HIGH-DIMENSIONAL TWO-SAMPLE COVARIANCE MATRIX TESTING VIA SUPER-DIAGONALS

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Supplementary Material

In this supplement document, we first provide some technical lemmas which will be used in the proofs of the main propositions and theorems. Next, we provide the proofs for Proposition 1 - 2 and Theorems 1 - 2. Throughout this supplement, constants which do not depend on n and p are denoted by $C, C_1, C_2, \ldots$. At last, We provide more simulation results of the proposed test method with the Benjamini and Hochberg (1995) procedure and for the Gamma distributed data.

S1. Lemmas 1 - 3

Lemma 1. *Under the Assumptions* (C1) - (C4), *there exists a positive constant C such that $h_q \leq Ck^{-\beta}$ for each q.*

Proof of Lemma 1. For the generic random vectors $\mathrm{X}_1 = (\mathrm{X}_{1,1}, \ldots, \mathrm{X}_{1,p})^\top$ and $\mathrm{X}_2 = (\mathrm{X}_{2,1}, \ldots, \mathrm{X}_{2,p})^\top$, under Assumptions (C3) and (C4), the Davydov's inequality implies that, for $i = 1, 2$,

$$|\sigma_{i,j,k}| \leq 12\|\mathrm{X}_{i,j}\|_4\|\mathrm{X}_{i,k}\|_4\left(\alpha_{\mathrm{X}_i}(|j-k|)\right)^{\frac{1}{2}} \leq 12M^{1/2}c|j-k|^{-\beta/2}, \tag{S1.1}$$

where $\|\mathrm{X}_{i,j}\|_r = (\mathrm{E}|\mathrm{X}_{i,j}|^r)^{1/r}$. Then, according to (S1.1), $(p-q)^{-1}\mathrm{S}_q = (p-q)^{-1}(\mathrm{D}_{1,q} + \mathrm{D}_{2,q} - 2\mathrm{D}_{c,q}) \leq Cq^{-\beta}$, where $C = 144Mc^2$. This completes the proof of Lemma 1. □

Lemma 2. *Under the Assumptions* (C1) - (C4), *for each q, there exist positive constants M_1 and M_2, such that*

$$\lambda_{max}(\Sigma_i) \leq M_1 \quad \textit{and} \quad \lambda_{max}(W_{i,q}) \leq M_2. \tag{S1.2}$$

Proof of Lemma 2. The upper bound for the eigenvalues of Σ_i can be derived directly by (S1.1) and the Gersgorins Theorem:

$$\lambda_{max}(\Sigma_i) \leq \max_j \sum_{k=1}^{p} |\sigma_{i,j,k}| \leq M_1.$$

The derivation of the upper bound for the eigenvalues of $W_{i,q}$ is similar but more involved. Let α_{Y_i} be the α-mixing coefficient of the sequence $\{\mathrm{Y}_{i,j}^{ss+q}\}_{s=1}^{p-q}$. Thus for each given q, $\alpha_{\mathrm{Y}_i}(k) \le \alpha_{\mathrm{Y}_i}(k-q)$ for $k > q$. We use the Davydov's inequality to obtain the upper bound for $\omega_{i,q}^{s_1,s_2}$. For $|s_1 - s_2| > q$,

$$|\omega_{i,q}^{s_1,s_2}| \le 12\|\mathrm{Y}_{i,j}^{s_1,s_1+q}\|_4\|\mathrm{Y}_{i,j}^{s_2,s_2+q}\|_4\left(\alpha_{\mathrm{Y}_i}(|s_1-s_2|)\right)^{1/2} \le C(|s_1-s_2|-q)^{\beta/2}. \quad \text{(S1.3)}$$

For $|s_1 - s_2| < q$,

$$\begin{aligned} |\omega_{i,q}^{s_1,s_2}| &= |\mathrm{Cov}(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_2}, \mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2+q}) + \sigma_{i,s_1,s_2}\sigma_{i,s_1+q,s_2+q} \\ &\quad -\sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}| \\ &\le C(q-|s_1-s_2|)^{\beta/2} + C|s_1-s_2|^{-2\beta} + Cq^{-2\beta}. \end{aligned} \quad \text{(S1.4)}$$

For $|s_1 - s_2| = q$,

$$|\omega_{i,q}^{s_1,s_1+q}| = \mathrm{E}(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}^2 X_{i,j,s_1+2q}) - \sigma_{i,s_1,s_1+q}^2 \le C_1. \quad \text{(S1.5)}$$

And for $s_1 = s_2$, we have

$$|\omega_{i,q}^{s_1,s_1}| = \mathrm{E}(\mathrm{X}_{i,j,s_1}^2\mathrm{X}_{i,j,s_1+q}^2) - \sigma_{i,s_1,s_1}^2 \le C_2. \quad \text{(S1.6)}$$

Thus, by the Gersgorins Theorem and (S1.3) - (S1.6), we have

$$\max_l \lambda_l(W_{i,q}) \le \max_{s_1}\sum_{s_2=1}^{p-q}|\omega_{i,q}^{s_1,s_2}| \le M_2.$$

This completes the proof. □

Lemma 3. *Under the Assumptions* (C1) - (C5)*, for $q = o(p)$ and $i = 1, 2$, $j = 1, \ldots, n_i$, then,*

$$\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\mathrm{E}(\mathrm{Y}_{i,j}^{s_1,s_1+q}\mathrm{Y}_{i,j}^{s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}) = O\left((p-q)^2\right).$$

Proof of Lemma 3. In order to show that

$$\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\mathrm{E}(\mathrm{Y}_{i,j}^{s_1,s_1+q}\mathrm{Y}_{i,j}^{s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}) = O\left((p-q)^2\right),$$

we will expand $\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\mathrm{Y}_{i,j}^{s_1,s_1+q}\mathrm{Y}_{i,j}^{s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}$ explicitly using Assumption (C4) and show that the expectation of each term is $O\left((p-q)^2\right)$. Since $\mathrm{Y}_{i,j}^{s_1,s_2} = \mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_2} - \sigma_{i,s_1,s_2}$, we have

$$\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\mathrm{E}(\mathrm{Y}_{i,j}^{s_1,s_1+q}\mathrm{Y}_{i,j}^{s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q})$$

$$= \quad \mathrm{M}_{i,q,a} - \sum_{k=1}^{4} \mathrm{M}_{i,q,bk} + \sum_{k=1}^{6} \mathrm{M}_{i,q,ck} - \sum_{k=1}^{4} \mathrm{M}_{i,q,dk} + \mathrm{M}_{i,q,e}, \tag{S1.7}$$

where

$$\mathrm{M}_{i,q,a} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \mathrm{E}\left(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}\right),$$

$$\begin{aligned}
\mathrm{M}_{i,q,b1} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_1+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}\right),\\
\mathrm{M}_{i,q,b2} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_2,s_2+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}\right),\\
\mathrm{M}_{i,q,b3} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_3,s_3+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}\right),\\
\mathrm{M}_{i,q,b4} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_4,s_4+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\right),
\end{aligned}$$

$$\begin{aligned}
\mathrm{M}_{i,q,c1} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}\right),\\
\mathrm{M}_{i,q,c2} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_1+q}\sigma_{i,s_3,s_3+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}\right),\\
\mathrm{M}_{i,q,c3} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_1+q}\sigma_{i,s_4,s_4+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\right),\\
\mathrm{M}_{i,q,c4} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_2,s_2+q}\sigma_{i,s_3,s_3+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}\right),\\
\mathrm{M}_{i,q,c5} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_2,s_2+q}\sigma_{i,s_4,s_4+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\right),\\
\mathrm{M}_{i,q,c6} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_3,s_3+q}\sigma_{i,s_4,s_4+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}\right),
\end{aligned}$$

$$\mathrm{M}_{i,q,d1} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}\sigma_{i,s_3,s_3+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}\right),$$

$$\begin{aligned}
\mathrm{M}_{i,q,d2} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}\sigma_{i,s_4,s_4+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\right),\\
\mathrm{M}_{i,q,d3} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_1+q}\sigma_{i,s_3,s_3+q}\sigma_{i,s_4,s_4+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}\right),\\
\mathrm{M}_{i,q,d4} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_2,s_2+q}\sigma_{i,s_3,s_3+q}\sigma_{i,s_4,s_4+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\right),
\end{aligned}$$

and

$$\mathrm{M}_{i,q,e} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}\sigma_{i,s_3,s_3+q}\sigma_{i,s_4,s_4+q}.$$

