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LACK OF FIT TEST FOR INFINITE VARIATION JUMPS AT HIGH FREQUENCIES

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Abstract: This paper is concerned with testing for infinite variation jumps, in addition to a continuous local martingale component, driven by Brownian motion using high-frequency data. We develop a lack of fit type test based on the empirical distribution of the "devolatized" increments. Under the null hypothesis that the jump component is of finite variation, the empirical process associated with the "devolatized" increments converges to a Gaussian process in the Skorohod topology. Under the alternative hypothesis that the jumps are of infinite variation, the empirical process explodes. Theoretical results and simulation show good performance on the size and power of the test. A financial data set is analyzed.

Key words and phrases: Infinite activity jumps, infinite variation jumps, Itô semimartingale.

1. Introduction

Itô's semimartingale is of vital importance in stochastic calculus and widely used in empirical studies in finance, environmental science, and other fields. Mathematically, it is defined on a filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{F}, P)$ and assumes the integral form

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + X_t^d, \tag{1.1}$$

where $\int_0^t b_s ds$ is the drift with b an optional and càdlàg process, $\int_0^t \sigma_s dW_s$ is a continuous local martingale with σ_s an adapted process and W a standard Brownian motion, and X^d is a pure-jump component with the jump activity index (JAI) β defined by

$$\beta = \inf\left\{r \left| \sum_{0\le s\le T} |\Delta_s X|^r < \infty \right.\right\}, \tag{1.2}$$

where $\Delta_s X = X_s - X_{s-}$ is the jump size of X at time s and T is a fixed time horizon. For the estimation of β, we refer to Aït-Sahalia and Jacod (2009a), Todorov and Tauchen (2010), Todorov and Tauchen (2011), and Jing et al. (2012). In this paper, we restrict Itô semimartingales to a deterministic JAI. β serves as an indicator of the activity of jumps contained in X^d. The larger the β, the more active the jumps. In particular, jump processes of finite activity (e.g., compound Poisson process), finite variation, and infinite variation correspond to $\beta = 0$, $0 < \beta < 1$ and $1 < \beta < 2$, respectively.

Model (1.1) contains the Brownian driving force in the continuous local martingale term and the jump driving force in X^d, including the infinite variation jump driving force ($\beta > 1$) and the finite variation jump driving force ($\beta < 1$). Inference on the types of driving force underlying high-frequency data or time series is of increasing interest in recent years. Barndorff-Nielsen and Shephard (2006), Fan and Wang (2008), and Aït-Sahalia and Jacod (2009b) derived tests for the existence of jumps. Aït-Sahalia and Jacod (2010), Todorov and Tauchen (2014), Jing, Kong and Liu (2012), Kong, Liu and Jing (2015), and Todorov (2015) established tests for the necessity of adding a Brownian force. Aït-Sahalia and Jacod (2011) studied whether the jump component is of finite activity or not when the Brownian force is present. See Jacod and Protter (2012) for recent developments.

In this paper, we investigate whether it is necessary to add an infinite variation jump term, in addition to a continuous local martingale, for the purpose of modeling high-frequency data. It is generally accepted that small jumps of infinite variation fluctuate rapidly and play the role of a Brownian motion. But many empirical studies, cf, Aït-Sahalia and Jacod (2009a), show evidence of infinite variation jumps in the presence of the diffusive process. This asks for a statistical method to validate the modeling assumption of no infinite variation jumps given a diffusive term. Statistically, this could be formulated as testing for the hypotheses on $\{\inf_{0\le s\le T}\sigma_s^2 > 0\}$,

$$H_0 : \beta \le 1 \quad vs. \quad H_1 : \beta > 1. \tag{1.3}$$

While much empirical evidence for the presence of infinite variation jumps is based on an estimate of JAI, it is not reliable to propose a test using a point estimator of it, since the estimation is challenging in the simultaneous presence of a diffusive term. For example, the best convergence rate in Bull (2016) is $n^{\beta/4}$, which is slow for $\beta \le 1$ and depends on the unknown β. In this paper, we develop a lack of fit type test based on the empirical distribution of the "devolatized" increments. Under H_0, the empirical process associated with the "devolatized" increments converges to a fully specified Gaussian process on any compact subset of R at a rate close to $\sqrt{n}$. Under H_1, it explodes. A comparison of the empirical distribution function with the Gaussian distribution function yields the "Kolmognov-Smirnov type" test statistic, which successfully differentiates the null and alternative hypotheses.

Thus we provide a theoretically sound test for the presence of infinite variation jumps in the presence of a diffusive term and a jump component of finite variation. This test has many nice properties. In addition to its good performance on size and power, the test becomes more powerful as JAI increases. This is surprising since, at first glance, the larger the β the more the jump process behaves like a Brownian motion in path regularities and hence seemingly harder to detect. The reason for the increasing power against β is essentially because of the frequency of small jumps or equivalently the jump intensity around the origin. See the remark below Assumption 1 for more details.

We establish the asymptotic theory of the empirical distribution of the "devolatized" increments of Itô semimartingales with infinitely active or even infinite variation jumps that is more desirable than that presented in Todorov and Tauchen (2014) allowing only for finitely many jumps. Proofs are nontrivial. Another distinctive feature of our paper is that we estimate the spot volatility based on the realized Laplace transform approach.

The rest of the paper is arranged as follows. In Section 2, we state the assumptions. Main results including the limit theorems of the empirical processes, and the size and power performance of the lack of fit test are presented in Section 3. Section 4 is devoted to monte carlo simulations and data analysis. Proofs are postponed to the supplement.

