Supplementary to " COPULA-BASED QUANTILE REGRESSION FOR LONGITUDINAL DATA"

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This supplementary file contains the technical proofs of Theorems 2.1 and 2.2 in the main paper, and some additional simulation results.

1 Technical assumptions and proofs

We make the following assumptions.

(A1) There exists a unique solution $\boldsymbol{\theta}_0 \in \Theta$ to the equation $\sum_{i=1}^{n} E_{\mathbf{u}}[\log c(u_{i1}, \cdots, u_{iJ}; \boldsymbol{\theta})] = 0$, where $\mathbf{u} = (u_1, \cdots, u_J)^T$, and $\Theta \subset \mathbb{R}^d$ is a convex and compact set.

(A2) $\boldsymbol{\beta}_0(u) \in \mathcal{B}$ is smooth for $0 < u < 1$, where $\mathcal{B} \subset \mathbb{R}^p$ is a convex and compact space.

(A3) The conditional probability density of Y given the covariate $\mathbf{X}$ at the τth quantile, $f\{\mathbf{x}_{ij}^T \boldsymbol{\beta}_0(\tau)|\mathbf{x}_{ij}\}$, is uniformly bounded away from both zero and infinity for all $\mathbf{x}$. Furthermore, there is a constant M such that the conditional densities satisfy the Lipschitz condition $|f(s|\mathbf{x}_{ij}) - f(t|\mathbf{x}_{ij})| \leq M|s-t|$, for all i and j.

(A4) The integral $\int_{\mathbf{u}} \sup_{\boldsymbol{\theta}\in\Theta} \|\partial \log\{c(u_1, \cdots, u_J; \boldsymbol{\theta})\}/\partial\boldsymbol{\theta}\| \, d\mathbf{u}$ exists.

(A5) The partial derivative $\sup_{1/(\kappa_n+1)\leq \mathbf{u}\leq \kappa_n/(\kappa_n+1)} \|\partial \log\{c(u_1, \cdots, u_J; \boldsymbol{\theta})\}/\partial\mathbf{u}\| \leq K\kappa_n^a$, where K and a are some positive constants.

(A6) The covariates satisfy $\sum_{i=1}^n \|\mathbf{x}_i\|^2 = O_p(n)$, and $\max_{1\leq i\leq n} \|\mathbf{x}_i\| = O_p(n^q)$ for some positive constant $q < 1/6$.

(A7) There exists a positive definite matrix $\mathbf{H}$ such that $n^{-1}\sum_{i=1}^n \mathbf{x}_i^T \boldsymbol{\Gamma}_i \{\mathbf{V}_i(\boldsymbol{\theta}_0)\}^{-1} \boldsymbol{\Gamma}_i \mathbf{x}_i \to \mathbf{H}^{-1}$ as $n \to \infty$.

(A8) The partial derivative $-n^{-1}\sum_{i=1}^n \mathbf{x}_i^T \boldsymbol{\Gamma}_i \{\mathbf{V}_i(\boldsymbol{\theta}_0)\}^{-1} E(\tilde{\boldsymbol{\Gamma}}_i|\mathbf{x}_i)\mathbf{x}_i$ converges to a positive definite matrix, where $\tilde{\boldsymbol{\Gamma}}_i = \text{diag}\left\{h_{i1}^{-1}\phi\left(\epsilon_{i1}(\tau)/h_{i1}\right), \cdots, h_{iJ}^{-1}\phi\left(\epsilon_{iJ}(\tau)/h_{iJ}\right)\right\}$, and $\epsilon_{ij}(\tau) = y_{ij} - \mathbf{x}_{ij}^T\boldsymbol{\beta}(\tau)$.

Assumptions (A1) and (A2), together with (A6), are commonly used in quantile regression literature. Assumption (A3) imposes the Lipschitz condition on probability densities of model errors, which is satisfied by many common distributions such as Gaussian and student's t distributions. Assumption (A3) also imposes the uniform bound for all the densities of model errors at the τth quantile. Assumptions (A4) and (A5) specify conditions on the copula function, which are satisfied by many commonly used copulas including Gaussian copula, t-copula, Frank copula and so on. We consider a restriction on the constant a in Assumption (A5) to ensure the consistency of the copula parameter estimator $\hat{\boldsymbol{\theta}}$. Assumptions (A7) and (A8) are needed to establish the asymptotic normality of $\hat{\boldsymbol{\beta}}(\tau)$ and $\hat{\boldsymbol{\beta}}_s(\tau)$.