According to Assumption (C4), (S1.7) can be explicitly written as follows:

$$\begin{aligned}
&\sum_{s_1,s_2,s_3,s_4=1}^{p-q} \mathrm{E}(\mathrm{Y}_{i,j}^{s_1,s_1+q}\mathrm{Y}_{i,j}^{s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}) \qquad \text{(S1.8)}\\
=& \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sum_{\ell_1,\ell_2}^{*}\sum_{\ell_3,\ell_4}^{*}\sum_{\ell_5,\ell_6}^{*}\sum_{\ell_7,\ell_8}^{*} \Gamma_{i,s_1,\ell_1}\Gamma_{i,s_1+q,\ell_2}\Gamma_{i,s_2,\ell_3}\Gamma_{i,s_2+q,\ell_4}\Gamma_{i,s_3,\ell_5}\Gamma_{i,s_3+q,\ell_6}\\
&\times\Gamma_{i,s_4,\ell_7}\Gamma_{i,s_4+q,\ell_8}\mathrm{E}\left(\prod_{k=1}^{8}\mathrm{Z}_{i,j,\ell_k}\right),
\end{aligned}$$

where $\sum^*$ denote the summation over mutually distinct indices. We consider all combinations of $\ell_1,\ldots,\ell_8$ in the set $\{(\ell_1,\ldots,\ell_8) : \ell_1 \neq \ell_2, \ell_3 \neq \ell_4, \ell_5 \neq \ell_6$ and $\ell_7 \neq \ell_8\}$ and will show that every term in (S1.8) is $O\left((p-q)^2\right)$. For simplicity of notations, we define

$$\begin{aligned}
f^{(i)}_{s_1,s_2,s_3,s_4} &= \sum_{\ell=1}^{m} \Gamma_{i,s_1,\ell}\Gamma_{i,s_2,\ell}\Gamma_{i,s_3,\ell}\Gamma_{i,s_4,\ell},\\
f^{(i)}_{s_1,s_2,s_3,s_4,s_5,s_6} &= \sum_{\ell=1}^{m} \Gamma_{i,s_1,\ell}\Gamma_{i,s_2,\ell}\Gamma_{i,s_3,\ell}\Gamma_{i,s_4,\ell}\Gamma_{i,s_5,\ell}\Gamma_{i,s_6,\ell}
\end{aligned}$$

and

$$f^{(i)}_{s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8} = \sum_{\ell=1}^{m} \Gamma_{i,s_1,\ell}\Gamma_{i,s_2,\ell}\Gamma_{i,s_3,\ell}\Gamma_{i,s_4,\ell}\Gamma_{i,s_5,\ell}\Gamma_{i,s_6,\ell}\Gamma_{i,s_7,\ell}\Gamma_{i,s_8,\ell}.$$

It can be shown that $\sigma_{i,s_1,s_2} = \sum_{\ell=1}^{m}\Gamma_{i,s_1,\ell}\Gamma_{i,s_2,\ell}$. Since $\mathrm{E}(\mathrm{Z}_{i,j,\ell}) = 0$, any term with a single ℓ_i in (S1.8) equals to zero. First of all, we consider cases where $k_1,\ldots,k_8$ consist of four pairs with distinct values. For example, if $\ell_1 = \ell_3$, $\ell_2 = \ell_4$, $\ell_5 = \ell_7$

and $\ell_6 = \ell_8$, we have

$$\sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sum_{\ell_1,\ell_2,\ell_3,\ell_4}^{*} \Gamma_{i,s_1,\ell_1}\Gamma_{i,s_2,\ell_1}\Gamma_{i,s_1+q,\ell_2}\Gamma_{i,s_2+q,\ell_2}\Gamma_{i,s_3,\ell_3}\Gamma_{i,s_4,\ell_3}\Gamma_{i,s_3+q,\ell_4}\Gamma_{i,s_4+q,\ell_4}$$
$$= \mathrm{F}_{i,q,a} - 2\mathrm{F}_{i,q,b1} - 2\mathrm{F}_{i,q,b2} - \mathrm{F}_{i,q,b3} - \mathrm{F}_{i,q,b4} + \sum_{k=1}^{3}\mathrm{F}_{i,q,ck} + 4\sum_{k=1}^{2}\mathrm{F}_{i,q,dk} - 6\mathrm{F}_{i,q,e}, \tag{S1.9}$$

where

$$\mathrm{F}_{i,q,a} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_2}\sigma_{i,s_1+q,s_2+q}\sigma_{i,s_3,s_4}\sigma_{i,s_3+q,s_4+q},$$

$$\mathrm{F}_{i,q,b1} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1+q,s_2+q}\sigma_{i,s_3,s_4}f^{(i)}_{s_1,s_2,s_3+q,s_4+q},$$

$$\mathrm{F}_{i,q,b2} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_2}\sigma_{i,s_1+q,s_2+q}f^{(i)}_{s_3,s_3+q,s_4,s_4+q},$$

$$\mathrm{F}_{i,q,b3} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1+q,s_2+q}\sigma_{i,s_3+q,s_4+q}f^{(i)}_{s_1,s_2,s_3,s_4},$$

$$\mathrm{F}_{i,q,b4} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_2}\sigma_{i,s_3,s_4}f^{(i)}_{s_1+q,s_2+q,s_3+q,s_4+q},$$

$$\mathrm{F}_{i,q,c1} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} f^{(i)}_{s_1,s_1+q,s_2,s_2+q}f^{(i)}_{s_3,s_3+q,s_4,s_4+q},$$

$$\mathrm{F}_{i,q,c2} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} f^{(i)}_{s_1,s_2,s_3,s_4}f^{(i)}_{s_1+q,s_2+q,s_3+q,s_4+q},$$

$$\mathrm{F}_{i,q,c3} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} f^{(i)}_{s_1,s_2,s_3+q,s_4+q}f^{(i)}_{s_1+q,s_2+q,s_3,s_4},$$

$$\mathrm{F}_{i,q,d1} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_2}f^{(i)}_{s_1+q,s_2+q,s_3,s_3+q,s_4,s_4+q},$$

$$\mathrm{F}_{i,q,d2} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1+q,s_2+q} f^{(i)}_{s_1,s_2,s_3,s_3+q,s_4,s_4+q},$$

and

$$\mathrm{F}_{i,q,e} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} f^{(i)}_{s_1,s_1+q,s_2,s_2+q,s_3,s_3+q,s_4,s_4+q}.$$

We are going to prove that every term in (S1.9) is $O\left((p-q)^2\right)$. According to Lemma 2, we have

$$|\mathrm{F}_{i,q,a}| \leq \left[\sum_{s_1,s_2=1}^{p} |\sigma_{i,s_1,s_2}|\right]^2 \leq C_3(p-q)^2. \tag{S1.10}$$

According to the Cauchy-Schwartz's inequality, the following inequality for $\mathrm{F}_{i,q,b1}$ holds:

$$|\mathrm{F}_{i,q,b1}| \leq \left(\sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma^2_{i,s_1+q,s_2+q}\sigma^2_{i,s_3,s_4}\right)^{1/2} \left(\sum_{s_1,s_2,s_3,s_4=1}^{p-q} (f^{(i)}_{s_1,s_2,s_3+q,s_4+q})^2\right)^{1/2}.$$

Similar to (S1.10), it can be shown that $\sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma^2_{i,s_1+q,s_2+q}\sigma^2_{i,s_3,s_4} = O\left((p-q)^2\right)$. And

$$\begin{aligned}\sum_{s_1,s_2,s_3,s_4=1}^{p-q} (f^{(i)}_{s_1,s_2,s_3+q,s_4+q})^2 &\leq \sum_{s_1,s_2,s_3,s_4=1}^{p} (f^{(i)}_{s_1,s_2,s_3,s_4})^2 = \sum_{\ell_1,\ell_2=1}^{m} A^4_{i,\ell_1\ell_2} = \mathrm{tr}\left((A_i \circ A_i)^2\right) \\ &\leq M^4 p = o\left((p-q)^2\right),\end{aligned}$$

where the $m \times m$ matrix $A_i = \Gamma_i^\top \Gamma_i$ has the same non-negative eigenvalues as Σ_i. Thus we have $\mathrm{F}_{i,q,b1} = O\left((p-q)^2\right)$. In the same way, $\mathrm{F}_{i,q,b3}$ and $\mathrm{F}_{i,q,b4}$ can also be shown to be $O\left((p-q)^2\right)$.