Throughout the paper, we assume that the available data set is $\{X_{t_i}; 0 \leq i \leq n\}$, discretely sampled from X, with $t_i = i\Delta_n$ with $\Delta_n = T/n$ for $0 \leq i \leq n$. We denote the jth one-step increment by $\Delta_j^n X = X_{t_j} - X_{t_{j-1}}, \ \ 1 \leq j \leq n$.

2. Basic Assumptions

First, we make an assumption on the pure-jump process X^d.

Assumption 1. 1. *If* $\beta > 1$,

$$X_t^d = \int_0^t \gamma_{s-}^+ dY_s^+ + \int_0^t \gamma_{s-}^- dY_s^- + \int_0^t \int_R \delta(s,z) p(ds,dz),$$

where Y^+ *and* Y^- *are independent Lévy processes with positive jumps and Lévy triplets* $(0, 0, F^{\pm})$, $\gamma^{\pm}$ *are two càdlàg adapted processes,* δ *is a càdlàg predictable process, and* p *is a Poisson random measure on* $R_+ \times R$ *with intensity* $q(dt, dx) = dt \otimes dx$. *For some* $r < 1$, *the Lévy measure satisfies*

$$\left|\overline{F}^{\pm}(x)-\frac{1}{x^{\beta}}\right| \equiv \left|F^{\pm}((x,\infty))-\frac{1}{x^{\beta}}\right| \leq g(x),\ x\in(0,\infty),$$

with $g(x)$ being a decreasing function s.t. $\int_0^1 x^{r-1}g(x)dx<\infty$ and $\int_1^{\infty} g(x)dx<\infty$.
2. *If $\beta \leq 1$, $\gamma^{+}=\gamma^{-}\equiv 0$.*

Thus when $\beta > 1$, X^d has two β-stable-like driving forces. The assumption on the Lévy measure of the stable-like driving processes is flexible enough, requiring that the Lévy density is equivalent to that of a stable Lévy process around the origin. The families of tempered stable processes with Lévy density $\left(e^{-cx}/x^{1+\beta}\right) I(x>0)$ and truncated stable processes with Lévy density $\left(1/x^{1+\beta}\right) I(x\in(0,c])$ are special examples. For them, the jumps have moments of any polynomial order. As the coefficient processes $\gamma_t^{\pm}$ can be either positive or negative, X^d allows for asymmetric jumps in its trajectory, which captures the empirical feature (downward jumps occur more frequently than upward jumps) in financial markets due to risk aversion. When the jump activity index is smaller than 1, X^d is completely nonparametric.

Assumption 2. 1. *σ_t is an Itô semimartingale of the form*

$$\sigma_t=\sigma_0+\int_0^t b_s^{\sigma}ds+\int_0^t H_s^{\sigma}dW_s+\int_0^t H_s'^{\sigma}dW_s'+\int_0^t\int_R \delta^{\sigma}(s,x)\tilde{p}(ds,dx),$$

where all the integrands are optional processes satisfying the integrable condition in Itô's sense, and $\tilde{p}$ is another Poisson random measure independent of W with intensity $\tilde{q}(dt,dx)=dt\otimes dx$. W and W' are independent Brownian motions that are independent of (p,Y^{+},Y^{-}).
2. *$\gamma_t^{\pm}$ are Itô semimartingales of the form*

$$\gamma_t^{\pm}=\gamma_0^{\pm}+\int_0^t b_s^{\gamma^{\pm}}ds+\int_0^t H_s^{\gamma^{\pm}}dW_s+\int_0^t H_s'^{\gamma^{\pm}}dW_s' + \int_0^t\int_R \delta^{\gamma^{\pm}}(s,x)\bar{p}(ds,dx),$$

where all the integrands are optional processes satisfying the integrable condition in Itô's sense, and $\bar{p}$ is another Poisson random measure independent of W with intensity $\bar{q}(dt,dx)=dt\otimes dx$.

Assumption 2 is standard in the literature. It allows for the "leverage" effect due to the common driving force W in X, σ, and $\gamma^{\pm}$. $\tilde{p}$ and $\bar{p}$ are two new Poisson random measures that may not necessarily be independent of each other, or independent of p.

Assumption 3. *There is a sequence τ_n of stopping times increasing to infinity, a sequence a'_n of numbers, and a nonnegative Lebesgue-integrable function J on R, such that the processes b, H^σ, $H^{\gamma^\pm}$, $\gamma^\pm$ are càdlàg adapted, the coefficients δ, δ^σ and $\delta^{\gamma^\pm}$ are predictable, the processes b^σ, H'^σ, $b^{\gamma^\pm}$ and $H'^{\gamma^\pm}$ are progressively measurable and, for some constant $r < 1$,*

$$
\begin{aligned}
&t < \tau_n \Rightarrow |\delta(t,z)|^r \wedge 1 \le a'_n J(z),\ \ |\delta^\sigma(t,z)| \wedge 1 \le a'_n J(z),\\
&|\delta^{\gamma^\pm}(t,z)| \wedge 1 \le a'_n J(z);\\
&t < \tau_n, V = b, b^\sigma, H^\sigma, H'^\sigma, b^{\gamma^\pm}, H^{\gamma^\pm}, H'^{\gamma^\pm}, |\sigma|, |\sigma|^{-1} \Rightarrow |V_t| \le a'_n,\\
&V = b, H^\sigma, H'^\sigma, \delta^\sigma, H^{\gamma^\pm}, H'^{\gamma^\pm}, \delta^{\gamma^\pm}\\
&\ \Rightarrow |E(V_{(t+s)\wedge\tau_n} - V_{t\wedge\tau_n}|\mathcal{F}_t)| + E(|V_{(t+s)\wedge\tau_n} - V_{t\wedge\tau_n}|^2|\mathcal{F}_t) \le a'_n s.
\end{aligned}
$$

If $\beta = 1$, $E([\delta\{(t+s)\wedge\tau_n, x\} - \delta(t\wedge\tau_n, x)]^2|\mathcal{F}_t) \le a'_n s^{1+\epsilon}$ uniformly for $x \in R$ and any $\epsilon > 0$.

