 h_n(Y)}\right\}}_{I_{n1}} - \underbrace{(\mathbb{P}_n - P)\left\{\int_L^Y h_n(x)dx\right\}}_{I_{n2}}$$

$$+\underbrace{P\left\{\frac{\delta h_n(Y)}{\lambda_n(Y)+\alpha_0 h_n(Y)}-\int_L^Y h_n(x)dx\right\}}_{I_{n3}}.$$

For the term I_{n1}, we firstly prove there exists a positive constant c and a positive integer N such that when $n>N$, $\|1/(\lambda_n+\alpha_0 h_n)\|_\infty \le c$. Given $\eta>0$, define the class $\mathcal{F}_{\eta,n}=\{\lambda:\ \lambda\in\psi_{l,\mathcal{T}},\ d(\lambda,\lambda_n)\le\eta\}$. There exists a positive integer N_1 such that when $n>N_1$, $\lambda_n+\alpha_0 h_n\in\mathcal{F}_{\eta,n}$. According to Condition 1, there exists a positive integer $N(>N_1)$ such that for $n>N$,

$$d(\lambda,\lambda_0)\le d(\lambda,\lambda_n)+d(\lambda_n,\lambda_0)\le\eta+O(N^{-\mu r})<2\eta,$$

where $\lambda\in\mathcal{F}_{\eta,n}, n>N_1$. Let $\mathcal{F}_\eta=\cup_{n\ge N}\mathcal{F}_{\eta,n}$ and then we have $\lambda_n+\alpha_0 h_n\in\mathcal{F}_\eta$, for $n>N$. Corollary 6.21 of Schumaker (2007, p. 227) shows that for any function $\lambda\in\mathcal{F}_\eta$, λ has uniformly bounded derivatives up to order $l-1$. Then according to Corollary 2.7.4 of van der Vaart and Wellner (1996, p. 158), we can find that given ε such that $0<\varepsilon\le\eta$, $\mathcal{F}_\eta$ can be covered by a set of ε-brackets $\{[\underline{\lambda}_k,\bar\lambda_k]:k=1,2,\ldots,(1/\varepsilon)^{c_0/l}\}$, where c_0 is a constant depending on l. For any $\lambda\in\mathcal{F}_\eta$, there exists a bracket $[\underline{\lambda}_k,\bar\lambda_k]$, such that $\underline{\lambda}_k(t)\le\lambda(t)\le\bar\lambda_k(t)$ for all $t\in[a,b]$, where $d^2(\underline{\lambda}_k,\bar\lambda_k)=\int|\underline{\lambda}_k-\bar\lambda_k|^2dF^*(t)\le\varepsilon^2$, $k=1,2,\ldots,(1/\varepsilon)^{c_0/l}$. Then we have

$$d(\underline{\lambda}_k,\lambda_0)\le d(\underline{\lambda}_k,\lambda)+d(\lambda,\lambda_n)+d(\lambda_n,\lambda_0)<\varepsilon+2\eta,$$

where $\lambda\in\mathcal{F}_\eta, n>N$. Then by the converse of Lemma 7.1 from Wellner and Zhang (2007, p. 2140), we get $\sup_{t\in[a,b]}|\underline{\lambda}_k-\lambda_0|\le c_1(\varepsilon+2\eta)^{2/3}$, c_1 is constant. Since λ_0 is positive and bounded on $[a,b]$, there exists $c_2>0$ such that $\underline{\lambda}_k>c_2>0$. Similarly , there exists a positive c_3 such that $\bar\lambda_k>c_3>0$. That means $\underline{\lambda}_k$ and $\bar\lambda_k$ have the positive lower bounds. Therefore, there exists a positive constant c such that $\|1/(\lambda_n+\alpha_0 h_n)\|_\infty\le c$, for $n>N$. Since $h_n\in\psi_{l,\mathcal{T}}$ with $\|h_n\|_2^2=O(n^{-2\nu r}+n^{-(1-\nu)/2})$, and $\|1/(\lambda_n+\alpha_0 h_n)\|_\infty\le c<\infty$ for $n>N$, we conclude from Lemma 11 of Huang (1998) that

$$\sup_{h_n\in\psi_{l,\mathcal{T}}}\frac{|(\mathbb{P}_n-P)(\delta h_n)/(\lambda_n+\alpha_0 h_n)|}{\|h_n\|_2}=O_P\left(n^{-(1-\nu)/2}\right).$$

Moreover, for the second term, we conclude similarly from Lemma 11 of Huang (1998) that

$$\sup_{h_n\in\psi_{l,\mathcal{T}}}\frac{\left|(\mathbb{P}_n-P)\{\int_L^Y h_n(x)dx\}\right|}{\{E[\int_L^Y h_n^2(t)dt]\}^{1/2}}=O_P\left(n^{-(1-\nu)/2}\right),$$

where $E[\int_L^Y h_n^2(t)dt]\asymp\|h_n\|_2^2$. Hence, $I_{n1}+I_{n2}=O_P\big(n^{(-(1-\nu))/2}(n^{-\nu r}+n^{-(1-\nu)/4})\big)$.