For $\mathrm{F}_{i,q,b2}$, since the following equality holds:

$$f^{(i)}_{s_3,s_3+q,s_4,s_4+q} = \frac{1}{\mathrm{E}(\mathrm{Z}^4_{i,j,k})}\omega^{s_3,s_4}_{i,q} - \sigma_{i,s_3,s_4}\sigma_{i,s_3+q,s_4+q} - \sigma_{i,s_3,s_4+q}\sigma_{i,s_3+q,s_4}, \tag{S1.11}$$

we have

$$\begin{aligned}|\mathrm{F}_{i,q,b2}| \;\leq\; & \frac{1}{\mathrm{E}(\mathrm{Z}^4_{i,j,k})}\left|\sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_2}\sigma_{i,s_1+q,s_2+q}\omega^{s_3,s_4}_{i,q}\right| \\ & + \left|\sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_2}\sigma_{i,s_1+q,s_2+q}\sigma_{i,s_3,s_4}\sigma_{i,s_3+q,s_4+q}\right|\end{aligned}$$

$$
\begin{aligned}
& + \left| \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_2}\sigma_{i,s_1+q,s_2+q}\sigma_{i,s_3,s_4+q}\sigma_{i,s_3+q,s_4} \right| \\
= \;& O\left((p-q)^2\right).
\end{aligned}
$$

Similarly, using equation (S1.11), we have

$$
\begin{aligned}
|\mathrm{F}_{i,q,c1}| \;\leq\; & \frac{1}{\mathrm{E}^2(\mathrm{Z}_{i,j,k}^4)} \left| \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \omega_{i,q}^{s_1,s_2}\omega_{i,q}^{s_3,s_4} \right| \\
& +2 \left| \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_2}\sigma_{i,s_1+q,s_2+q}\sigma_{i,s_3,s_4+q}\sigma_{i,s_3+q,s_4} \right| \\
& - \frac{2}{\mathrm{E}(\mathrm{Z}_{i,j,k}^4)} \left| \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \omega_{i,q}^{s_1,s_2}\sigma_{i,s_3+q,s_4}\sigma_{i,s_3,s_4+q} \right| \\
& - \frac{2}{\mathrm{E}(\mathrm{Z}_{i,j,k}^4)} \left| \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \omega_{i,q}^{s_1,s_2}\sigma_{i,s_3,s_4}\sigma_{i,s_3+q,s_4+q} \right| \\
& + \left| \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_2}\sigma_{i,s_1+q,s_2+q}\sigma_{i,s_3,s_4}\sigma_{i,s_3+q,s_4+q} \right| \\
& + \left| \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_2+q}\sigma_{i,s_1+q,s_2}\sigma_{i,s_3,s_4+q}\sigma_{i,s_3+q,s_4} \right| \\
= \;& O\left((p-q)^2\right).
\end{aligned}
$$

The derivation for $\mathrm{F}_{i,q,c2}$ and $\mathrm{F}_{i,q,c3}$ are also similar. Using the Cauchy-Schwartz's inequality, we have

$$
\begin{aligned}
|\mathrm{F}_{i,q,c2}| \;\leq\; & \left(\sum_{s_1,s_2,s_3,s_4=1}^{p-q} (f_{s_1,s_2,s_3,s_4}^{(i)})^2 \right)^{1/2} \left(\sum_{s_1,s_2,s_3,s_4=1}^{p-q} (f_{s_1+q,s_2+q,s_3+q,s_4+q}^{(i)})^2 \right)^{1/2} \\
\leq\; & \sum_{s_1,s_2,s_3,s_4=1}^{p} (f_{s_1,s_2,s_3,s_4}^{(i)})^2 \leq= \mathrm{tr}\left((A_i \circ A_i)^2\right) = o\left((p-q)^2\right).
\end{aligned}
$$

Furthermore, for $\mathrm{F}_{i,q,d1}$ and $\mathrm{F}_{i,q,d2}$, we have

$$
\begin{aligned}
|\mathrm{F}_{i,q,d1}| \;\leq\; & M_1 \sum_{\ell=1}^{m} \left\{ \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \Gamma_{i,s_1+q,\ell}^2 \Gamma_{i,s_3,\ell}^2 \Gamma_{i,s_4,\ell}^2 \right\}^{1/2} \\
& \times \left\{ \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \Gamma_{i,s_2+q,\ell}^2 \Gamma_{i,s_3+q,\ell}^2 \Gamma_{i,s_4+q,\ell}^2 \right\}^{1/2}
\end{aligned}
$$

$$\leq \quad C_4 p(p-q) = O\left((p-q)^2\right)$$

And the last term

$$\begin{aligned} |\mathrm{F}_{i,q,e}| &\leq \sum_{\ell=1}^{m} \sum_{s_1,s_2,s_3,s_4=1}^{p} \Gamma_{i,s_1,\ell}^2 \Gamma_{i,s_2,\ell}^2 \Gamma_{i,s_3,\ell}^2 \Gamma_{i,s_4,\ell}^2 \\ &= \operatorname{tr}(A_i \circ A_i) = o\left((p-q)^2\right). \end{aligned}$$

Thus, (S1.9) is shown to be $O\left((p-q)^2\right)$. Likewise, other combinations where $\ell_1, \ldots, \ell_8$ consists of four distinct pairs can be shown to be $O\left((p-q)^2\right)$.

Define

$$f_{s_1,s_2,s_3}^{(i)} = \sum_{\ell=1}^{m} \Gamma_{i,s_1,\ell} \Gamma_{i,s_2,\ell} \Gamma_{i,s_3,\ell}$$

and

$$f_{s_1,s_2,s_3,s_4,s_5}^{(i)} = \sum_{\ell=1}^{m} \Gamma_{i,s_1,\ell} \Gamma_{i,s_2,\ell} \Gamma_{i,s_3,\ell} \Gamma_{i,s_4,\ell} \Gamma_{i,s_5,\ell}.$$

We consider the case if $\ell_1 = \ell_4$, $\ell_2 = \ell_5 = \ell_7$ and $\ell_3 = \ell_6 = \ell_8$ and we have

$$\begin{aligned} &\sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sum_{\ell_1,\ell_2,\ell_3}^{*} \Gamma_{i,s_1,\ell_1} \Gamma_{i,s_2+q,\ell_1} \Gamma_{i,s_1+q,\ell_2} \Gamma_{i,s_3,\ell_2} \Gamma_{i,s_4,\ell_2} \Gamma_{i,s_2,\ell_3} \Gamma_{i,s_3+q,\ell_3} \Gamma_{i,s_4+q,\ell_3} \\ = &\sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_2+q} f_{s_2,s_3+q,s_4+q}^{(i)} f_{s_1+q,s_3,s_4}^{(i)} - \sum_{s_1,s_2,s_3,s_4=1}^{p-q} f_{s_1,s_2+q,s_2,s_3+q,s_4+q}^{(i)} f_{s_1+q,s_3,s_4}^{(i)} \\ &- \sum_{s_1,s_2,s_3,s_4=1}^{p-q} f_{s_1,s_1+q,s_2+q,s_3,s_4}^{(i)} f_{s_2,s_3+q,s_4+q}^{(i)} + \sum_{s_1,s_2,s_3,s_4=1}^{p-q} 2 f_{s_1,s_1+q,s_2,s_2+q,s_3,s_3+q,s_4,s_4+q}^{(i)}. \end{aligned} \tag{S1.12}$$