Lemma 1. *Under Assumptions (A1)–(A6), if* $\kappa_n^{2a+1}/n^{1-2q} \to 0$ *and* $\kappa_n \to \infty$ *as* $n \to \infty$*, we have* $\hat{\boldsymbol{\theta}} \xrightarrow{p} \boldsymbol{\theta}_0$.

Proof. Consider any real number y. Let $u \in (0,1)$ and $v \in (0,1)$ be quantile levels such that $y = \mathbf{x}^T\boldsymbol{\beta}_0(u)$ and $y = \mathbf{x}^T\tilde{\boldsymbol{\beta}}(v)$, where the quantile process $\tilde{\boldsymbol{\beta}}(\cdot)$ is constructed by the linear interpolation of the initial estimates $\tilde{\boldsymbol{\beta}}(\tau_k)$ given in (2.3). It then follows that

$$v - u = \frac{1}{\mathbf{x}^T\boldsymbol{\beta}_0'(\tau^*)}\{\mathbf{x}^T\tilde{\boldsymbol{\beta}}(u) - \mathbf{x}^T\boldsymbol{\beta}_0(u)\}, \tag{S.1}$$

where $\tau^* \in (0,1)$. By the proof of Theorem 1 of Wei, Ma and Carroll (2012) and Assumption (A6), we obtain

$$\sup_{1/(\kappa_n+1)\le\tau\le\kappa_n/(\kappa_n+1)} |\mathbf{x}^T\{\tilde{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\}| = O_p(\kappa_n^{1/2}n^{q-1/2}). \tag{S.2}$$

Hence, by Assumption (A3), we have $\sup_{1/(\kappa_n+1)\le u\le\kappa_n/(\kappa_n+1)} |v-u| = O_p(\kappa_n^{1/2}n^{q-1/2})$. It then follows from Assumptions (A1), (A3) and (A5) that

$$\sup_{\boldsymbol{\theta}} |\log c(\tilde{u}_1, \cdots, \tilde{u}_J; \boldsymbol{\theta}) - \log c(u_1, \cdots, u_J; \boldsymbol{\theta})| = o_p(\kappa_n^{a+1/2}/n^{1/2-q}),$$

where $\tilde{u}_j = (\mathbf{x}^T\tilde{\boldsymbol{\beta}})^{-1}(y)$ and $u_j = (\mathbf{x}^T\boldsymbol{\beta}_0)^{-1}(y)$. Thus, if $\kappa_n^{2a+1}/n^{1-2q} \to 0$ as $n \to \infty$, we have

$$\sup_{\boldsymbol{\theta}} \left| n^{-1}\sum_{i=1}^{n} \{\log c(\tilde{u}_1, \cdots, \tilde{u}_J; \boldsymbol{\theta}) - \log c(u_1, \cdots, u_J; \boldsymbol{\theta})\} \right| = o_p(1).$$

Under Assumption (A1), by Theorems 2.7.11 and 2.4.1 of van der Varrt and Wellner (1996), $\sup_{\boldsymbol{\theta}} |n^{-1}\sum_{i=1}^{n}\{\log c(u_1,\cdots,u_J;\boldsymbol{\theta}) - E[\log c(u_1,\cdots,u_J;\boldsymbol{\theta})]\}| = o_p(1)$. Furthermore, by Theorem 2.10 of Kosorok (2008), we have $\hat{\boldsymbol{\theta}} \stackrel{p}{\longrightarrow} \boldsymbol{\theta}_0$. $\diamond$

Lemma 2. *Under the conditions of Lemma 1, for any given quantile level* $\tau \in (0,1)$*, we have* $\hat{\boldsymbol{\beta}}(\tau) \stackrel{p}{\longrightarrow} \boldsymbol{\beta}_0(\tau)$.

Proof. Note that

$$\mathbf{V}_i(\hat{\boldsymbol{\theta}}) = \begin{pmatrix} \tau(1-\tau) & \hat{\lambda}_\tau(\tilde{u}_{i1},\tilde{u}_{i2}) & \cdots & \hat{\lambda}_\tau(\tilde{u}_{i1},\tilde{u}_{iJ}) \\ \hat{\lambda}_\tau(\tilde{u}_{i2},\tilde{u}_{i1}) & \tau(1-\tau) & \cdots & \hat{\lambda}_\tau(\tilde{u}_{i2},\tilde{u}_{iJ}) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\lambda}_\tau(\tilde{u}_{iJ},\tilde{u}_{i1}) & \cdots & \hat{\lambda}_\tau(\tilde{u}_{iJ},\tilde{u}_{i,J-1}) & \tau(1-\tau) \end{pmatrix},$$