For the third term, since λ_0 is the maximum of $M(\lambda)$, the first derivative is zero at λ_0. Then we have

$$P\left\{\frac{\delta h_n(Y)}{\lambda_0(Y)} - \int_L^Y h_n(x)dx\right\} = 0,$$

and by adding and subtracting terms,

$$I_{n3} = P\left\{\frac{\delta h_n(Y)}{\lambda_n(Y) + \alpha_0 h_n(Y)} - \frac{\delta h_n(Y)}{\lambda_0(Y)}\right\}.$$

Define $m(s) = 1/(\lambda_0 + s\Delta)$, where $\Delta = \lambda_n - \lambda_0 + \alpha_0 h_n,\ 0 \le s \le 1$. By the Taylor expansion, there exists $\theta \in (0,1)$ such that

$$m(s) = m(0) + m'(\theta)s = \frac{1}{\lambda_0} + \left(-\frac{\Delta}{(\lambda_0 + \theta\Delta)^2}\right)s.$$

Therefore,

$$I_{n2} \le E\left\{-\frac{\Delta}{(\lambda_0 + \theta\ eta^2}\right\}h_n \lesssim -Eh_n^2 = O(n^{-2\nu r} + n^{-(1-\nu)/2}).$$

Since $n^{-2\nu r} + n^{-(1-\nu)/2} > n^{-1/2} > n^{-(1-\nu)}$, we have

$$H_n'(\alpha_0) \le O_P\left(n^{-(1-\nu)/(2)}(n^{-\nu r} + n^{-(1-\nu)/4})\right) - O(n^{-2\nu r} + n^{-(1-\nu)/2}) < 0,$$

except on an event with probability converging to zero. The same arguments show that $H_n'(-\alpha_0) > 0$ with probability converging to 1.

Proof of Theorem 2 (Rate of Convergence)

Denote $m_\lambda(X) = \delta \ln \lambda(Y) - \int_L^Y \lambda(u)du$ and define $M(\lambda) = Pm_\lambda(X)$ and $\mathbb{M}_n(\lambda) = \mathbb{P}_n m_\lambda(X)$. Then the log-likelihood function can be written as $n\mathbb{P}_n m_\lambda(X)$. Given $\eta > 0$, define the class

$$\mathscr{F}_\eta = \{\lambda | \lambda \in \psi_{l,\mathcal{T}}, d(\lambda, \lambda_0) \le \eta\}.$$

By the result of Theorem 1, $\hat{\lambda}_n \in \mathscr{F}_\eta$ for sufficiently large n. For $\eta > 0$ and any $\varepsilon < \eta$,

$$\log N_{[\,]}\{\varepsilon, \psi_{l,\mathcal{T}}, L_2(P)\} \le cq_n \log\left(\frac{\eta}{\varepsilon}\right),\ J_{[\,]}\{\eta, \psi_{l,\mathcal{T}}, L_2(P)\} \le c_0 q_n^{1/2}\eta,$$

where $q_n = m_n + l$ is the number of spline base functions, c and c_0 are constants (Shen and Wong (1994, p. 597)). Therefore for each $\lambda \in \mathscr{F}_\eta$, there exists a bracket $[\underline{\lambda}_k, \bar{\lambda}_k]$, such that

$$\underline{\lambda}_k(t) \le \lambda(t) \le \bar{\lambda}_k(t)$$

for all $t \in [a,b]$, where $d^2(\underline{\lambda}_k, \bar{\lambda}_k) = \int |\underline{\lambda}_k - \bar{\lambda}_k|^2 dF^*(t) \le \varepsilon^2$, $k = 1, 2, \ldots, (\eta/\varepsilon)^{cq_n}$. Moreover, $\underline{\lambda}_k$ and $\bar{\lambda}_k$ are bounded on $[a,b]$ and have positive lower bounds.

Since λ_0 is the maximum of $M(\lambda)$, the first derivative is zero at λ_0 and the second derivative is negative definite. According to the Taylor expansion,

$$M(\lambda) = M(\lambda_0) + 0 + \frac{M''(\lambda_0)}{2}(\lambda - \lambda_0)^2 + o(\lambda - \lambda_0)^2.$$

Thus, for $\lambda \in \mathscr{F}_\eta$, $M(\lambda_0) - M(\lambda) \gtrsim d^2(\lambda, \lambda_0)$. Next, define the class

$$\mathcal{M}_\eta = \{m_\lambda - m_{\lambda_0} : \lambda \in \mathscr{F}_\eta\}.$$

Let

$$\underline{\mathfrak{m}}_k(X) = \delta \ln \underline{\lambda}_k(Y) - \int_L^Y \bar{\lambda}_k(x)dx - m_{\lambda_0}(x)$$

and

$$\bar{\mathfrak{m}}_k(X) = \delta \ln \bar{\lambda}_k(Y) - \int_L^Y \underline{\lambda}_k(x)dx - m_{\lambda_0}(x).$$