The first term in (S1.12) can be shown to be $O\left((p-q)^2\right)$ as follows

$$\begin{aligned} &\sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_2+q} f_{s_2,s_3+q,s_4+q}^{(i)} f_{s_1+q,s_3,s_4}^{(i)} \\ \leq \quad & M_1 \left(\sum_{s_1,s_2,s_3,s_4=1}^{p-q} (f_{s_2,s_3+q,s_4+q}^{(i)})^2 \right)^{1/2} \left(\sum_{s_1,s_2,s_3,s_4=1}^{p-q} (f_{s_1+q,s_3,s_4}^{(i)})^2 \right)^{1/2} \\ \leq \quad & M_1 (p-q) \sum_{\ell_1,\ell_2=1}^{m} A_{i,\ell_1\ell_2}^3 \leq C_5 (p-q) \operatorname{tr}(A_i^2) = O\left((p-q)^2\right). \end{aligned}$$

And similar results hold for the other three terms. Thus, (S1.12) is $O\left((p-q)^2\right)$. Other combinations with $\ell_1, \ldots, \ell_8$ taking three distinct values can also be shown as

$O\left((p-q)^2\right)$.

Furthermore, we consider cases where $\ell_1, \ldots, \ell_8$ are divided into two groups with distinct values. For instance, if $\ell_1 = \ell_3 = \ell_5 = \ell_7$ and $\ell_2 = \ell_4 = \ell_6 = \ell_8$,

$$\begin{aligned}
&\sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sum_{\ell_1,\ell_2}^{*} \Gamma_{i,s_1,\ell_1}\Gamma_{i,s_2,\ell_1}\Gamma_{i,s_3,\ell_1}\Gamma_{i,s_4,\ell_1}\Gamma_{i,s_1+q,\ell_2}\Gamma_{i,s_2+q,\ell_2}\Gamma_{i,s_3+q,\ell_2}\Gamma_{i,s_4+q,\ell_2} \\
= &\sum_{s_1,s_2,s_3,s_4=1}^{p-q} f^{(i)}_{s_1,s_2,s_3,s_4} f^{(i)}_{s_1+q,s_2+q,s_3+q,s_4+q} - \sum_{s_1,s_2,s_3,s_4=1}^{p-q} f^{(i)}_{s_1,s_1+q,s_2,s_2+q,s_3,s_3+q,s_4,s_4+q},
\end{aligned}$$

Similar to the proof of (S1.9), we can show that the two terms in the above equation are both $O\left((p-q)^2\right)$. This completes the proof of Lemma 3.

$\square$

S2. Proof of the main results

Proof of Proposition 1. Recall that we define the following in the proof of Proposition 1 in the Appendix.

$$\begin{aligned}
\mathrm{A}^{(1)}_{i,nq} &= \sum_{s=1}^{p-q} \frac{1}{P^2_{n_i}} \sum_{j,k}^{*} (\mathrm{X}_{i,j,s}\mathrm{X}_{i,j,s+q})(\mathrm{X}_{i,k,s}\mathrm{X}_{i,k,s+q}), \\
\mathrm{A}^{(2)}_{i,nq} &= \sum_{s=1}^{p-q} \frac{1}{P^3_{n_i}} \sum_{j,k,l}^{*} \mathrm{X}_{i,j,s}\mathrm{X}_{i,k,s+q}(\mathrm{X}_{i,l,s}\mathrm{X}_{i,l,s+q}), \\
\mathrm{A}^{(3)}_{i,nq} &= \sum_{s=1}^{p-q} \frac{1}{P^4_{n_i}} \sum_{j,k,l,t}^{*} \mathrm{X}_{i,j,s}\mathrm{X}_{i,k,s+q}\mathrm{X}_{i,l,s}\mathrm{X}_{i,t,s+q},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{A}^{(1)}_{c,nq} &= \sum_{s=1}^{p-q} \frac{1}{n_1 n_2} \sum_{j=1}^{n_1}\sum_{k=1}^{n_2} (\mathrm{X}_{1,j,s}\mathrm{X}_{1,j,s+q})(\mathrm{X}_{2,k,s}\mathrm{X}_{2,k,s+q}), \\
\mathrm{A}^{(2)}_{c,nq} &= \sum_{s=1}^{p-q} \frac{1}{n_1 n_2 (n_1-1)} \sum_{j,k}^{*}\sum_{l=1}^{n_2} \mathrm{X}_{1,j,s}\mathrm{X}_{1,k,s+q}(\mathrm{X}_{2,l,s}\mathrm{X}_{2,l,s+q}), \\
\mathrm{A}^{(3)}_{c,nq} &= \sum_{s=1}^{p-q} \frac{1}{n_1 n_2 (n_2-1)} \sum_{l=1}^{n_1}\sum_{j,k}^{*} (\mathrm{X}_{1,l,s}\mathrm{X}_{1,l,s+q})\mathrm{X}_{2,j,s}\mathrm{X}_{2,k,s+q}, \\
\mathrm{A}^{(4)}_{c,nq} &= \sum_{s=1}^{p-q} \frac{1}{n_1 n_2 (n_1-1)(n_2-1)} \sum_{j,k}^{*}\sum_{l,t}^{*} \mathrm{X}_{1,j,s}\mathrm{X}_{1,k,s+q}\mathrm{X}_{2,l,s}\mathrm{X}_{2,t,s+q}.
\end{aligned}$$

Since $\hat{\mathrm{D}}_{1,nq}$, $\hat{\mathrm{D}}_{2,nq}$ and $\hat{\mathrm{D}}_{c,nq}$ are unbiased and location invariant, we assume $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = 0$ without loss of generality from now on. It will be shown that $\mathrm{A}^{(1)}_{i,nq}$ and $\mathrm{A}^{(1)}_{c,nq}$ are the leading terms of $\hat{\mathrm{D}}_{i,nq}$ and $\hat{\mathrm{D}}_{c,nq}$ shortly after. Therefore, in order to derive the variance of $\hat{\mathrm{S}}_{nq}$, it suffices to derive the variances of $\mathrm{A}^{(1)}_{i,nq}$ and $\mathrm{A}^{(1)}_{c,nq}$, $i = 1, 2$, as well as the covariances among them.

First of all, the variance of the U-statistic $\mathrm{A}^{(1)}_{i,nq}$ can be derived via the Hoeffding decompositions (Hoeffding , 1948), which was proved to be valid in the high-dimensional setting by Zhong and Chen (2011). For $j = 1, \ldots, n_i$, let $\mathbf{B}_{i,j}(q) = (\mathrm{X}_{i,j,1}\mathrm{X}_{i,j,1+q}, \ldots, \mathrm{X}_{i,j,p-q}\mathrm{X}_{i,j,p})^\top$. Define the $\mathbb{R}^{p-q} \times \mathbb{R}^{p-q} \to \mathbb{R}$ kernel function $H(x_1, x_2) = x_1^\top x_2$, $x_1, x_2 \in \mathbb{R}^{p-q}$. Hence, $\mathrm{A}^{(1)}_{i,nq}$ can be written in the standard U-statistics form as follows:

$$\mathrm{A}^{(1)}_{i,nq} = \frac{1}{\binom{n_i}{2}} \sum_{C_{n,2}} H(\mathbf{B}_{i,j}(q), \mathbf{B}_{i,k}(q)),$$

where $\binom{n_i}{s} = \frac{n_i!}{(n_i-s)!s!}$ and $C_{n_i,s}$ refers to all distinct combinations of $\{j_1, \ldots, j_s\}$ from $\{1, 2, \ldots, n_i\}\}$.