Assumption 3 is rather general. It is satisfied by the multifactor stochastic volatility model that is widely used in financial econometrics, e.g., the popular affine jump diffusion model in Duffie, Pan and Singleton (2000). This assumption also requires that the jumps in σ and $\gamma^\pm$ be of finite variation.

3. Methodology and Main Results

3.1. Motivational example

Our test statistic is based on the empirical process of the "devolatized" increments. As a motivation for constructing the empirical process, consider a simple example of an Itô process, $X_t = \sigma_t^0 W_t + \gamma_{\alpha,t} Y_{\alpha,t} + \gamma_{\beta,t} Y_{\beta,t}$, where σ_t^0, $\gamma_{\alpha,t}$, and $\gamma_{\beta,t}$ are three deterministic smooth functions, and $Y_{\alpha,t}$ and $Y_{\beta,t}$ are, respectively, symmetric α and β stable Lévy processes with $\alpha < 1$ and $\beta > 1$ representing jump processes of finite variation and infinite variation, respectively. By smoothness and self-similarity, we have

$$\frac{\Delta_j^n X}{\sqrt{\Delta_n}} \approx \sigma^0_{j\Delta_n} \mathcal{N}_j(0,1) + \gamma_{\alpha,j\Delta_n} \Delta_n^{1/\alpha - 1/2} S_j^\alpha + \gamma_{\beta,j\Delta_n} \Delta_n^{1/\beta-1/2} S_j^\beta, \tag{3.1}$$

where the $\mathcal{N}_j(0,1)$'s, S_j^α's, and S_j^β's form sequences of i.i.d. standard normal, symmetric α and β stable variables, respectively. From (3.1), the function $P((\Delta_j^n X)/\sqrt{\Delta_n} > x)$ is not time invariant due to time varying of σ_t^0. So, it is beneficial to standardize the increments by estimated spot volatilities.

For fixed x, the function $P(\Delta_j^n X/(|\sigma^0_{j\Delta_n}|\sqrt{\Delta_n}) > x) \sim 1 - \Phi(x)$ where $\Phi(x)$ is the c.d.f. of standard normal random variable and hence is mainly due to the

Brownian motion. The α-stable Lévy motion is completely dominated by the other two terms and reduces to 0 even faster than $\sqrt{\Delta_n}$. Although the β-stable Lévy motion is negligible, its rate converging to 0 is slower than $\sqrt{\Delta_n}$ resulting in a non-negligible bias ('signal' under H_1) that cannot be balanced by the asymptotic variance (typically of order close to $\sqrt{\Delta_n}$). So under the null hypothesis, the devolatized increments look like those of a standard Brownian motion in distribution. Under H_1, the unbalanced bias term due to $\gamma_{\beta,j\Delta_n}\Delta_n^{1/\beta-1/2}S_j^\beta$ stands out, resulting in good power which increases against β.

The tail for $x \sim c\Delta_n^{-\varpi}$, for some constants $c > 0$ and $0 < \varpi < 1/2$, is dominated by β (α under H_0) stable Lévy motion if it is present, since $P(\Delta_j^n Y_\beta/\sqrt{\Delta_n} > x) \sim c' x^{-\beta}\Delta_n^{1-\beta/2}$ ($c' x^{-\alpha}\Delta_n^{1-\alpha/2}$ under H_0) for some constant $c' > 0$, while $P(\Delta_j^n W/\sqrt{\Delta_n} > x) \sim (1/x)\phi(x)$, where $\phi(x)$ is the p.d.f. of the standard normal distribution. So it is better not to consider too large values of x, otherwise size may not be safeguarded.

3.2. Method of devolatizing the increments

To estimate the spot volatility, we use the local method and thus split the interval into non-overlapping shrinking blocks with block lengths equal to $2v_n$, consisting of $2k_n$ intervals of length Δ_n, where k_n is some integer depending on n. Aggregation of local estimates is widely used in other contexts, cf, Mykland and Zhang (2009) Mykland, Shephard and Shephard (2012), and Todorov and Tauchen (2012). Returning to our simple example of an Itô process, the characteristic function of the **symmetrized** increment is

$$\begin{aligned}
&E\left[\exp\left\{\sqrt{-1}u\frac{X_{t+2\Delta_n} - X_{t+\Delta_n} - (X_{t+\Delta_n} - X_t)}{\sqrt{\Delta_n}}\right\}\right] \\
&\approx E\cos\left\{u\frac{X_{t+2\Delta_n} - X_{t+\Delta_n} - (X_{t+\Delta_n} - X_t)}{\sqrt{\Delta_n}}\right\} \\
&\approx \exp\left(-u^2\sigma_t^{02} - 2|u\gamma_{\alpha,t}|^\alpha\Delta_n^{1-\alpha/2} - 2|u\gamma_{\beta,t}|^\beta\Delta_n^{1-\beta/2}\right).
\end{aligned}$$