Clearly, the class $\mathcal{M}_\eta$ is covered by the set $[\underline{\mathfrak{m}}_k, \bar{\mathfrak{m}}_k]$, $k = 1, 2, \ldots, (\eta/\varepsilon)^{cq_n}$. To prove the uniformly bounded class $\mathcal{M}_\eta$ is a Donsker class, we need to show that (equicontinuity condition)

$$\|\bar{\mathfrak{m}}_k - \underline{\mathfrak{m}}_k\|_2^2 \lesssim \varepsilon^2.$$

Let

$$f = \bar{\mathfrak{m}}_k - \underline{\mathfrak{m}}_k = \underbrace{\delta\left(\log \bar{\lambda}_k - \log \underline{\lambda}_k\right)}_{I_1} + \underbrace{\int_L^Y \left(\bar{\lambda}_k - \underline{\lambda}_k\right) dx}_{I_2}.$$

Since $\sup_{t\in[a,b]} |\underline{\lambda}_k - \lambda_0| \le \varepsilon_1$ and $\sup_{t\in[a,b]} |\bar{\lambda}_k - \lambda_0| \le \varepsilon_2$ by converse theorem of Lemma 7.1 from Wellner and Zhang (2007, p. 2140), the boundedness of Y, L, δ and λ_0 yields the boundedness of I_1 and I_2 on $[a, b]$. Then according to Cauchy Schwartz inequality,

$$|f|^2 \le 2\left\{\delta^2\left(\log\frac{\bar{\lambda}_k}{\underline{\lambda}_k}\right)^2 + \left[\int_L^Y (\bar{\lambda}_k - \underline{\lambda}_k)dx\right]^2\right\}.$$

By taylor expansion,

$$\log\frac{\bar{\lambda}_k}{\underline{\lambda}_k} = \frac{1}{\theta}\left(\bar{\lambda}_k - \underline{\lambda}_k\right),$$

where θ between $\bar{\lambda}_k$ and $\underline{\lambda}_k$. Since $\bar{\lambda}_k$ and $\underline{\lambda}_k$ are bounded functions on $[a, b]$, there exists a constant c_1 such that

$$\log\frac{\bar{\lambda}_k}{\underline{\lambda}_k} < c_1\left(\bar{\lambda}_k - \underline{\lambda}_k\right).$$

Therefore

$$P(|f|^2) \lesssim d^2(\bar{\lambda}_k, \underline{\lambda}_k) \le \varepsilon^2.$$

Then according to the Lemma 3.4.2 of van der Vaart and Wellner (1996, p. 324), we obtain

$$E_P\|n^{1/2}(\mathbb{P}-P)\|_{\mathcal{M}_\eta} \lesssim J_{[\,]}(\eta,\mathcal{M}_\eta,L_2(P))\left\{1+\frac{J_{[\,]}(\eta,\mathcal{M}_\eta,L_2(P))}{\eta^2 n^{1/2}}\right\}. \qquad (A.1)$$

The right-hand side of (A.1) yields $\phi_n(\eta) = c_2(q_n^{(1/2)}\eta + q_n/n^{1/2})$. It is easy to see that $\phi(\eta)/\eta$ is decreasing in η, and

$$r_n^2\phi\left(\frac{1}{r_n}\right) = r_n q_n^{1/2} + \frac{r_n^2 q_n}{n^{1/2}} \le n^{1/2}$$

yields $r_n = n^{(1-\nu)/2}$, where $0<\nu<1/2$. Hence, $n^{(1-\nu)/2}d(\hat{\lambda}_n,\lambda_0) = O_p(1)$ by Theorem 3.4.1 of van der Vaart and Wellner (1996, p. 322). If $\nu = r/(2r+1)$, the rate of convergence of $\hat{\lambda}_n$ is $r/(2r+1)$, which is the same as the optimal rate in nonparametric regression.

Proof of Theorem 3 (Asymptotic Normality)

According to Theorem 1 of Zhao and Zhang (2017, p. 933), we need the following conditions to establish the asymptotic normality.

- A1. $\sqrt{n}(\mathbb{P}_n - P)(\phi(\hat{\lambda}_n;X)[h] - \phi(\lambda_0;X)[h]) = o_p(1)$.
- A2. $\sqrt{n}(G_n - G)(\lambda_0)[h]$ convergences in distribution to a tight Gaussian process on $l^\infty(\mathcal{H}_r)$.
- A3. $G(\lambda_0)[h] = 0$ and $G_n(\hat{\lambda}_n)[h] = o_p(n^{-1/2})$.
- A4. $G(\lambda)[h]$ is the Fréchet-differentiable at λ_0 with a continuous derivative, denoted by $\dot{G}_{\lambda_0}[h]$.
- A5. $G(\hat{\lambda}_n)[h] - G(\lambda_0)[h] - \dot{G}_{\lambda_0}(\hat{\lambda}_n - \lambda_0)[h] = o_p(n^{-1/2})$, where $\dot{G}_{\lambda_0}(\hat{\lambda}_n - \lambda_0)[h]$ is the directional derivative at λ_0 in the direction $(\lambda-\lambda_0)$.