where $\hat{\lambda}_\tau(\tilde{u}_1,\tilde{u}_2) = C_{j,k}(\tau,\tau;\hat{\boldsymbol{\theta}}) - \tau^2$. By (S.1) and (S.2), we have $\max_i \|\mathbf{V}_i(\boldsymbol{\theta}_0) - \mathbf{V}_i(\hat{\boldsymbol{\theta}})\| = O_p(\kappa_n^{1/2} n^{q-1/2})$. From now on, we will omit the difference between $\mathbf{V}_i(\boldsymbol{\theta}_0)$ and $\mathbf{V}_i(\hat{\boldsymbol{\theta}})$. Hence, we can directly consider the estimation equation

$$\hat{U}_n\{\boldsymbol{\beta}(\tau)\} = n^{-1}\sum_{i=1}^{n}\mathbf{x}_i^T\hat{\boldsymbol{\Gamma}}_i\{\mathbf{V}_i(\boldsymbol{\theta}_0)\}^{-1}\psi_\tau\{\mathbf{y}_i - \mathbf{x}_i^T\boldsymbol{\beta}(\tau)\} \approx 0, \tag{S.3}$$

where $\hat{\boldsymbol{\Gamma}}_i = \text{diag}\{\hat{s}_{i1},\ldots,\hat{s}_{iJ}\}$. With similar arguments as in the proof of Theorem 1 of Jung (1996), we can show that $\hat{\boldsymbol{\beta}}(\tau) \stackrel{p}{\longrightarrow} \boldsymbol{\beta}_0(\tau)$. $\diamond$

Proof of Theorem 2.1. Similar to the proof of Theorem 1 of Jung (1996), we can show that $n^{1/2}\left\{\hat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}_0(\tau)\right\} \xrightarrow{d} N(0,\mathbf{H})$, as Assumptions (A3), (A6) and (A7) hold and n converges to infinity. Similar as in the proof of Lemma 2, we can get $\hat{\boldsymbol{\beta}}_s(\tau) \xrightarrow{p} \boldsymbol{\beta}_0(\tau)$.

For convenience, we omit τ in various expressions in the following. We consider the Taylor expansion of the score $\tilde{U}_n(\boldsymbol{\beta},\mathbf{H})$ in a neighbour of the true parameter $\boldsymbol{\beta}_0$ as follows.

$$\tilde{U}_n(\boldsymbol{\beta},\mathbf{H}) = \tilde{U}_n(\boldsymbol{\beta}_0,\mathbf{H}) + \frac{\partial \tilde{U}_n(\boldsymbol{\beta},\mathbf{H})}{\partial \boldsymbol{\beta}}\Big|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}(\boldsymbol{\beta}-\boldsymbol{\beta}_0),$$

where $\boldsymbol{\beta}^*$ lies between $\boldsymbol{\beta}$ and $\boldsymbol{\beta}_0$. By Assumption (A6), we can further obtain

$$\tilde{U}_n(\boldsymbol{\beta},\mathbf{H}) = \tilde{U}_n(\boldsymbol{\beta}_0,\mathbf{H}) + \frac{\partial \tilde{U}_n(\boldsymbol{\beta}_0,\mathbf{H})}{\partial \boldsymbol{\beta}_0}(\boldsymbol{\beta}-\boldsymbol{\beta}_0) + O(\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|^2). \tag{S.4}$$

Let $\mathbf{W}_i = \mathbf{x}_i^T \boldsymbol{\Gamma}_i\{\mathbf{V}_i(\boldsymbol{\theta}_0)\}^{-1}$. We consider

$$n^{1/2}\{\tilde{U}_n(\boldsymbol{\beta}_0,\mathbf{H}) - U_n(\boldsymbol{\beta}_0)\} = n^{1/2}\sum_{i=1}^n \mathbf{W}_i \begin{pmatrix} sgn(-\epsilon_{i1}(\tau)/h_{i1})\Phi(-|\epsilon_{i1}(\tau)/h_{i1}|) \\ \vdots \\ sgn(-\epsilon_{iJ}(\tau)/h_{iJ})\Phi(-|\epsilon_{iJ}(\tau)/h_{iJ}|) \end{pmatrix},$$

where $\epsilon_{ij}(\tau) = y_{ij} - \mathbf{x}_{ij}^T\boldsymbol{\beta}(\tau)$. By Assumption (A3), we have