This suggests computing the sample analogue in the jth shrinking block,

$$\begin{aligned}
L_j(u) &= \frac{1}{k_n}\sum_{l=1}^{k_n}\cos\left(u\frac{\Delta_{2jk_n+2l}^n X - \Delta_{2jk_n+2l-1}^n X}{\sqrt{\Delta_n}}\right), \\
c_j(u) &= -\frac{1}{u^2}\log\left\{L_j(u)\vee\frac{c}{\sqrt{k_n}}\right\}, \quad 0 \le j \le \left\{\frac{n}{(2k_n)}\right\} - 1,
\end{aligned}$$

where the lower threshold $c/\sqrt{k_n}$ is to assure that the logarithmic function makes sense. Our local estimate of $\sigma_j^2 \equiv \sigma_{2jk_n\Delta_n}^2$ is

$$\hat{\sigma}_j^2(u) = c_j(u) - \frac{1}{u^2 k_n}[\sinh\{u^2 c_j(u)\}]^2,$$

where the subtracted term is used to correct the bias due to the jumps. This local estimate was used in Jacod and Todorov (2014) to get an efficient estimator of the integrated volatility, and in Kong, Liu and Jing (2015) to test for the presence of a Brownian force. There are some other methods to estimate spot volatility, cf, Todorov and Tauchen (2014), Jacod and Rosenbaum (2013), Fan and Wang (2007), Li and Xiu (2016), and Li, Liu and Xiu (2017). For long-memory volatility models that are driven by fractional Brownian motion, we refer to Comte and Renault (1996) and Comte and Renault (1998). The major advantage of this Laplace-transform-based local estimator is that it can easily separate the effect of the Brownian force and the stable-like driving force.

For properly chosen m_n and u_n, the empirical distribution function of the devolatized increments is defined as

$$\hat{F}_n(u_n, \tau) = \frac{1}{\{n/(2k_n)\}m_n} \sum_{j=1}^{[n/(2k_n)]} \sum_{i=2jk_n+1}^{2jk_n+m_n} I\left\{\frac{\Delta_i^n X}{\sqrt{\hat{\sigma}_{j-1}^2(u_n)\Delta_n}} \le \tau\right\}, \qquad (3.2)$$

for $\tau \in R$.

3.3. Empirical processes and their limiting properties

We need some notation. Let $\tilde{\Phi}_{j,i}^n(\tau)$ be the c.d.f. of

$$\frac{\Delta_{2jk_n+i}^n W}{\sqrt{\Delta_n}} + \frac{\gamma_{t_{2jv_n+(i-1)\Delta_n}}^{+}\Delta_{2jk_n+i}^n Y^+ + \gamma_{t_{2jv_n+(i-1)\Delta_n}}^{-}\Delta_{2jk_n+i}^n Y^-}{|\sigma_{2(j-1)k_n\Delta_n}|\sqrt{\Delta_n}},$$

conditional on $\mathcal{F}_{2jv_n+(i-1)\Delta_n}$, and

$$\overline{\Phi}_n(\tau) = \frac{1}{[n/(2k_n)]m_n} \sum_{j=1}^{[n/(2k_n)]} \sum_{i=1}^{m_n} \tilde{\Phi}_{j,i}^n(\tau).$$

When the JAI of X^d is no larger than 1, $\bar{\Phi}_n(\tau)$ reduces to $\Phi(\tau)$.

Consider the empirical process

$$\hat{Y}_n(\tau) = \sqrt{\frac{n}{2k_n}m_n}\left\{\hat{F}_n(u_n, \tau) - \Phi(\tau)\right\}.$$

As notation, let

$$U_t(u) = \exp\left(-u^2\sigma_t^2 - 2\Delta_n^{1-\beta/2}u^\beta a_t\right), \text{where } a_t = \chi(\beta)(|\gamma_t^+|^\beta + |\gamma_t^-|^\beta)$$

with $\chi(\beta) = \int_0^\infty y^{-\beta}\sin(y)dy$, and $\xi_j(u) = L_j(u)/U_{2jk_n\Delta_n}(u) - 1$. Take

$$\hat{Z}_1^n(\tau) = \frac{1}{\{n/(2k_n)\}m_n} \sum_{j=1}^{[n/(2k_n)]-1} \sum_{i=1}^{m_n} \left\{ I\left(\frac{\Delta_{2jk_n+i}^n W}{\sqrt{\Delta_n}} \le \tau \right) - \Phi(\tau) \right\},$$

$$\hat{Z}_2^n(\tau) = \frac{1}{\{n/(2k_n)\}} \sum_{j=1}^{\{n/(2k_n)\}-1} \left\{ \frac{1}{2} \tau \Phi'(\tau) \frac{\xi_{j-1}(u_n)}{u_n^2 \sigma_{2k_n(j-1)\Delta_n}^2} \right\}.$$

We first state a functional central limit theorem of Donsker's type on the empirical process when $\beta \le 1$.

Theorem 1. *Suppose k_n, $u_n \downarrow 0$, m_n, and Δ_n satisfy $k_n \Delta_n^{1/2} \to 0$,*

$$k_n \Delta_n^{1/2-\epsilon} \to \infty,\ \sup_n \frac{k_n \Delta_n^{1/2}}{u_n^4} < \infty,\ \frac{k_n}{m_n (\log n)^4} \to \infty,\ \frac{k_n \Delta_n^\epsilon}{m_n} \to 0, \tag{3.3}$$

for any $\epsilon > 0$, and $\beta \le 1$ ($\gamma^+ = \gamma^- \equiv 0$). Under Assumptions 1-3, we have that $\hat{Y}_n(\tau)$ and

$$\sqrt{\frac{n}{2k_n} m_n} \left[\hat{Z}_1^n(\tau) + \hat{Z}_2^n(\tau) - \frac{1}{4Tk_n} \tau^2 \left\{ \Phi''(\tau) - \Phi'(\tau) \right\} \right] \tag{3.4}$$

converge weakly to the same Gaussian process in Skorohod's topology for all $\tau \in \mathcal{A}_c$ where $\mathcal{A}_c$ is any compact subset of R, and