Then we need to verify conditions A1-A5 above.

For A1, given $\varepsilon>0$, define the class

$$\mathcal{G}_n(\varepsilon)[h] = \{\phi(\lambda;X)[h] - \phi(\lambda_0;X)[h] : \lambda \in \psi_{l,\mathcal{T}} \text{ such that } d(\lambda,\lambda_0)\le\varepsilon, h\in\mathcal{H}_r\}.$$

Let

$$\underline{g}_k(X) = \frac{\delta}{\bar{\lambda}_k}h - \frac{\delta}{\lambda_0}h$$

and

$$\bar{g}_k(X) = \frac{\delta}{\underline{\lambda}_k}h - \frac{\delta}{\lambda_0}h,$$

where $\underline{\lambda}_k$ and $\bar{\lambda}_k$ are similar defined in the proof of Theorem 2. Clearly, the class

$\mathcal{G}_n(\varepsilon)[h]$ is covered by the set $[\underline{g}_k, \bar{g}_k]$, $k = 1, 2, \ldots, (\eta/\varepsilon)^{cq_n}$. Let

$$f = \bar{g}_k - \underline{g}_k = \frac{\delta}{\underline{\lambda}_k} h - \frac{\delta}{\bar{\lambda}_k} h = \delta h(Y) \frac{\bar{\lambda}_k - \underline{\lambda}_k}{\bar{\lambda}_k \underline{\lambda}_k}.$$

By the Cauchy-Schawartz inequality,

$$Pf^2 = P\left[\delta h(Y) \frac{\bar{\lambda}_k - \underline{\lambda}_k}{\bar{\lambda}_k \underline{\lambda}_k}\right]^2 \lesssim P\left[\frac{1}{\bar{\lambda}_k \underline{\lambda}_k}\right]^2 \left(\bar{\lambda}_k - \underline{\lambda}_k\right)^2,$$

where the last inequality bolds due to $h \in \mathcal{H}_r$. Due to the result of Theorem 1, we can find $\lambda \in \psi_{l,\mathcal{T}}$ such that $d(\lambda, \lambda_0) \leq \varepsilon$. Therefore, $d(\underline{\lambda}_k, \lambda_0) \leq d(\underline{\lambda}_k, \lambda) + d(\lambda, \lambda_0) < 2\varepsilon$. Then by converse theorem of Lemma 7.1 from Wellner and Zhang (2007, p. 2140), we get $\sup_{t\in[a,b]} |\underline{\lambda}_k - \lambda_0| \leq c_1 \varepsilon^{2/3}$, c_1 is constant. Since λ_0 is positive and bounded on $[a, b]$, there exists a constant $c_2 > 0$ such that $\underline{\lambda}_k > c_2 > 0$. Similarly as $\bar{\lambda}_k$. So $\bar{\lambda}_k$ and $\underline{\lambda}_k$ have the positive lower bounds. Furthermore, using the fact that $\bar{\lambda}_k$and $\underline{\lambda}_k$ have the positive lower bounds, we have

$$Pf^2 \lesssim P\left(\bar{\lambda}_k - \underline{\lambda}_k\right)^2 \lesssim d^2\left(\bar{\lambda}_k - \underline{\lambda}_k\right) \leq \varepsilon^2.$$

Then according to the Lemma 3.4.2 of van der Vaart and Wellner (1996, p. 324), we obtain

$$E_P\|n^{1/2}(\mathbb{P} - P)\|_{\mathcal{G}_n(\varepsilon)[h]} \lesssim J_{[\,]}(\varepsilon, \mathcal{G}_n(\varepsilon)[h], L_2(P)) \left\{1 + \frac{J_{[\,]}(\varepsilon, \mathcal{G}_n(\varepsilon)[h], L_2(P))}{\varepsilon^2 n^{1/2}}\right\}. \tag{A.2}$$

Theorem 1 shows that $d(\hat{\lambda}_n, \lambda_0) \to 0$ almost surely. Hence that by converse theorem of Lemma 7.1 from Wellner and Zhang (2007, p. 2140), we have

$$\sup_{t\in[a,b]} |\hat{\lambda}_n(t) - \lambda_0(t)| \to 0 \text{ almost surely.}$$