$$
\begin{aligned}
& |E\{sgn(-\epsilon_{ij}(\tau)/h_{ij})\Psi(-|\epsilon_{ij}(\tau)/h_{ij}|)\}| \\
= & \left| h_{ij}\int_{-\infty}^{\infty} \Phi(-|t|)\{2I(t \le 0) - 1\} f(h_{ij}t|\mathbf{x}_{ij})dt \right| \\
\le & \left| h_{ij}\int_{-\infty}^{\infty} \Phi(-|t|)\{2I(t \le 0) - 1\} s_{ij}dt \right| + Mh_{ij}^2 \int_{-\infty}^{\infty} |t|\Phi(-|t|)dt \\
= & Mh_{ij}^2/2,
\end{aligned}
$$

where $s_{ij} = f\{\mathbf{x}_{ij}^T\boldsymbol{\beta}(\tau)|\mathbf{x}_{ij}\}$, and the last equality follows from the fact that $\int_{-\infty}^{\infty}\Phi(-|t|)\{2I(t \le 0) - 1\}s_{ij}dt = 0$ and $\int_{-\infty}^{\infty}|t|\Phi(-|t|)dt = 1/2$. It then follows from Assumption (A6) that

$$
n^{1/2}E\{\tilde{U}_n(\boldsymbol{\beta}_0, \mathbf{H}) - U_n(\boldsymbol{\beta}_0)\} \le n^{-1/2}M \sup_{1\le i\le n} \|\mathbf{x}_i\|^3/2 = o(1). \qquad \text{(S.5)}
$$

Note that

$$
\begin{aligned}
& \text{Var}\{sgn(-\epsilon_{ij}(\tau)/h_{ij})\Phi(-|\epsilon_{ij}(\tau)/h_{ij}|)\} \\
\le & E\{\Phi^2(-|\epsilon_{ij}(\tau)/h_{ij}|)\} \\
= & h_{ij}\int_{-\infty}^{\infty} \Phi^2(-|t|) f(h_{ij}t|\mathbf{x}_{ij})dt \\
\le & h_{ij}\left\{\int_{-\infty}^{\infty} \Phi^2(-|t|) f(t|\mathbf{x}_{ij})dt + \int_{-\infty}^{\infty} \Phi^2(-|t|) \left| f(h_{ij}t|\mathbf{x}_{ij}) - f(t|\mathbf{x}_{ij})\right| dt\right\} \\
\le & h_{ij}(K_1 + |1 - h_{ij}|K_2),
\end{aligned}
$$

where K_1 and K_2 are some constants. Since $h_{ij} = O(n^{-1/2+q})$ holds uniformly for all i and j by Assumption (A6), where $0 < q < 1/6$, thus, by Assumption (A6), we have

$$\begin{aligned}
&\text{Cov}\Big(n^{1/2}\{\tilde{U}_n(\boldsymbol{\beta}_0, \mathbf{H}) - U_n(\boldsymbol{\beta}_0)\}\Big) \\
= \;& n^{-1}\sum_{i=1}^{n}\mathbf{W}_i\text{Cov}\left(\begin{array}{c} sgn(-\epsilon_{i1}(\tau)/h_{i1})\Phi(-|\epsilon_{i1}(\tau)/h_{i1}|) \\ \vdots \\ sgn(-\epsilon_{iJ}(\tau)/h_{iJ})\Phi(-|\epsilon_{iJ}(\tau)/h_{iJ}|) \end{array}\right)\mathbf{W}_i^T \xrightarrow{p} 0.
\end{aligned}$$

It then follows from Chebyshev's inequality and (S.5) that

$$n^{1/2}\{\tilde{U}_n(\boldsymbol{\beta}_0, \mathbf{H}) - U_n(\boldsymbol{\beta}_0)\} = o_p(1). \tag{S.6}$$

In addition, note that

$$\begin{aligned}
\left|h_{ij}^{-1}E\phi(\epsilon_{ij}(\tau)/h_{ij}|\mathbf{x}_{ij}) - s_{ij}\right| &\leq \left|\int_{-\infty}^{\infty}\phi(t)f(h_{ij}t|\mathbf{x}_{ij})dt - s_{ij}\right| \\
&\leq Mh_{ij}\int_{-\infty}^{\infty}|t|\phi(t)dt,
\end{aligned}$$

where the last inequality follows from Assumption (A3). Thus, by Assumption (A6) and the fact that $h_{ij} = \sqrt{\mathbf{x}_{ij}^T\mathbf{H}\mathbf{x}_{ij}/n}$, we obtain

$$\max_{1\leq i\leq n}\left\|\tilde{\boldsymbol{\Gamma}}_i - \boldsymbol{\Gamma}_i\right\| = o_p(1). \tag{S.7}$$