$$\left\{ \sqrt{\frac{n}{2k_n} m_n} \hat{Z}_1^n(\tau), \sqrt{\frac{n}{2k_n} 2k_n} \hat{Z}_2^n(\tau) \right\} \Rightarrow \{Z_1(\tau), Z_2(\tau)\}, \tag{3.5}$$

functionally in $\tau \in \mathcal{A}_c$, in the sense of the Skorohod topology, where $Z_1(\tau)$ and $Z_2(\tau)$ are two independent centered Gaussian processes with covariance functions

$$Cov\{Z_1(\tau_1), Z_1(\tau_2)\} = \Phi(\tau_1 \wedge \tau_2) - \Phi(\tau_1)\Phi(\tau_2),\ \tau_1, \tau_2 \in R, \tag{3.6}$$

$$Cov\{Z_2(\tau_1), Z_2(\tau_2)\} = \frac{\tau_1 \Phi'(\tau_1) \tau_2 \Phi'(\tau_2)}{T}. \tag{3.7}$$

When $\beta > 1$, we do not have the functional central limit theorem, but we do have a pointwise central limit theorem for $\hat{Y}_n(\tau)$. This is already enough in the context of detecting infinite variation jumps.

Theorem 2. *If $\beta > 1$ and the other conditions in Theorem 1 hold, then*

$$\begin{aligned} \hat{Y}_n(\tau) = & \sqrt{\frac{n}{2k_n} m_n} \left[\hat{Z}_1^n(\tau) + \hat{Z}_2^n(\tau) - \frac{1}{4Tk_n} \tau^2 \{ \Phi''(\tau) - \Phi'(\tau) \} \right] \\ & + \sqrt{\frac{n}{2k_n} m_n} \left(\overline{\Phi}_n(\tau) - \Phi(\tau) \right) \\ & + \frac{m_n^{1/2} \tau \Phi'(\tau) u_n^{\beta-2} \Delta_n^{1-\beta/2}}{\{n/(2k_n)\}^{1/2}} \sum_{j=1}^{\{n/(2k_n)\}} \frac{a_{j\Delta_n}}{\sigma_{(j-1)\Delta_n}^2} + o_p(1), \end{aligned} \tag{3.8}$$

pointwise in $\tau \in R$.

Remark 1. In Theorem 2, $\overline{\Phi}_n(\tau)$ depends on the mixed distribution of the Brownian force and the β-stable-like Lévy process. In the limiting sense, under the conditions in Theorem 2, we have (proved in the supplement),

$$\begin{aligned}&\overline{\Phi}_n(\tau)\\&=\Phi(\tau)-\frac{\Phi'(\tau)\Delta_n^{1/\beta-1/2}}{(n/2k_n)m_n}\sum_{j=1}^{\{n/(2k_n)\}}\sum_{i=1}^{m_n}\frac{\gamma_{j,i-1}^{-}EY_1^{-}+\gamma_{j,i-1}^{+}EY_1^{+}}{|\sigma_{2(j-1)k_n\Delta_n}|}+O_p(\Delta_n^{1-\beta/2-\epsilon})\\&=\Phi(\tau)-\frac{\Phi'(\tau)\Delta_n^{1/\beta-1/2}}{T}\int_0^T\frac{\gamma_s^{+}EY_1^{+}+\gamma_s^{-}EY_1^{-}}{|\sigma_s|}ds+O_p(\Delta_n^{(1-\beta/2-\epsilon)\wedge(1/\beta-1/4+\epsilon)}).\end{aligned}\tag{3.9}$$

There are then two major sources of the bias for the empirical process. One is the increment of the driving jump process, $\Delta_{2jk_n+i}^n Y^{\pm}$, which distorts the empirical distribution function away from $\Phi(\tau)$, see (3.9); The second is the estimation error of the instantaneous volatility using the Laplace-transform-based procedure, see (3.8). If the jump component is of finite variation, both biases vanish asymptotically, but explode otherwise. If $\beta > 1$, the first bias dominates the second in the explosive rate.

Remark 2. (3.3) provides guidelines for choosing k_n, m_n, and u_n. If one sets $u_n = c_1/\log n$, $k_n = [c_2\sqrt{n}/(\log n)^4]$, and $m_n = [c_1 k_n/(\log n)^{4+\epsilon}]$ for some $\epsilon > 0$, (3.3) is satisfied. This implies that the convergence rate of the empirical distribution function of the "devolatized" increments is almost $\sqrt{n}$.

For completeness, though irrelevant to the testing background here, we have a result for $\hat{F}_n(u_n,\tau)$ when the Brownian force does not exist, but the β-stable-like driving process is present.

Theorem 3. *Suppose Assumptions* 1-3 *hold, except that* $|\sigma|\equiv 0$ *and* $|\gamma^{+}|^{-1}$ *and* $|\gamma^{-}|^{-1}$ *are strictly positive. If* (3.3) *holds,*

$$\hat{F}_n(u_n,\tau)\to^P 1,\tag{3.10}$$

pointwise in $\tau \in R$.