Define $\zeta_1 = \mathrm{Var}\left(\mathrm{E}(H(\mathbf{B}_{i,j}(q), \mathbf{B}_{i,k}(q))|\mathbf{B}_{i,j}(q))\right)$ and $\zeta_2 = \mathrm{Var}(H(\mathbf{B}_{i,j}(q), \mathbf{B}_{i,k}(q)))$. Then, standard derivations show that

$$\zeta_1 = \sum_{s_1,s_2=1}^{p-q} \sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}\omega^{s_1,s_2}_{i,q} = J_{i,q}^\top W_{i,q} J_{i,q},$$

and

$$\zeta_2 = \sum_{s_1,s_2=1}^{p-q} (\omega^{s_1,s_2}_{i,q})^2 + 2\sum_{s_1,s_2=1}^{p-q} \sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}\omega^{s_1,s_2}_{i,q} = 2J_{i,q}^\top W_{i,q} J_{i,q} + \mathrm{tr}(W_{i,q}^2).$$

According to the Hoeffding decomposition, we have

$$\mathrm{Var}(\mathrm{A}^{(1)}_{i,nq}) = \binom{n_i}{2}^{-1} \sum_{c=1}^{2} \binom{2}{c}\binom{n-2}{2-c}\zeta_c = \frac{4}{n_i} J_{i,q}^\top W_{i,q} J_{i,q} + \frac{2}{n_i(n_i-1)}\mathrm{tr}(W_{i,q}^2). \tag{S2.1}$$

Similarly, we get

$$\mathrm{Var}(\mathrm{A}^{(1)}_{c,nq}) = \frac{1}{n_1} J_{2,q}^\top W_{1,q} J_{2,q} + \frac{1}{n_2}\sum_{s_1,s_2=1}^{p-q} J_{1,q}^\top W_{2,q} J_{1,q} + \frac{1}{n_1 n_2}\mathrm{tr}(W_{1,q}W_{2,q}). \tag{S2.2}$$

and

$$\mathrm{Cov}(\mathrm{A}^{(1)}_{c,nq}, \mathrm{A}^{(1)}_{1,nq}) = \frac{2}{n_1} J_{1,q}^\top W_{1,q} J_{2,q}, \qquad \mathrm{Cov}(\mathrm{A}^{(1)}_{c,nq}, \mathrm{A}^{(1)}_{2,nq}) = \frac{2}{n_2} J_{1,q}^\top W_{2,q} J_{2,q}.$$

Then, the leading terms of the variance of $\hat{S}_{nq}$ are

$$\begin{aligned}\mathrm{Var}(\hat{\mathrm{S}}_{nq}) &= \mathrm{Var}(\mathrm{A}_{1,nq}^{(1)}) + \mathrm{Var}(\mathrm{A}_{2,nq}^{(1)}) + 4\mathrm{Var}(\mathrm{A}_{c,nq}^{(1)}) \\ &\quad -4\mathrm{Cov}(\mathrm{A}_{c,nq}^{(1)}, \mathrm{A}_{1,nq}^{(1)}) - 4\mathrm{Cov}(\mathrm{A}_{c,nq}^{(1)}, \mathrm{A}_{2,nq}^{(1)}) \\ &= \frac{4}{n_1} J_q^\top W_{1,q} J_q + \frac{4}{n_2} J_q^\top W_{2,q} J_q + \frac{2}{n_1(n_1-1)} \mathrm{tr}(W_{1,q}^2) \\ &\quad + \frac{2}{n_2(n_2-1)} \mathrm{tr}(W_{2,q}^2) + \frac{4}{n_1 n_2} \mathrm{tr}(W_{1,q} W_{2,q}). \qquad \text{(S2.3)}\end{aligned}$$

Thus, it remains to show that the remaining terms $\mathrm{A}_{i,nq}^{(2)}$, $\mathrm{A}_{i,nq}^{(3)}$, $\mathrm{A}_{c,nq}^{(2)}$, $\mathrm{A}_{c,nq}^{(3)}$ and $\mathrm{A}_{c,nq}^{(4)}$ are at a smaller order than the leading order terms of $\hat{\mathrm{S}}_{nq}$. We first consider the general case that $h_q \asymp cq^{-\beta}$ and establish the order of $\mathrm{A}_{i,nq}^{(1)}$ and $\mathrm{A}_{c,nq}^{(1)}$. In (S2.1), ignoring the non-stochastic constants, the first term equals to

$$\sum_{s_1,s_2=1}^{p-q} \sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}\omega_{i,q}^{s_1,s_2} \asymp Cq^{-\beta} \sum_{s_1,s_2=1}^{p-q} |\omega_{i,q}^{s_1,s_2}| \asymp C(p-q)q^{-\beta}.$$

And it can be shown that $\mathrm{tr}(W_{i,q}^2) \asymp (p-q)$ for the second term in (S2.1), according to Lemma 2 and Assumption (C5).

Notice that the order of $\mathrm{A}_{i,nq}^{(1)}$ depends on q. In the following three different regimes, the leading order of $\mathrm{A}_{i,nq}^{(1)}$ is:

(i) If $q = o(n_i^{1/\beta})$, the first term of in (S2.1) is the leading term;

(ii) If $q^{-\beta} \asymp 1/n_i$, both terms in (S2.1) are of the same order;

(iii) If $n_i^{1/\beta} = o(q)$ and $q = o(p)$, the second term in (S2.1) is the leading term.

In order to show that $\mathrm{A}_{i,nq}^{(2)}$ and $\mathrm{A}_{i,nq}^{(3)}$ are at a smaller order than $\mathrm{A}_{i,nq}^{(1)}$, it is sufficient to show that the variances of $\mathrm{A}_{i,nq}^{(2)}$ and $\mathrm{A}_{i,nq}^{(3)}$ are both at a smaller order of $\mathrm{Var}(\mathrm{A}_{i,nq}^{(1)})$. Similarly, the variance of $\mathrm{A}_{i,nq}^{(2)}$ and $\mathrm{A}_{i,nq}^{(3)}$ can be derived using the Hoeffding decomposition. Write $\mathrm{A}_{i,nq}^{(2)} = \frac{1}{P_{n_i}^3} \sum_{j,k,l}^* \phi(j,k,l)$, where

$$\phi(j,k,l) = \sum_{s=1}^{p-q} \mathrm{X}_{i,j,s}\mathrm{X}_{i,l,s+q}\mathrm{X}_{i,k,s}\mathrm{X}_{i,k,s+q}.$$

Symmetrizing ϕ by

$$H_2(\mathbf{X}_{i,j}, \mathbf{X}_{i,k}, \mathbf{X}_{i,l}) = \frac{1}{6}\left[\phi(j,k,l) + \phi(j,l,k) + \phi(k,j,l) + \phi(k,l,j) + \phi(l,j,k) + \phi(l,k,j)\right],$$

and we can write $\mathrm{A}_{i,nq}^{(2)}$ in the standard U-statistic form with the symmetric kernel function H_2 such that

$$\mathrm{A}_{i,nq}^{(2)} = \frac{1}{\binom{n_i}{3}} \sum_{C_{n_i,3}} H_2(\mathrm{X}_{j_1}, \mathrm{X}_{j_2}, \mathrm{X}_{j_3}).$$