By Assumption (A8) and the law of large numbers, we have

$$-n^{-1}\sum_{i=1}^{n}\left[\mathbf{x}_i^T\mathbf{\Gamma}_i\{\mathbf{V}_i(\boldsymbol{\theta}_0)\}^{-1}\left\{\tilde{\mathbf{\Gamma}}_i - E\left(\tilde{\mathbf{\Gamma}}_i|\mathbf{x}_i\right)\right\}\mathbf{x}_i\right] \xrightarrow{p} 0.$$

It follows from Assumption (A8) that the solution to (2.7) is unique, so the Newton-Raphson algorithm leads to the unique solution $\hat{\boldsymbol{\beta}}_s(\tau)$ since the initial estimator $\tilde{\boldsymbol{\beta}}(\tau)$ is consistent to the true parameter $\boldsymbol{\beta}_0(\tau)$. Hence, by (S.4), we have $n^{1/2}\left\{\hat{\boldsymbol{\beta}}_s(\tau)-\boldsymbol{\beta}_0(\tau)\right\} \xrightarrow{d} N(0,\mathbf{H})$. By (S.7), we obtain that for any $\hat{\mathbf{H}} = O_p(1)$,

$$\frac{\partial \tilde{U}_n(\mathbf{b},\hat{\mathbf{H}})}{\partial \mathbf{b}}\Big|_{\mathbf{b}=\hat{\boldsymbol{\beta}}_s(\tau)} \xrightarrow{p} \mathbf{H}^{-1}.$$

In addition, by (S.6), for any $\hat{\mathbf{H}} = O_p(1)$, we have

$$A(\boldsymbol{\beta}_0,\hat{\mathbf{H}}) - n^{-1}\sum_{i=1}^{n}\mathbf{x}_i^T\mathbf{\Gamma}_i\{\mathbf{V}_i(\hat{\boldsymbol{\theta}})\}^{-1}\boldsymbol{\psi}_\tau(\mathbf{y}_i-\mathbf{x}_i^T\boldsymbol{\beta}_0)\boldsymbol{\psi}_\tau^T(\mathbf{y}_i-\mathbf{x}_i^T\boldsymbol{\beta}_0)\{\mathbf{V}_i(\hat{\boldsymbol{\theta}})\}^{-1}\mathbf{\Gamma}_i\mathbf{x}_i \xrightarrow{p} 0.$$

It then follows that $A(\boldsymbol{\beta}_0,\hat{\mathbf{H}}) \xrightarrow{p} \mathbf{H}^{-1}$. Therefore, we obtain $\hat{\mathbf{H}} \xrightarrow{p} \mathbf{H}$.$\diamond$

Proof of Theorem 2.2. The result directly follows from the continuity of the function

$$\frac{\partial^{J-1}C(u_1,\cdots,u_{J-1},u_J;\boldsymbol{\theta})}{\partial u_1\cdots\partial u_{J-1}}$$

with respect to both u_J and the parameter $\boldsymbol{\theta}$, and the consistency of the estimators $\hat{\boldsymbol{\beta}}_s(\cdot)$ and $\hat{\boldsymbol{\theta}}$. $\diamond$

2 Additional simulation results

Table S1 summarizes the relative efficiency with respect to the working independence (WI) estimator $\tilde{\beta}(\tau)$ and coverage probability of 95% confidence intervals from different methods at $\tau = 0.5$ in Cases 1–3, where the AL and CQR methods are based on Gaussian copula with an exchangeable correlation structure.

Tables S2–S4 summarize the relative efficiency with respect to the WI estimator and coverage probability of 95% confidence intervals from the copula regression methods based on Gaussian copula with AR(1) and unspecified correlation structures in Cases 1–3, respectively.

Table S5 contains the mean prediction error, coverage probability and mean length of 90% prediction intervals from different methods in Cases 1–3 with correlation parameter $\varrho = 0.3$ and 0.8.

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Table S1: Relative efficiency with respect to the working independence estimator $\tilde{\beta}(\tau)$ and coverage probability of 95% confidence intervals from different methods at $\tau = 0.5$ in Cases 1–3. The copula-based methods are based on Gaussian copula with an exchangeable correlation structure.