3.4. Test statistics

Tests for infinite variation jumps via estimating the jump activity index directly with simultaneous presence of diffusive process suffer from the difficulty of separating the continuous term and the jump term, and typically some semi-parametric assumption for the jumps under H_0 are needed, see for example, Aït-Sahalia and Jacod (2009a) and Jing et al. (2012). Theorems 1-2, as well as their remarks, motivate us to propose a test statistic of the "Kolmogrov-Smirnov" type as

$$\mathcal{T}^n_{\mathcal{A}_c} \equiv \sqrt{\frac{n}{2k_n} m_n} \sup_{\tau \in \mathcal{A}_c} \left| \hat{F}(u_n, \tau) - \Phi(\tau) \right|, \tag{3.11}$$

where $\mathcal{A}_c$ is a compact subset in R. Under H_0, by Theorem 1,

$$\mathcal{T}^n_{\mathcal{A}_c} =^{\mathcal{L}} \sqrt{\frac{n}{2k_n} m_n} \sup_{\tau \in \mathcal{A}_c} \left| \hat{Z}^n_1(\tau) + \hat{Z}^n_2(\tau) - \frac{\tau^2(\Phi''(\tau) - \Phi'(\tau))}{4k_n T} \right| + o_p(1). \tag{3.12}$$

Then by (3.12) and Theorem 1, we can approximate the $(1-\alpha)$ quantile of the null distribution by that of the distribution of

$$\sup_{\tau \in \mathcal{A}_c} \left| Z_1(\tau) + \sqrt{\frac{m_n}{2k_n}} Z_2(\tau) - \frac{\sqrt{(n/2k_n) m_n}}{4k_n T} \tau^2 \{\Phi''(\tau) - \Phi'(\tau)\} \right|, \tag{3.13}$$

which is denoted by $Q_n(\alpha, \mathcal{A}_c)$; This can be estimated via simulation. The test is

$$\mathcal{T}^n_{\mathcal{A}_c} > Q_n(\alpha, \mathcal{A}_c) \Leftrightarrow \text{Rejecting } H_0. \tag{3.14}$$

Theorem 4. *Under the conditions in Theorem* 2*, we have*

$$P(\mathcal{T}^n_{\mathcal{A}_c} > Q_n(\alpha, \mathcal{A}_c) \mid \beta \le 1) \to \alpha, \tag{3.15}$$

and on $\{\int_0^T (\gamma^+_s E Y^+_1 + \gamma^-_s E Y^-_1)/|\sigma_s| ds \neq 0\}$,

$$P(\mathcal{T}^n_{\mathcal{A}_c} > Q_n(\alpha, \mathcal{A}_c) \mid \beta > 1) \to 1. \tag{3.16}$$

As seen in (3.9), the larger the β, the higher the detection power.

4. Numerical Studies

4.1. Monte carlo experiments

In this section, we report on simulation studies to check the finite sample performance of the test. We generated data from a stochastic volatility model, 5,000 times,

$$X_t = X_0 + \int_0^t \sqrt{c_s} dW_s + 0.5 Y_t, \quad 0 \le t \le T, \tag{4.1}$$

$$c_t = c_0 + \int_0^t 0.03(1.0 - c_s) ds + 0.15 \int_0^t \sqrt{c_s} dW'_s, \tag{4.2}$$

where Y_t is a skewed β-stable Lévy process, with the negative jumps appearing twice as often as the positive jumps to capture the market stylized feature caused by risk aversion. The volatility c_t is a square root diffusion process widely used in financial applications. The parameters in c_t are specified as in Jacod and Todorov (2014). To incorporate the leverage effect, we set $corr(dW, dW') = -0.5$. We also set $u_n = 0.5$ in estimating c_t. To illustrate the effect of the microstructure noise and find a good sampling frequency, we added a noise term to X at the observation times, to observe

$$\tilde{X}_{t_i} = X_{t_i} + \epsilon_{t_i}, \text{ where } \epsilon_{t_i} \sim \mathcal{N}(0, 0.035).$$

In the simulation studies, our tests were carried out on the data set $\{\tilde{X}_{t_i}; i = 1, \ldots, n\}$. We took $\mathcal{A}_c$ as a set of grid points in $[-1, 1]$ with step length 0.2. On the same $\mathcal{A}_c$, we used the monte carlo method to find $Q_n(\alpha, \mathcal{A}_c)$ for $\alpha = 0.05$ and 0.10.

We first sampled the data every 5, 10 or 30 seconds, or one minute in a single day ($T = 1$). Correspondingly the sample sizes were 4,680, 2,340, 780 and 390, respectively. The conditions in Theorem 1 imply that finite sample k_n should be smaller than $\sqrt{n}$ and m_n should be smaller than k_n. Hence for the sampling frequencies mentioned above we set the pairs of (k_n, m_n) to be $(60, 28)$, $(40, 20)$, $(26, 18)$, and $(18, 12)$ with k_n/m_n ranging from 1.5 to 2.15 with an increasing trend. Figure 1 displays the power functions of the test against β, from which we observe the following.

Due to the bias caused by the microstructure noise, our test cannot control type I error when the sampling frequency is as high as every 5 seconds; when the sampling frequencies are equal to or below every 10 seconds, our test is quite robust to the microstructure noise. We observe that for $\beta < 1$, the three curves in both panels are close to the nominal level, consistent with Theorem 4; for $\beta > 1$, the four curves in both panels increase, demonstrating that the larger the β, the more powerful our test, consistent with Remark 1; and as the sample sizes increase, the performance of our test improves, consistent with Theorem 4.

To check the performance of the test for sparser sampling schemes, we generated five days ($T = 5$) of one-minute and five-minute data and ten days ($T = 10$) of five-minute data from the noise contaminated model. We set (k_n, m_n) as $(39, 20)$, $(12, 8)$, and $(25, 15)$. The power functions are plotted in the left panel of Figure 2. Except for five days of one-minute data, the power functions indicate over-rejection of our test due to the increase of discretization error as Δ_n increases. This suggests why the power function for five days of one-minute data performs better than that for five days of five-minute data. The power function for five days of five-minute data is below that for ten days of five-minute data, possibly due to the adverse effect of the microstructure noise. This and the findings in Figure 1 show that sampling up to half a minute in a single day or every one minute in five days is safe for killing the noise in testing for infinite variation