By the Hoeffding variance decomposition again, we have

$$\begin{aligned} \mathrm{Var}(\mathrm{A}_{i,nq}^{(2)}) &= \binom{n_i}{3}^{-1} \sum_{c=1}^{3} \binom{3}{c} \binom{n-3}{3-c} \zeta_{2c} \\ &= \frac{n_i - 3}{P_{n_i}^3}(2b_{i,q,a1} + b_{i,q,a2}) + \frac{1}{P_{n_i}^3}(\sum_{j=1}^{4} b_{i,q,bj} + \sum_{j=1}^{2} b_{i,q,cj}), \end{aligned} \quad \text{(S2.4)}$$

where

$$b_{i,q,a1} = \sum_{s_1,s_2=1}^{p-q} \sigma_{i,s_1,s_2}\sigma_{i,s_1+q,s_2+q}\sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}, \quad \text{(S2.5)}$$

$$b_{i,q,a2} = \sum_{s_1,s_2=1}^{p-q} \sigma_{i,s_1,s_2+q}\sigma_{i,s_1+q,s_2}\sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q},$$

$$b_{i,q,b1} = \sum_{s_1,s_2=1}^{p-q} \sigma_{i,s_1+q,s_2+q}\mathrm{E}(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2})\mathrm{E}(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}),$$

$$b_{i,q,b2} = \sum_{s_1,s_2=1}^{p-q} \sigma_{i,s_1+q,s_2}\mathrm{E}(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2+q})\mathrm{E}(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}),$$

$$b_{i,q,b3} = \sum_{s_1,s_2=1}^{p-q} \sigma_{i,s_1,s_2}\mathrm{E}(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2+q})\mathrm{E}(\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}),$$

$$b_{i,q,b4} = \sum_{s_1,s_2=1}^{p-q} \sigma_{i,s_1,s_2+q}\mathrm{E}(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2})\mathrm{E}(\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}),$$

$$b_{i,q,c1} = \sum_{s_1,s_2=1}^{p-q} \sigma_{i,s_1,s_2}\sigma_{i,s_1+q,s_2+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}\right),$$

and

$$b_{i,q,c2} = \sum_{s_1,s_2=1}^{p-q} \sigma_{i,s_1,s_2+q}\sigma_{i,s_1+q,s_2}\mathrm{E}\left(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}\right).$$

It suffices to show that each term in (S2.4) is a smaller order of $\mathrm{Var}(\mathrm{A}_{i,nq}^{(1)})$. It can be shown that the first term in (S2.4)

$$\frac{n_i - 3}{P_{n_i}^3} b_{i,q,a1} \le \frac{M_1 q^{-\beta}}{n_i^2} \sum_{s_1,s_2=1}^{p-q} |\sigma_{i,s_1,s_2}| \le \frac{M_1 (p-q) q^{-\beta}}{n_i^2}. \tag{S2.6}$$

For q which is in Regimes (i) and (ii), $\mathrm{Var}(\mathrm{A}_{i,nq}^{(1)}) \asymp (p-q)q^{-\beta}/n_i$ thus (S2.6) is at a smaller order of $\mathrm{Var}(\mathrm{A}_{i,nq}^{(1)})$. For larger q in Regime (iii), since $\mathrm{Var}(\mathrm{A}_{i,nq}^{(1)}) \asymp (p-q)/n_i^2$ and $q^{-\beta} = o(1)$, (S2.6) is also at a smaller order of $\mathrm{Var}(\mathrm{A}_{i,nq}^{(1)})$. The same result holds for $b_{i,q,a2}$. Next, we show that $b_{i,q,b1} = o\left(\mathrm{Var}(\mathrm{A}_{i,nq}^{(1)})\right)$ and the rest of $b_{i,q,bj}$, $j = 2, 3, 4$, can be shown in the same way. For $s_1 < s_2$, we have

$$\mathrm{E}(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}) \le 12\|\mathrm{X}_{i,j,s_1}\|_4 \|\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}\|_4 \alpha_{\mathrm{X}_i}(|s_2 - s_1|)^{1/2} \le C_7 \alpha_{\mathrm{X}_i}(|s_2 - s_1|)^{1/2}.$$

Similarly, for $s_2 < s_1$,

$$\mathrm{E}(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2}) \le 12\|\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\|_4 \|\mathrm{X}_{i,j,s_2}\|_4 \alpha_{\mathrm{X}_i}(|s_1 - s_2|)^{1/2} \le C_8 \alpha_{\mathrm{X}_i}(|s_1 - s_2|)^{1/2}.$$

Then, it follows that

$$\begin{aligned} b_{i,q,b1} &= \sum_{s=1}^{p-q} \sigma_{i,s+q,s+q} \mathrm{E}^2(\mathrm{X}_{i,j,s}^2 \mathrm{X}_{i,j,s+q}) \\ &\quad + \sum_{|s_2 - s_1|=1}^{p-q-1} \sigma_{i,s_1+q,s_2+q} \mathrm{E}(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{X}_{i,j,s_2}) \mathrm{E}(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}) \\ &= O(p-q). \end{aligned}$$

Therefore, $b_{i,q,b1}/P_{n_i}^3$ is a smaller order of $\mathrm{Var}(\mathrm{A}_{i,nq}^{(1)})$ for any $q = o(p)$. It also can be shown that $b_{2,q,c1}$ and $b_{2,q,c2}$ are both at a smaller order than $\mathrm{Var}(\mathrm{A}_{i,nq}^{(1)})$ for any $q = o(p)$. Employing the same method, we can show that $\mathrm{Var}(\mathrm{A}_{i,nq}^{(3)}) = o(\mathrm{Var}(\mathrm{A}_{i,nq}^{(1)}))$, since $\mathrm{Var}(\mathrm{A}_{i,nq}^{(3)})$ is at a even smaller order than $\mathrm{Var}(\mathrm{A}_{i,nq}^{(2)})$. It is noticed that the covariances between $\mathrm{A}_{i,nq}^{(1)}$, $\mathrm{A}_{i,nq}^{(2)}$ and $\mathrm{A}_{i,nq}^{(3)}$ are all ignorable comparing with $\mathrm{Var}(\mathrm{A}_{i,nq}^{(1)})$ by Cauchy-Schwartz inequality. Thus, $\mathrm{A}_{i,nq}^{(1)}$ is the leading order term of $\hat{\mathrm{D}}_{i,nq}$.

Similarly, it can be shown that for q which is in Regimes (i) and (ii), $\mathrm{Var}(\mathrm{A}_{c,nq}^{(1)}) \asymp (p-q)q^{-\beta}/n$ and for larger q in Regime (iii), since $\mathrm{Var}(\mathrm{A}_{c,nq}^{(1)}) \asymp (p-q)/n^2$. Now we show that the variances of $\mathrm{A}_{c,nq}^{(2)}$, $\mathrm{A}_{c,nq}^{(3)}$ and $\mathrm{A}_{c,nq}^{(4)}$ are at a smaller order than $\mathrm{Var}(\mathrm{A}_{c,nq}^{(1)})$. Standard derivation shows that

$$\mathrm{Var}(\mathrm{A}_{c,nq}^{(2)}) = \frac{1}{n_1 n_2 (n_1 - 1)} \sum_{s_1,s_2=1}^{p-q} \sigma_{1,s_1,s_2} \sigma_{1,s_1+q,s_2+q} \omega_{2,q}^{s_1,s_2}$$

$$
\begin{aligned}
&+\frac{1}{n_1 n_2(n_1-1)}\sum_{s_1,s_2=1}^{p-q}\sigma_{1,s_1,s_2+q}\sigma_{1,s_1+q,s_2}\omega_{2,q}^{s_1,s_2}\\
&+\frac{1}{n_1(n_1-1)}\sum_{s_1,s_2=1}^{p-q}\sigma_{1,s_1,s_2}\sigma_{1,s_1+q,s_2+q}\sigma_{2,s_1,s_1+q}\sigma_{2,s_2,s_2+q}\\
&+\frac{1}{n_1(n_1-1)}\sum_{s_1,s_2=1}^{p-q}\sigma_{1,s_1,s_2+q}\sigma_{1,s_1+q,s_2}\sigma_{2,s_1,s_1+q}\sigma_{2,s_2,s_2+q} \quad \text{(S2.7)}
\end{aligned}
$$