Moreover, Theorem 2 shows that $n^{r/(2r+1)}\|\hat{\lambda}_n - \lambda_0\|_2 = O_p(1)$ with $r > 1$. Therefore we have $\phi(\hat{\lambda}_n; X)[h] - \phi(\lambda_0; X)[h] \in \mathcal{G}_n(\varepsilon_n)[h]$ with $\varepsilon_n = O(n^{-r/(1+2r)})$. Moreover, for any $\phi(\lambda; X)[h] - \phi(\lambda_0; X)[h] \in \mathcal{G}_n(\varepsilon_n)[h]$, exists $M > 0$, such that

$$P(\phi(\lambda; X)[h] - \phi(\lambda_0; X)[h])^2 \lesssim \varepsilon_n^2 \text{ and } \sup_{h\in\mathcal{H}_r} |\phi(\lambda; X)[h] - \phi(\lambda_0; X)[h]| < M.$$

Hence, we have

$$\begin{aligned}
&E_P\|n^{1/2}(\mathbb{P} - P)\|_{\mathcal{G}_n(\varepsilon_n)[h]} \\
&\lesssim J_{[\,]}(\varepsilon_n, \mathcal{G}_n(\varepsilon_n)[h], L_2(P)) \left\{1 + \frac{J_{[\,]}(\varepsilon_n, \mathcal{G}_n(\varepsilon_n)[h], L_2(P))}{{\varepsilon_n}^2 n^{1/2}}\right\} \\
&\lesssim q_n^{1/2}\varepsilon_n + q_n n^{-1/2} \\
&= O(n^{1/2(1+2r) - r/(1+2r)}) + O(n^{1/(1+2r) - 1/2}) \\
&= o(1).
\end{aligned}$$

Therefore, we have

$$\sqrt{n}(\mathbb{P}_n - P)(\phi(\hat{\lambda}_n; X)[h] - \phi(\lambda_0; X)[h]) = o_p(1)$$

uniformly in h.

For A2, since $\mathcal{H}_r$ is a Donsker class and the function $\phi(\lambda_0; X)[h]$ is a bounded Lipschitz function with respect to $\mathcal{H}_r$, we have the class $\{\phi(\lambda_0; X)[h] : h \in \mathcal{H}_r\}$ is Donsker (van der Vaart and Wellner (1996, Thm. 2.10.6, p. 192)). Then based on Theorem 3.10.12 (van der Vaart and Wellner (1996, p. 407)), $\sqrt{n}(G_n - G)(\lambda_0)[h]$ convergences in distribution to a tight Gaussian process on $l^\infty(\mathcal{H}_r)$.

To prove the third part A3, clearly $G(\lambda_0)[h] = 0$. Note that $\hat{\lambda}_n = \sum_{j=1}^{q_n} \hat{\alpha}_j B_j(t)$ satisfies the following score function

$$n^{-1} \sum_{i=1}^{n} \left\{ \frac{\delta_i B_j(Y_i)}{\hat{\lambda}_n(Y_i)} - \int_{L_i}^{Y_i} B_j(x) dx \right\} = 0,\ j = 1, \ldots, q_n.$$

Thus, for any $h_n = \sum_{j=1}^{q_n} \alpha_j B_j \in \varphi_{l,\mathcal{T}}$, we have

$$n^{-1} \sum_{i=1}^{n} \left\{ \frac{\delta_i h_n(Y_i)}{\hat{\lambda}_n(Y_i)} - \int_{L_i}^{Y_i} h_n(x) dx \right\} = 0,$$

that is, $G_n(\hat{\lambda}_n)[h_n] = 0$ for any $h_n \in \varphi_{l,\mathcal{T}}$. Moreover, for any $h \in \mathcal{H}_r$, there exists $h_n \in \varphi_{l,\mathcal{T}}$ such that $\|h - h_n\|_\infty = O(n^{-r\nu})$. Therefore, we have

$$\begin{aligned}
&G_n(\hat{\lambda}_n)[h] \\
&= G_n(\hat{\lambda}_n)[h - h_n] \\
&= \Big\{ G_n(\hat{\lambda}_n)[h - h_n] - G_n(\lambda_0)[h - h_n] \Big\} + G_n(\lambda_0)[h - h_n] - G_0(\lambda_0)[h - h_n] \\
&= n^{-1} \sum_{i=1}^{n} \delta_i \left\{ \frac{1}{\hat{\lambda}_n(Y_i)} - \frac{1}{\lambda_0(Y_i)} \right\} [h(Y_i) - h_n(Y_i)] + \left(G_n - G\right)(\lambda_0)[h - h_n] \\
&\lesssim d(\hat{\lambda}_n, \lambda_0)\|h - h_n\|_\infty + o_p(n^{-1/2}) \\
&= o_p(n^{-1/2}),
\end{aligned}$$

where the proof of A2 leads to that $(G_n - G)(\lambda_0)[h - h_n]$ convergences in distribution to a tight Gaussian process.