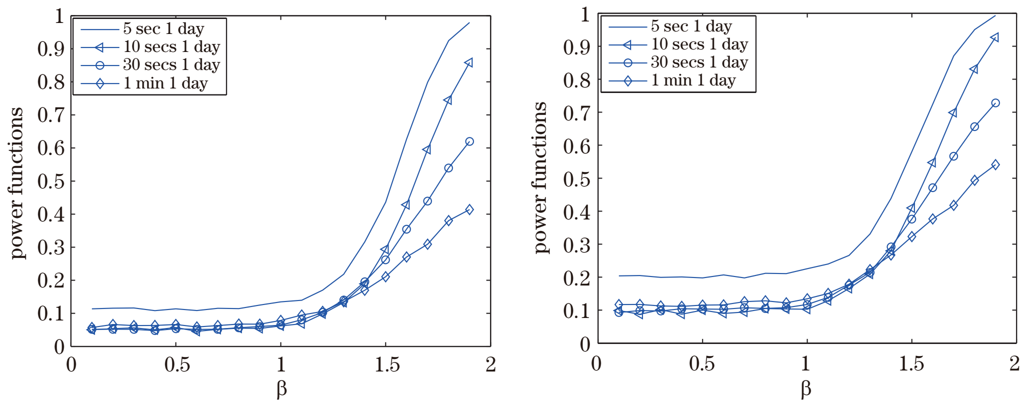

Figure 1. The power functions of the test for infinite variation jumps for five-second (solid), ten-second (left triangle), thirty-second (circle), and one-minute (square) data in a single day ($T = 1$); Left panel: $\alpha = 0.05$; Right panel: $\alpha = 0.10$.

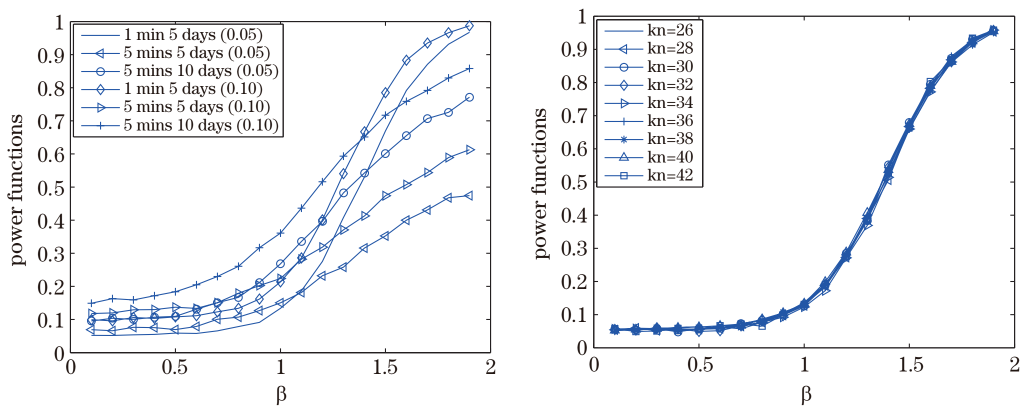

Figure 2. Left panel: the power functions of the test for infinite variation jumps for one-minute data in five days (solid for $\alpha = 0.05$ and diamond for $\alpha = 0.10$, $T = 5$), five-minute data in five days (left triangle for $\alpha = 0.05$ and right triangle for $\alpha = 0.10$, $T = 5$) and ten days (circle for $\alpha = 0.05$ and plus for $\alpha = 0.10$, $T = 10$); Right panel: stability analysis of the power function against k_n for five days of one-minute data; $\alpha = 0.05$.

jumps.

We did a sensitivity study on the tuning parameter k_n and m_n. We let $k_n = 26, 28, 30, 32, 34, 36, 38, 40, 42$ (m_n changes correspondingly as 13, 15, 17, 19, 21, 23, 25, 23, 22). For each k_n, we plot a power function in the right panel of Figure 2. From the figure, it is not easy to tell the differences among the power functions, suggesting that our test is not sensitive to the choice of k_n and m_n.

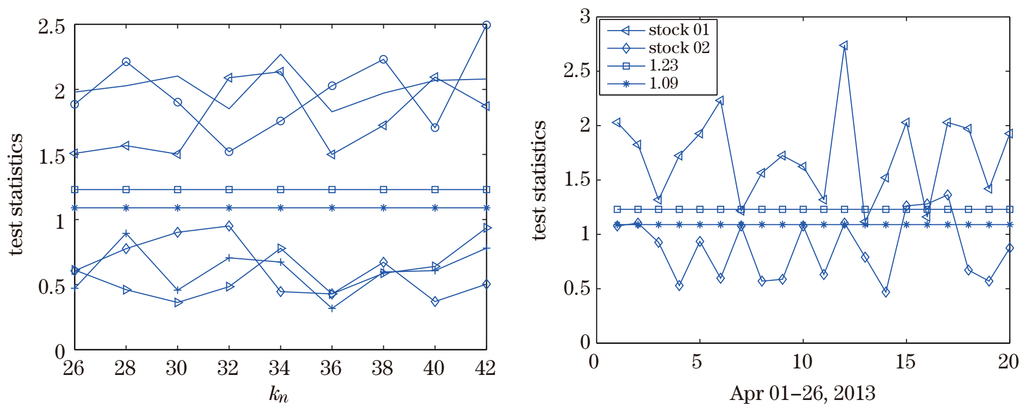

Figure 3. Left panel: the observed test statistics against $k_n = 26$, 28, 30, 32, 34, 36, 38, 40, 42 for stock 01 in Apr 8th (circle), Apr 11th (left triangle), and Apr 24 (solid) in 2013, and for stock 02 in Apr 8th (plus), Apr 11th (right triangle), and Apr 24 (diamond) in 2013; Right panel: test statistics in 20 consecutive trading days from Apr 1st to 26th in 2013 for stock 01 (left triangle) and stock 02 (diamond); The upper 0.05, and 0.10 quantiles are 1.23 (square) and 1.09 (star; two digital decimals are left for all k_n), respectively.