For smaller q which is in Regimes (i) and (ii), $\mathrm{Var}(\mathrm{A}_{c,nq}^{(2)})$ is ar a smaller order than $\mathrm{Var}(\mathrm{A}_{c,nq}^{(1)})$ since there are additional n_i in the denominator of each component in (S2.7). On the other hand, for larger q in Regimes (iii), the first two terms in (S2.7) is at a smaller order of $n^{-2}(p-q)$. Note that

$$
\sum_{s_1,s_2=1}^{p-q}\sigma_{1,s_1,s_2}\sigma_{1,s_1+q,s_2+q}\sigma_{2,s_1,s_1+q}\sigma_{2,s_2,s_2+q} \le M_1^2\left(\sum_{s=1}^{p-q}\sigma_{2,s,s+q}\right)^2 = o(p-q).
$$

Thus, the latter two terms in (S2.7) is also of smaller order than $n^{-2}(p-q)$. Hence, $\mathrm{Var}(\mathrm{A}_{c,nq}^{(2)}) = o(\mathrm{Var}(\mathrm{A}_{c,nq}^{(1)}))$ for any $q = o(p)$. The specific form of $\mathrm{Var}(\mathrm{A}_{c,nq}^{(3)})$ and $\mathrm{Var}(\mathrm{A}_{c,nq}^{(4)})$ are provided as follows.

$$
\begin{aligned}
\mathrm{Var}(\mathrm{A}_{c,nq}^{(3)}) = &\frac{1}{n_1 n_2(n_2-1)}\sum_{s_1,s_2=1}^{p-q}\sigma_{2,s_1,s_2}\sigma_{2,s_1+q,s_2+q}\omega_{1,q}^{s_1,s_2}\\
&+\frac{1}{n_1 n_2(n_2-1)}\sum_{s_1,s_2=1}^{p-q}\sigma_{2,s_1,s_2+q}\sigma_{2,s_1+q,s_2}\omega_{1,q}^{s_1,s_2}\\
&+\frac{1}{n_2(n_2-1)}\sum_{s_1,s_2=1}^{p-q}\sigma_{1,s_1,s_1+q}\sigma_{1,s_2,s_2+q}\sigma_{2,s_1,s_2}\sigma_{2,s_1+q,s_2+q}\\
&+\frac{1}{n_2(n_2-1)}\sum_{s_1,s_2=1}^{p-q}\sigma_{1,s_1,s_1+q}\sigma_{1,s_2,s_2+q}\sigma_{2,s_1,s_2+q}\sigma_{2,s_1+q,s_2+q}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}(\mathrm{A}_{c,nq}^{(4)}) = &\frac{1}{P_{n_1}^2 P_{n_2}^2}\sum_{s_1,s_2=1}^{p-q}\left(\sigma_{1,s_1,s_2}\sigma_{1,s_1+q,s_2+q}+\sigma_{1,s_1,s_2+q}\sigma_{1,s_1+q,s_2}\right)\\
&\times\left(\sigma_{2,s_1,s_2}\sigma_{2,s_1+q,s_2+q}+\sigma_{2,s_1,s_2+q}\sigma_{2,s_1+q,s_2}\right)
\end{aligned}
$$

Similarly, they can be shown to be at a smaller order than $\mathrm{Var}(\mathrm{A}_{c,nq}^{(1)})$. Thus, $\mathrm{A}_{c,nq}^{(1)}$ is the leading order term of $\hat{\mathrm{D}}_{c,nq}$. This completes the proof of Proposition 1. $\square$

Proof of Proposition 2. We prove the ratio consistency of $\mathrm{R}_{i,nq}$, $i = 1, 2$, and $\mathrm{R}_{c,nq}$ by deriving their means and variances, respectively. Write the sample covariance estimator $\hat{\sigma}_{h,s,s+q}$ as the standard U-statistic form:

$$\hat{\sigma}_{i,s,s+q} = \frac{1}{n_i}\sum_{j=1}^{n_i} \mathrm{X}_{i,j,s}\mathrm{X}_{i,j,s+q} - \frac{1}{P_{n_i}^2}\sum_{j,k}^{*} \mathrm{X}_{i,j,s}\mathrm{X}_{i,k,s+q}, \tag{S2.8}$$

To be consistent with the notations before, $\sum_{i,j}^{*}$ indicates the summation over distinct subscripts j and k. And the expectation of $\mathrm{R}_{i,nq}$ can be written as

$$\mathrm{E}\left(\mathrm{R}_{i,nq}\right) = \sum_{s_1,s_2=1}^{p-q} \mathrm{E}^2(\hat{\mathrm{Y}}_{i,j}^{s_1,s_1+q}\hat{\mathrm{Y}}_{i,j}^{s_2,s_2+q}). \tag{S2.9}$$

Derivations show that

$$\mathrm{E}(\hat{\mathrm{Y}}_{i,j}^{s_1,s_1+q}\hat{\mathrm{Y}}_{i,j}^{s_2,s_2+q}) - \omega_{i,q}^{s_1,s_2} = -\frac{1}{n_i}\omega_{i,q}^{s_1,s_2} + \frac{1}{P_{n_i}^2}\sigma_{i,s_1,s_2}\sigma_{i,s_1+q,s_2+q} + \frac{1}{P_{n_i}^2}\sigma_{i,s_1,s_2+q}\sigma_{i,s_1+q,s_2}. \tag{S2.10}$$

It can be shown that

$$\sum_{s_1,s_2=1}^{p-q} \sigma_{i,s_1,s_2}^2\sigma_{i,s_1+q,s_2+q}^2 \le M_1^2\mathrm{tr}(\Sigma_i^2) = O(p-q),$$

and

$$\begin{aligned}
&\sum_{s_1,s_2=1}^{p-q} \sigma_{i,s_1,s_2+q}^2\sigma_{i,s_1+q,s_2}^2 \\
= \quad & \left\{\sum_{s=1}^{p-q}\sigma_{i,s,s+q}^4\right\} + \left\{2\sum_{s=1}^{p-q-1}\sigma_{i,s,s+1+q}^2\sigma_{i,s+q,s+1}^2\right\} + \left\{2\sum_{s=1}^{p-q-2}\sigma_{i,s,s+2+q}^2\sigma_{i,s+q,s+2}^2\right\} \\
& + \cdots + \left\{2\sum_{s=1}^{p-2q}\sigma_{i,s,s+2q}^2\sigma_{i,s+q,s+q}^2\right\} + \cdots + \left\{2\sigma_{i,1,p}^2\sigma_{i,1+q,p-q}^2\right\} \\
\le \quad & M^2\left\{\sum_{s=1}^{p-q}\sigma_{i,s,s+q}^2 + 2\sum_{s=1}^{p-q-1}\sigma_{i,s,s+1+q}^2 + \cdots + 2\sum_{s=1}^{p-2q}\sigma_{i,s,s+2q}^2 + \cdots + 2\sigma_{i,1,p}^2\right\} \\
\le \quad & 2M^2(p-q)\sum_{k\ge q} h_i(q) = O(p-q).
\end{aligned}$$

Combining with Assumption (C5) and substituting (S2.10) into (S2.9), we have

$$\mathrm{E}\left(\mathrm{R}_{i,nq}\right) = \mathrm{tr}(W_{i,q}^2)\left\{1 + O\left(\frac{1}{n_i^2}\right)\right\}.$$