For A4, by the assumption of smoothness, $G(\lambda)[h]$ is the Fr$\acute{e}$chet-differentiable at λ_0 with a continuous derivative, denoted by $\dot{G}_{\lambda_0}[h]$. Moreover, the directional derivative $\dot{G}_{\lambda_0}(\hat{\lambda}_n - \lambda_0)[h]$ at λ_0 in the direction $(\lambda - \lambda_0)$ can be defined as

$$\dot{G}_{\lambda_0}(\hat{\lambda}_n - \lambda_0)[h] = \bigtriangledown_{(\lambda-\lambda_0)} G(\lambda_0)[h] = \lim_{\varepsilon \to 0} \frac{G(\lambda_0 + \varepsilon(\lambda - \lambda_0))[h] - G(\lambda_0)[h]}{\varepsilon}$$

$$= -P\left[\delta h(Y)\frac{\lambda(Y)-\lambda_0(Y)}{\lambda_0^2(Y)}\right]$$
$$= -\int \frac{h(t)}{\lambda_0^2(t)}(\lambda(t)-\lambda_0(t))dF^*(t),$$

where $F^*(t) = P(L \le T \le C, T \le t)$.

Then for A5, we can prove

$$\begin{aligned}
&G(\hat{\lambda}_n)[h] - G(\lambda_0)[h] - \dot{G}_{\lambda_0}(\hat{\lambda}_n - \lambda_0)[h] \\
&= P\left[\delta h(Y)\left(\frac{1}{\hat{\lambda}_n(Y)} - \frac{1}{\lambda_0(Y)}\right)\right] + P\left[\delta h(Y)\frac{\hat{\lambda}_n(Y)-\lambda_0(Y)}{\lambda_0^2(Y)}\right] \\
&= P\left[\frac{\delta h(Y)}{\hat{\lambda}_n(Y)\lambda_0^2(Y)}\left\{\hat{\lambda}_n(Y)-\lambda_0(Y)\right\}^2\right] \\
&= O_p(d^2(\hat{\lambda}_n,\lambda_0)) = O_p(n^{-2r/(1+2r)}) = o_p(n^{-1/2}).
\end{aligned}$$

Thus it follows from Theorem 1 (Zhao and Zhang (2017, p. 934)) that

$$\sqrt{n}\int \frac{h(t)}{\lambda_0^2(t)}(\lambda(t)-\lambda_0(t))dF^*(t) = -\sqrt{n}\dot{G}_{\lambda_0}(\hat{\lambda}_n - \lambda_0)[h] = \sqrt{n}(G_n - G)(\lambda_0)[h] + o_p(1).$$

Proof of Theorem 4

We first note that U_n can be rewritten as

$$\begin{aligned}
U_n &= \frac{\sqrt{n}}{\sum_{i=1}^n \delta_i}\sum_{i=1}^n \delta_i W_n(Y_i)\left\{\hat{\lambda}_n^{(1)}(Y_i) - \hat{\lambda}_n^{(2)}(Y_i)\right\} \\
&= \sqrt{n}\mathbb{P}_n\left[W_n^{(k)}(Y)\left\{\hat{\lambda}_n^{(1)}(Y) - \hat{\lambda}_n^{(2)}(Y)\right\}\right] \\
&= \sqrt{n}\mathbb{P}_n\left[W_n^{(k)}(Y)\left\{\hat{\lambda}_n^{(1)}(Y) - \lambda_0(Y)\right\}\right] \\
&\quad - \sqrt{n}\mathbb{P}_n\left[W_n^{(k)}(Y)\left\{\hat{\lambda}_n^{(2)}(Y) - \lambda_0(Y)\right\}\right] \\
&= U_n^{(1)} - U_n^{(2)}.
\end{aligned}$$

Then we define $U = [U_n^{(1)}, U_n^{(2)}]$ and note that $U_n^{(l)}$ can be written as

$$U_n^{(l)} = U_{1n}^{(l)} + U_{2n}^{(l)} + \sqrt{\frac{n}{n_l}}U_{3n}^{(l)},$$

where, for $l = 1, 2$,

$$\begin{aligned}
U_{1n}^{(l)} &= \sqrt{n}(\mathbb{P}_n - P)\left[W_n^{(k)}(Y)\left\{\hat{\lambda}_n^{(l)}(Y) - \lambda_0(Y)\right\}\right], \\
U_{2n}^{(l)} &= \sqrt{n}P\left[\left(W_n^{(k)}(Y) - W(Y)\right)\left\{\hat{\lambda}_n^{(l)}(Y) - \lambda_0(Y)\right\}\right],
\end{aligned}$$

$$U_{3n}^{(l)} = \sqrt{n_l} P \left[W(Y) \left\{ \hat{\lambda}_n^{(l)}(Y) - \lambda_0(Y) \right\} \right].$$