			RE			CovP		
Case	ϱ	Method	$\beta_0(\tau)$	$\beta_1(\tau)$	$\beta_2(\tau)$	$\beta_0(\tau)$	$\beta_1(\tau)$	$\beta_2(\tau)$
1	0.30	LZ	1.09	1.06	1.06	0.94	0.94	0.94
		AL	0.44	0.86	1.10	0.55	0.56	0.69
		CQR	1.04	1.07	1.26	0.94	0.94	0.92
	0.50	LZ	1.09	1.12	1.16	0.94	0.94	0.94
		AL	0.21	0.54	1.32	0.27	0.41	0.71
		CQR	1.08	1.11	1.29	0.95	0.92	0.91
	0.80	LZ	1.14	1.40	1.68	0.93	0.93	0.94
		AL	0.07	0.17	2.71	0.05	0.03	0.82
		CQR	1.18	1.33	1.79	0.94	0.92	0.93
2	0.30	LZ	1.09	1.14	1.08	0.91	0.94	0.94
		AL	0.08	1.34	1.50	0.02	0.72	0.78
		CQR	1.07	1.11	1.18	0.94	0.93	0.94
	0.50	LZ	1.09	1.27	1.23	0.91	0.94	0.93
		AL	0.05	2.12	1.91	0.01	0.81	0.81
		CQR	1.13	1.31	1.32	0.95	0.93	0.95
	0.80	LZ	1.10	2.01	2.21	0.92	0.94	0.94
		AL	0.03	5.02	5.16	0.00	0.90	0.90
		CQR	1.25	2.01	2.10	0.95	0.94	0.94
3	0.30	LZ	1.05	1.04	1.03	0.92	0.95	0.94
		AL	0.77	0.93	1.07	0.87	0.84	0.89
		CQR	1.05	1.04	1.26	0.94	0.94	0.93
	0.50	LZ	1.07	1.08	1.09	0.94	0.93	0.94
		AL	0.52	0.94	1.12	0.78	0.82	0.88
		CQR	1.07	1.08	1.26	0.94	0.93	0.92
	0.80	LZ	1.07	1.28	1.47	0.92	0.94	0.95
		AL	0.28	1.19	1.66	0.47	0.82	0.93
		CQR	1.16	1.32	1.68	0.95	0.91	0.93

RE: relative efficiency; CovP: coverage probability of 95% confidence intervals; LZ: the method from Leng and Zhang (2014) based on quadratic inference assuming an exchangeable working correlation structure; AL: copula regression method assuming asymmetric Laplace marginal distributions; CQR: the proposed copula-based quantile regression method. Both AL and CQR are based on Gaussian copula with exchangeable correlation structure.

Table S2: Relative efficiency with respect to $\tilde{\beta}(\tau)$ and coverage probability of 95% confidence intervals from the copula regression methods based on Gaussian copula with different correlation structures in Case 1.

			RE			CovP		
ϱ	τ	Method	$\beta_0(\tau)$	$\beta_1(\tau)$	$\beta_2(\tau)$	$\beta_0(\tau)$	$\beta_1(\tau)$	$\beta_2(\tau)$
0.30	0.25	AL (AR1)	0.51	0.88	1.22	0.57	0.59	0.72
		AL (UN)	0.48	0.85	1.22	0.54	0.59	0.70
		CQR (AR1)	1.04	1.07	1.43	0.95	0.94	0.95
		CQR (UN)	1.05	1.09	1.29	0.95	0.94	0.93
	0.50	AL (AR1)	0.96	1.12	1.20	0.80	0.72	0.76
		AL (UN)	0.95	1.09	1.19	0.79	0.72	0.77
		CQR (AR1)	1.12	1.05	1.27	0.95	0.93	0.94
		CQR (UN)	1.11	1.08	1.28	0.95	0.94	0.94
0.50	0.25	AL (AR1)	0.25	0.48	1.65	0.32	0.34	0.76
		AL (UN)	0.23	0.47	1.62	0.29	0.31	0.76
		CQR (AR1)	1.12	1.18	1.36	0.95	0.93	0.91
		CQR (UN)	1.11	1.17	1.36	0.95	0.92	0.91
	0.50	AL (AR1)	0.86	1.29	1.58	0.76	0.73	0.81
		AL (UN)	0.85	1.27	1.56	0.76	0.74	0.82
		CQR (AR1)	1.08	1.17	1.45	0.95	0.94	0.94
		CQR (UN)	1.08	1.17	1.46	0.95	0.94	0.94
0.80	0.25	AL (AR1)	0.08	0.16	3.86	0.04	0.02	0.87
		AL (UN)	0.07	0.15	3.82	0.03	0.02	0.88
		CQR (AR1)	1.24	1.42	2.08	0.95	0.93	0.92
		CQR (UN)	1.24	1.41	2.05	0.95	0.93	0.92
	0.50	AL (AR1)	0.83	2.11	3.87	0.69	0.71	0.91
		AL (UN)	0.81	2.11	3.79	0.70	0.74	0.92
		CQR (AR1)	1.19	1.68	2.29	0.96	0.95	0.93
		CQR (UN)	1.19	1.65	2.28	0.96	0.95	0.93

RE: relative efficiency; CovP: coverage probability of 95% confidence intervals; AL: copula regression method assuming asymmetric Laplace marginal distributions; CQR: the proposed copula-based quantile regression method; AR1: first-order autoregressive correlation structure; UN: unstructured correlation.