4.2. Data analysis

We implemented our test on two constituent stocks of the S&P 500 index. The test was first carried on the every-one-minute data sets of Apr 8th, 11th, and 24th in 2013 for stock 01 and stock 02 ($T = 1$ and $n = 390$). We plotted the observed test statistics against different values of $k_n = 42$, 40, 38, 36, 34, 32, 30, 28, 26 (correspondingly $m_n = 15$, 13, 13, 12, 11, 10, 10, 9, 9). We set other parameters as in the simulation studies. The results are illustrated in the left panel of Figure 3. The finding is that for all three days and any value of k_n, there is strong evidence of infinite variation jumps in the continuous-time price dynamics of stock 01. Such evidences is not significant for stock 02.

To study the existence of infinite variation jumps across different days, we did the test on 20 consecutive trading days from Apr 1st to 26th in 2013, for both stocks. The daily test statistics are plotted in the right panel of Figure 3. We find that in all days the existence of infinite variation jumps is significant (above upper 0.10 quantile) for stock 01 while for stock 02 only 20% of the days present strong evidence of infinite variation jumps.

Supplementary Materials

The supplement contains the proofs of the main results and some auxiliary lemmas that are of interest.

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References

Aït-Sahalia, Y. and Jacod, J. (2009a). Estimating the degree of activity of jumps in high-frequency data. *Ann. Statist.* **37**, 2202-2244.

Aït-Sahalia, Y. and Jacod, J. (2009b). Testing for jumps in discretely observed process. *Ann. Statist.* **37**, 184-222.

Aït-Sahalia, Y. and Jacod, J. (2010). Is Brownian motion necessary to model high-frequency data? *Ann. Statist.* **38**, 3093-3128.

Aït-Sahalia, Y. and Jacod, J. (2011). Testing whether jumps have finite or infinite activity. *Ann. Statist.* **39**, 1689-1719.

Barndorff-Nielsen, O. E. and Shephard, N. (2006). Econometrics of testing for jumps in financial economics using bipower variation. *J. Finan. Economet.* **2**, 1-48.

Billingsley, P. (1968). *Convergence of Probability Measures*. John Wiley & Sons, Inc.

Bull, A. (2016). Near-optimal estimation of jump activity in semimartingales. *Ann. Statist.* **44**, 58-86.

Comte, F. and Renault, E. (1996). Long memory continuous time models. *J. Econometrics* **73**, 101-149.

Comte, F. and Renault, E. (1998). Long memory in continuous-time stochastic volatility models. *Math. Finance* **8**, 291-323.

Duffie, D., Pan, J. and Singleton, K. (2000). Transform analysis and asset pricing for affine-jump diffusions. *Econometrica* **68**, 1343-1376.

Fan, J. and Wang, Y. (2007). Multi-scale jump and volatility analysis for high-frequency fincancial data. *J. Amer. Statist. Assoc.* **102**, 1349-1362.

Fan, J. and Wang, Y. (2008). Spot volatility estimation for high-frequency data. *Stat. Interface* **1**, 279-288.

Jacod, J. and Protter, P. (2012). *Discretization of Processes*. Springer, Heidelberg.

Jacod, J. and Rosenbaum, M. (2013). Quarticity and other functionals of volatility: efficient estimation. *Ann. Statist.* **41**, 1462-1484.

Jacod, J. and Todorov, V. (2014). Efficient estimation of integrated volatility in presence of infinite variation jumps. *Ann. Statist.* **42**, 1029-1069.

Jing, B. Y., Kong, X. B., Liu, Z. and Mykland, P. (2012). On the jump activity index on semimartingales. *J. Econometrics* **166**, 213-223.

Jing, B. Y., Kong, X. B. and Liu, Z. (2012). Modeling high-frequency financial data by pure jump models. *Ann. Statist.* **40**, 759-784.

Kong, X. B., Liu, Z. and Jing, B. Y. (2015). Testing for pure-jump processes for high-frequency data. *Ann. Statist.* **43**, 847-877.

Li, J. and Xiu, D. (2016). Generalized methods of integrated moments for high-frequency data. *Econometrica* **84**, 1613-1633.

Li, J., Liu, Y. and Xiu, D. (2017). Efficient estimation of integrated volatility functionals via multiscale jackknife. *Technical Report*.

Mykland, P. A., Shephard, N. and Shephard, K. (2012). Efficient and feasible inference for the components of financial variation using blocked multipower variation. *Technical report*.

Mykland, P. A. and Zhang, L. (2009). Inference for continuous semimartingales observed at high frequncy. *Econometrica* **77**, 1403-1445.

Todorov, V. (2015). Jump activity estimation for pure-jump semimartingales via self-normalized statsitics. *Ann. Statist.* **43**, 1831-1864.

Todorov, V. and Tauchen, G. (2010). Activity signature functions for high-frequency data analysis. *J. Econometrics* **154**, 125-138.

Todorov, V. and Tauchen, G. (2011). Limit theorems for power variation of pure-jump processes with application to activity estimation. *Ann. Appl. Probab.* **21**, 546-588.

Todorov, V. and Tauchen, G. (2012). The realized Laplace transform of volatility. *Econometrica* **80**, 1105-1127.

Todorov, V. and Tauchen, G. (2014). Limit theorems for the empirical distribution function of scaled increments of Itô semimartingales at high frequencies. *Ann. Appl. Probab.* **24**, 1850-1888.

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