Firstly consider $U_{1n}^{(l)} = \sqrt{n}(P_n - P) \left[W_n^{(k)}(Y) \{ \hat{\lambda}_n^{(l)}(Y) - \lambda_0(Y) \} \right]$. Set

$$\mathcal{G} = \{ \xi : [0, b] \to [0, \tau] \},$$

where τ is the uniform upper bound of weight process $W_n^{(k)}$, $k = 1, 2, 3, 4$. Let

$$\psi_\lambda(\xi, \mathbf{D}) = \xi(Y)\{\lambda(Y) - \lambda_0(Y)\},$$

where $\xi \in \mathcal{G}$, $\lambda \in \mathscr{F}_\eta$ and $\mathscr{F}_\eta = \{\lambda | \lambda \in \psi_{l,\mathcal{T}}, d(\lambda, \lambda_0) \leq \eta\}$. For a fixed $\xi \in \mathcal{G}$, let

$$\Psi_\eta(\xi) = \{ \psi_\lambda(\xi, \mathbf{D}) : \lambda \in \mathscr{F}_\eta \},$$

where $\eta > 0$. By the conclusion of Theorem 1, $\hat{\lambda}_n^{(l)} \in \mathscr{F}_\eta$ for any $\eta > 0$ and sufficiently large n. Note that it follows from Corollary 2.7.2 of van der Vaart and Wellner (1996, p. 157) that

$$N_{[\,]}(\varepsilon, \mathscr{F}_\eta, L_2(P)) \leq e^{c_1/\varepsilon^{1/2}},$$

for some constant c_1. Then, we have

$$N_{[\,]}(\varepsilon, \Psi_\eta(\xi), L_2(P)) \leq e^{c_1/\varepsilon^{1/2}}.$$

It can be easily shown that $|\psi_\lambda(\xi, \mathbf{D})| \lesssim \eta$, and $P\psi_\lambda^2(\xi, \mathbf{D}) \lesssim \eta^2$. Thus,

$$J_{[\,]}(\eta, \Psi_\eta(\xi), L_2(P)) = \int_0^\eta \sqrt{\log N_{[\,]}(\varepsilon \|\psi\|_{P,2}, \Psi_\eta(\xi), L_2(P)) + 1} d\varepsilon \lesssim \eta.$$

Hence, from Theorem 2.14.2 of van der Vaart and Wellner (1996), we have

$$\begin{aligned} & E^* \left\{ \sup_{\psi_\lambda(\xi, X) \in \Psi_\eta(\xi)} \left| \sqrt{n}(\mathbb{P}_n - P)\psi_\lambda(\xi, X) \right| \right\} \\ & \lesssim \Big[J_{[\,]}(\eta, \Psi_\eta(\xi), L_2(P)) \|\psi\|_{P,2} + \sqrt{n} P\psi\{\psi > \sqrt{n} a(\eta)\} \\ & \quad + \|\psi\|_{P,2} \sqrt{\log N_{[\,]}(\eta \|\psi\|_{P,2}, \Psi_\eta(\xi), L_2(P)) + 1} \Big], \end{aligned}$$

where

$$a(\eta) = \frac{\eta \|\psi\|_{P,2}}{\sqrt{\log N_{[\,]}(\eta \|\psi\|_{P,2}, \Psi_\eta(\xi), L_2(P)) + 1}}.$$

Then, it is easily shown that

$$\limsup_{n \to \infty} E^* \left\{ \sup_{\psi_\lambda(\xi, X) \in \Psi_\eta(\xi)} \left| \sqrt{n}(\mathbb{P}_n - P)\psi_\lambda(\xi, X) \right| \right\} \lesssim \eta^{1/2}.$$

It follows from $d(\hat{\lambda}_n^{(l)}, \lambda_0) \stackrel{a.s.}{\to} 0$ that

$$\limsup_{n \to \infty} E \left\{ |\sqrt{n}(\mathbb{P}_n - P)\psi_{\hat{\lambda}_n^{(l)}}(W_n^{(k)}, X)| \right\} \lesssim \eta^{1/2}.$$

Let $\eta \to 0$ to see

$$\lim_{n\to\infty} E\left\{|\sqrt{n}(\mathbb{P}_n - P)\psi_{\hat{\lambda}_n^{(l)}}(W_n^{(k)}, X)|\right\} = 0,$$

which yields $U_{1n}^{(l)} = o_p(1)$.

Next consider $U_{2n}^{(l)} = \sqrt{n}P\left[(W_n^{(k)}(Y) - W(Y))\left\{\hat{\lambda}_n^{(l)}(Y) - \lambda_0(Y)\right\}\right]$.

$$\begin{aligned}
U_{2n}^{(l)} &= \sqrt{n}P\left\{\left(W_n^{(k)}(Y) - W(Y)\right)\left[\hat{\lambda}_n^{(l)}(Y) - \lambda_0(Y)\right]\right\} \\
&\le \sqrt{n}\int \left|W_n^{(k)}(t) - W(t)\right|\left|\hat{\lambda}_n^{(l)}(Y) - \lambda_0(Y)\right| dF^*(t) \\
&\lesssim \sqrt{n}\left\{\int_0^b \left(W_n^{(k)}(t) - W(t)\right)^2 dF^*(t)\right\}^{1/2} \\
&\quad \left\{\int_0^b \left(\hat{\lambda}_n^{(l)}(Y) - \lambda_0(Y)\right)^2 dF^*(t)\right\}^{1/2}.
\end{aligned}$$

Since

$$\left[\int_a^b \left\{W_n^{(k)}(t) - W(t)\right\}^2 dt\right]^{1/2} = o_p\left(n^{-1/(2(1+2r))}\right)$$

and

$$\left\{\int_0^b \left(\hat{\lambda}_n^{(l)}(Y) - \lambda_0(Y)\right)^2 dF^*(t)\right\}^{1/2} = O_p\left(n^{-r/(1+2r)}\right),$$

we have $U_{2n}^{(l)} = o_p(1)$.