Table S3: Relative efficiency with respect to $\tilde{\beta}(\tau)$ and coverage probability of 95% confidence intervals from the copula regression methods based on Gaussian copula with different correlation structures in Case 2.

			RE			CovP		
ϱ	τ	Method	$\beta_0(\tau)$	$\beta_1(\tau)$	$\beta_2(\tau)$	$\beta_0(\tau)$	$\beta_1(\tau)$	$\beta_2(\tau)$
0.30	0.25	AL (AR1)	0.20	1.19	1.36	0.19	0.73	0.73
		AL (UN)	0.08	1.34	1.50	0.03	0.73	0.77
		CQR (AR1)	1.05	1.08	1.15	0.94	0.93	0.94
		CQR (UN)	1.07	1.11	1.17	0.94	0.93	0.94
	0.50	AL (AR1)	0.96	1.10	1.14	0.75	0.83	0.80
		AL (UN)	0.90	1.17	1.20	0.71	0.82	0.80
		CQR (AR1)	1.08	1.08	1.13	0.94	0.93	0.94
		CQR (UN)	1.08	1.10	1.15	0.94	0.93	0.94
0.50	0.25	AL (AR1)	0.09	1.69	1.66	0.01	0.78	0.78
		AL (UN)	0.05	2.09	1.90	0.00	0.79	0.82
		CQR (AR1)	1.10	1.25	1.26	0.95	0.93	0.94
		CQR (UN)	1.13	1.31	1.31	0.94	0.94	0.95
	0.50	AL (AR1)	0.98	1.35	1.45	0.73	0.82	0.84
		AL (UN)	0.91	1.52	1.66	0.72	0.85	0.83
		CQR (AR1)	1.08	1.23	1.31	0.95	0.94	0.94
		CQR (UN)	1.09	1.30	1.37	0.95	0.94	0.94
0.80	0.25	AL (AR1)	0.04	4.06	4.18	0.00	0.87	0.88
		AL (UN)	0.03	4.96	5.17	0.00	0.90	0.89
		CQR (AR1)	1.22	1.85	1.97	0.95	0.94	0.94
		CQR (UN)	1.25	2.01	2.10	0.94	0.95	0.93
	0.50	AL (AR1)	0.90	2.90	3.16	0.64	0.88	0.90
		AL (UN)	0.86	3.66	3.91	0.63	0.90	0.93
		CQR (AR1)	1.15	1.90	1.97	0.95	0.94	0.93
		CQR (UN)	1.17	2.07	2.15	0.96	0.94	0.92

RE: relative efficiency; CovP: coverage probability of 95% confidence intervals; AL: copula regression method assuming asymmetric Laplace marginal distributions; CQR: the proposed copula-based quantile regression method; AR1: first-order autoregressive correlation structure; UN: unstructured correlation.

Table S4: Relative efficiency with respect to $\tilde{\beta}(\tau)$ and coverage probability of 95% confidence intervals from the copula regression methods based on Gaussian copula with different correlation structures in Case 3.

			RE			CovP		
ϱ	τ	Method	$\beta_0(\tau)$	$\beta_1(\tau)$	$\beta_2(\tau)$	$\beta_0(\tau)$	$\beta_1(\tau)$	$\beta_2(\tau)$
0.30	0.25	AL (AR1)	0.84	0.95	1.07	0.87	0.83	0.91
		AL (UN)	0.79	0.93	1.07	0.85	0.82	0.89
		CQR (AR1)	1.08	1.06	1.36	0.94	0.95	0.95
		CQR (UN)	1.05	1.05	1.27	0.94	0.93	0.93
	0.50	AL (AR1)	0.11	0.56	1.23	0.16	0.59	0.82
		AL (UN)	0.06	0.35	1.34	0.04	0.42	0.84
		CQR (AR1)	1.10	0.95	1.21	0.96	0.94	0.93
		CQR (UN)	1.06	1.05	1.26	0.95	0.94	0.93
0.50	0.25	AL (AR1)	0.66	1.04	1.18	0.81	0.85	0.89
		AL (UN)	0.57	1.00	1.18	0.80	0.85	0.89
		CQR (AR1)	1.10	1.10	1.31	0.95	0.94	0.92
		CQR (UN)	1.08	1.09	1.29	0.94	0.94	0.92
	0.50	AL (AR1)	0.05	0.31	1.49	0.02	0.31	0.84
		AL (UN)	0.03	0.22	1.56	0.02	0.20	0.86
		CQR (AR1)	1.05	1.10	1.33	0.95	0.93	0.95
		CQR (UN)	1.07	1.10	1.34	0.95	0.93	0.95
0.80	0.25	AL (AR1)	0.34	0.94	1.98	0.55	0.75	0.93
		AL (UN)	0.30	1.04	1.93	0.48	0.77	0.92
		CQR (AR1)	1.21	1.37	1.84	0.95	0.93	0.93
		CQR (UN)	1.20	1.36	1.83	0.95	0.93	0.93
	0.50	AL (AR1)	0.02	0.23	3.46	0.01	0.13	0.94
		AL (UN)	0.02	0.19	3.52	0.00	0.05	0.92
		CQR (AR1)	1.18	1.44	1.96	0.95	0.94	0.93
		CQR (UN)	1.18	1.43	1.97	0.95	0.94	0.93

RE: relative efficiency; CovP: coverage probability of 95% confidence intervals; AL: copula regression method assuming asymmetric Laplace marginal distributions; CQR: the proposed copula-based quantile regression method; AR1: first-order autoregressive correlation structure; UN: unstructured correlation.

Table S5: The mean prediction error at $\tau = 0.25$ and 0.5, and coverage probability and mean length of 90% prediction intervals from different methods in Cases 1–3 with $\varrho = 0.3$ and 0.8. The values in the parentheses are standard errors.

			MPE			
Case	ϱ	Method	$\tau = 0.25$	$\tau = 0.5$	CovP	ML
1	0.3	QR	0.990 (0.041)	0.593 (0.023)	0.898	4.944 (0.074)
		LZ	0.987 (0.040)	0.592 (0.023)	0.894	4.924 (0.074)
		AL	0.961 (0.039)	0.580 (0.022)	0.896	4.873 (0.073)
		CQR	0.967 (0.039)	0.582 (0.022)	0.918	4.856 (0.073)
	0.8	QR	0.985 (0.043)	0.603 (0.023)	0.902	4.948 (0.074)
		LZ	0.981 (0.043)	0.601 (0.023)	0.898	4.898 (0.073)
		AL	0.712 (0.030)	0.450 (0.017)	0.894	3.794 (0.055)
		CQR	0.727 (0.030)	0.425 (0.017)	0.907	3.716 (0.060)
2	0.3	QR	0.847 (0.039)	0.551 (0.026)	0.884	4.706 (0.012)
		LZ	0.845 (0.039)	0.550 (0.026)	0.880	4.665 (0.011)
		AL	0.760 (0.037)	0.505 (0.025)	0.821	3.347 (0.029)
		CQR	0.783 (0.036)	0.509 (0.025)	0.896	4.736 (0.065)
	0.8	QR	0.849 (0.039)	0.542 (0.027)	0.898	4.730 (0.014)
		LZ	0.848 (0.039)	0.544 (0.027)	0.894	4.645 (0.013)
		AL	0.468 (0.023)	0.284 (0.014)	0.871	2.265 (0.029)
		CQR	0.522 (0.024)	0.324 (0.027)	0.922	3.463 (0.093)
3	0.3	QR	1.254 (0.081)	0.765 (0.047)	0.904	7.536 (0.116)
		LZ	1.247 (0.081)	0.762 (0.047)	0.910	7.511 (0.116)
		AL	1.209 (0.075)	0.770 (0.043)	0.540	5.496 (0.074)
		CQR	1.242 (0.080)	0.761 (0.046)	0.920	7.644 (0.136)
	0.8	QR	1.331 (0.096)	0.828 (0.057)	0.902	7.542 (0.117)
		LZ	1.318 (0.095)	0.822 (0.057)	0.899	7.461 (0.116)
		AL	1.023 (0.070)	0.672 (0.041)	0.771	4.476 (0.095)
		CQR	1.128 (0.083)	0.787 (0.089)	0.920	7.364 (0.392)

MPE: mean prediction error; CovP: coverage probability of 90% prediction intervals; ML: mean length of 90% prediction intervals; QR: conventional quantile regression method that uses only covariate information for prediction; LZ: prediction based on covariates and the quantile coefficient estimator in Leng and Zhang (2014); AL: the copula regression based on asymmetric Laplace marginals; CQR: the proposed copula-based quantile regression method.