From the result of Theorem 3, we have , for $l = 1, 2$,

$$U_{3n}^{(l)} = \sqrt{n_l}(\mathbb{P}_{n_l} - P)\left[\phi(\lambda_0; X)[h]\right] + o_p(1) = Z_n^{(l)} + o_p(1),$$

where $\mathbb{P}_{n_l} f = (1/n_l)\sum_{i\in S_l} f(Z_i)$ and S_l denotes the set of indices for subjects in group l, $l = 1, 2$. Moreover, $Z_n^{(l)}$'s converge to U_w in distribution as $n \to \infty$, where U_w has a normal distribution with mean zero and variance $\sigma^2 = E\left[\phi^2(\lambda_0; X)[h]\right]$. Evidently, $Z_n^{(l)}$'s are independent and identically distributed, because $\mathbb{P}_{n_l}$ is the empirical measure based on group l respectively. Hence, we have

$$U_n = \sqrt{\frac{n}{n_1}}Z_n^{(1)} - \sqrt{\frac{n}{n_2}}Z_n^{(2)} + o_p(1),$$

where U_n convergences in distribution to $N(0, 1/(p(1-p))\sigma^2)$. Thus it follows that U_n has an asymptotic normal distribution $N(0, \sigma_w^2)$, where

$$\sigma_w^2 = \frac{1}{p(1-p)}E\{\phi^2(\lambda_0; X)[h_w]\}.$$

To show that $\hat{\sigma}_w^2 - \sigma_w^2 = o_p(1)$. We set ${\sigma_w}^2 = P\phi^2(\lambda_0; X)[h_w]$ and $\hat{\sigma}_w^2 =$

$\mathbb{P}_n\phi^2(\hat{\lambda}_n; X)[\hat{h}_w]$. Note that

$$\begin{aligned}\hat{\sigma}_w^2 - \sigma_w^2 &= \mathbb{P}_n\phi^2(\hat{\lambda}_n; X)[W_n^{(k)}\hat{\lambda}_n^2] - P\phi^2(\lambda_0; X)[W\lambda_0^2] \\ &= \mathbb{P}_n\left\{\phi^2(\hat{\lambda}_n; X)[W_n^{(k)}\hat{\lambda}_n^2] - \phi^2(\lambda_0; X)[W_n^{(k)}\lambda_0^2]\right\} \\ &\quad + (\mathbb{P}_n - P)\phi^2(\lambda_0; X)[W\lambda_0^2] \\ &\quad + \mathbb{P}_n\left\{\phi^2(\lambda_0; X)[W_n^{(k)}\lambda_0^2] - \phi^2(\lambda_0; X)[W\lambda_0^2]\right\}.\end{aligned}$$

It can be easily shown that

$$\mathbb{P}_n\left\{\phi^2(\hat{\lambda}_n; X)[W_n^{(k)}\hat{\lambda}_n^2] - \phi^2(\lambda_0; X)[W_n^{(k)}\lambda_0^2]\right\} = o_p(1)$$

and

$$(\mathbb{P}_n - P)\phi^2(\lambda_0; X)[W\lambda_0^2] = o_p(1).$$

On the other hand, based on the conditions imposed on W_n and W, we have

$$\left|\phi(\lambda_0; X)[W_n^{(k)}\lambda_0^2] - \phi(\lambda_0; X)[W\lambda_0^2]\right| = \left|\phi(\lambda_0; X)[(W_n^{(k)} - W)\lambda_0^2]\right| = o_p(1),$$

and

$$\left|\phi(\lambda_0; X)[W_n^{(k)}\lambda_0^2] + \phi(\lambda_0; X)[W\lambda_0^2]\right| = \left|\phi(\lambda_0; X)[(W_n^{(k)} + W)\lambda_0^2]\right| = O(1).$$

The above two displays imply that

$$\begin{aligned}\left|\phi^2(\lambda_0; X)[W_n^{(k)}\lambda_0^2] - \phi^2(\lambda_0; X)[W\lambda_0^2]\right| &\lesssim \left|\phi(\lambda_0; X)[W_n^{(k)}\lambda_0^2] - \phi(\lambda_0; X)[W\lambda_0^2]\right| \\ &= o_p(1).\end{aligned}$$

Therefore,

$$\mathbb{P}_n\left\{\left|\phi^2(\lambda_0; X)[W_n^{(k)}\lambda_0^2] - \phi^2(\lambda_0; X)[W\lambda_0^2]\right|\right\} = o_p(1).$$

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