Thus, it is sufficient to show that $\mathrm{Var}\left(\mathrm{R}_{i,nq}\right) = o\left((p-q)^2\right)$. Note that

$$\hat{\mathrm{Y}}_{i,j}^{s,s+q} = \mathrm{Y}_{i,j}^{s,s+q} + (\sigma_{i,s,s+q} - \hat{\sigma}_{i,s,s+q}), \tag{S2.11}$$

and we have

$$\begin{aligned}
\mathrm{E}\left(\mathrm{R}_{i,nq}^2\right) &= \frac{1}{(P_{n_i}^2)^2}\mathrm{E}\left\{\sum_{j_1,k_1}^{*}\sum_{j_2,k_2}^{*}(\mathbf{Y}_{i,j_1}(q)^\top\mathbf{Y}_{i,k_1}(q))^2(\mathbf{Y}_{i,j_2}(q)^\top\mathbf{Y}_{i,k_2}(q))^2\right\} \\
&= \frac{1}{(P_{n_i}^2)^2}\sum_{j,k}^{*}\mathrm{E}\left(\mathbf{Y}_{i,j}(q)^\top\mathbf{Y}_{i,k}(q)\right)^4 \\
&\quad + \frac{2}{(P_{n_i}^2)^2}\sum_{j_1,j_2,k}^{*}\mathrm{E}\left(\mathbf{Y}_{i,k}(q)^\top\mathbf{Y}_{i,j_1}(q)\right)^2\left(\mathbf{Y}_{i,k}(q)^\top\mathbf{Y}_{i,j_2}(q)\right)^2 \\
&\quad + \frac{1}{(P_{n_i}^2)^2}\sum_{j_1,j_2,k_1,k_2}^{*}\mathrm{E}\left(\mathbf{Y}_{i,j_1}(q)^\top\mathbf{Y}_{i,k_1}(q)\right)^2\left(\mathbf{Y}_{i,j_2}(q)^\top\mathbf{Y}_{i,k_2}(q)\right)^2 \\
&\hat{=} \mathrm{T}_{i,1} + 2\mathrm{T}_{i,2} + \mathrm{T}_{i,3}, \text{ say,}
\end{aligned}$$

where

$$\mathrm{T}_{i,1} = \frac{1}{P_{n_i}^2}\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\mathrm{E}^2\left(\hat{\mathrm{Y}}_{i,j}^{s_1,s_1+q}\hat{\mathrm{Y}}_{i,j}^{s_2,s_2+q}\hat{\mathrm{Y}}_{i,j}^{s_3,s_3+q}\hat{\mathrm{Y}}_{i,j}^{s_4,s_4+q}\right),$$

$$\begin{aligned}
\mathrm{T}_{i,2} &= \frac{P_{n_i}^3}{(P_{n_i}^2)^2}\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\mathrm{E}\left(\hat{\mathrm{Y}}_{i,j}^{s_1,s_1+q}\hat{\mathrm{Y}}_{i,j}^{s_2,s_2+q}\hat{\mathrm{Y}}_{i,j}^{s_3,s_3+q}\hat{\mathrm{Y}}_{i,j}^{s_4,s_4+q}\right) \\
&\qquad\qquad \times\mathrm{E}\left(\hat{\mathrm{Y}}_{i,j}^{s_1,s_1+q}\hat{\mathrm{Y}}_{i,j}^{s_2,s_2+q}\right)\mathrm{E}\left(\hat{\mathrm{Y}}_{i,j}^{s_3,s_3+q}\hat{\mathrm{Y}}_{i,j}^{s_4,s_4+q}\right) \\
&\le \frac{C_8}{n_i}\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\mathrm{E}\left(\hat{\mathrm{Y}}_{i,j}^{s_1,s_1+q}\hat{\mathrm{Y}}_{i,j}^{s_2,s_2+q}\hat{\mathrm{Y}}_{i,j}^{s_3,s_3+q}\hat{\mathrm{Y}}_{i,j}^{s_4,s_4+q}\right). \qquad \text{(S2.12)}
\end{aligned}$$

and

$$\mathrm{T}_{i,3} = \frac{P_{n_i}^4}{(P_{n_i}^2)^2}\left[\sum_{s_1,s_2=1}^{p-q}\mathrm{E}^2\left(\hat{\mathrm{Y}}_{i,j}^{s_1,s_1+q}\hat{\mathrm{Y}}_{i,j}^{s_2,s_2+q}\right)\right]^2$$

Since $\mathrm{T}_{i,3}$ is a smaller order of $(p-q)^2$ after subtracting $\mathrm{E}^2(\mathrm{R}_{i,nq})$, we will focus on analyzing $\mathrm{T}_{i,2}$. The derivation of $\mathrm{T}_{i,1}$ is exactly the same without using the Cauchy-Schwartz inequality as the proof of Lemma 3.

Substituting (S2.11) into (S2.12), we have

$$\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\mathrm{E}\left(\hat{\mathrm{Y}}_{i,j}^{s_1,s_1+q}\hat{\mathrm{Y}}_{i,j}^{s_2,s_2+q}\hat{\mathrm{Y}}_{i,j}^{s_3,s_3+q}\hat{\mathrm{Y}}_{i,j}^{s_4,s_4+q}\right) = \mathrm{C}_{i,q,a} - \sum_{j=1}^{4}\mathrm{C}_{i,q,bj} + \sum_{j=1}^{6}\mathrm{C}_{i,q,cj} + \mathrm{C}_{i,q,d}. \qquad \text{(S2.13)}$$

where

$$\mathrm{C}_{i,q,a} = \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \mathrm{E}\left(\mathrm{Y}_{i,j}^{s_1,s_1+q}\mathrm{Y}_{i,j}^{s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right),$$

$$\begin{aligned}
\mathrm{C}_{i,q,b1} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} (\sigma_{i,s_1,s_1+q} - \hat{\sigma}_{i,s_1,s_1+q})\mathrm{E}\left(\mathrm{Y}_{i,j}^{s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right),\\
\mathrm{C}_{i,q,b2} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} (\sigma_{i,s_2,s_2+q} - \hat{\sigma}_{i,s_2,s_2+q})\mathrm{E}\left(\mathrm{Y}_{i,j}^{s_1,s_1+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right),\\
\mathrm{C}_{i,q,b3} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} (\sigma_{i,s_3,s_3+q} - \hat{\sigma}_{i,s_3,s_3+q})\mathrm{E}\left(\mathrm{Y}_{i,j}^{s_1,s_1+q}\mathrm{Y}_{i,j}^{s_2,s_2+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right),\\
\mathrm{C}_{i,q,b4} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} (\sigma_{i,s_3,s_3+q} - \hat{\sigma}_{i,s_4,s_4+q})\mathrm{E}\left(\mathrm{Y}_{i,j}^{s_1,s_1+q}\mathrm{Y}_{i,j}^{s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\right),
\end{aligned}$$

$$\begin{aligned}
\mathrm{C}_{i,q,c1} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} (\sigma_{i,s_1,s_1+q} - \hat{\sigma}_{i,s_1,s_1+q})(\sigma_{i,s_2,s_2+q} - \hat{\sigma}_{i,s_2,s_2+q})\mathrm{E}\left(\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right),\\
\mathrm{C}_{i,q,c2} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} (\sigma_{i,s_1,s_1+q} - \hat{\sigma}_{i,s_1,s_1+q})(\sigma_{i,s_3,s_3+q} - \hat{\sigma}_{i,s_3,s_3+q})\mathrm{E}\left(\mathrm{Y}_{i,j}^{s_2,s_2+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right),\\
\mathrm{C}_{i,q,c3} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} (\sigma_{i,s_1,s_1+q} - \hat{\sigma}_{i,s_1,s_1+q})(\sigma_{i,s_4,s_4+q} - \hat{\sigma}_{i,s_4,s_4+q})\mathrm{E}\left(\mathrm{Y}_{i,j}^{s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\right),\\
\mathrm{C}_{i,q,c4} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} (\sigma_{i,s_2,s_2+q} - \hat{\sigma}_{i,s_2,s_2+q})(\sigma_{i,s_3,s_3+q} - \hat{\sigma}_{i,s_3,s_3+q})\mathrm{E}\left(\mathrm{Y}_{i,j}^{s_1,s_1+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right),\\
\mathrm{C}_{i,q,c5} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} (\sigma_{i,s_2,s_2+q} - \hat{\sigma}_{i,s_2,s_2+q})(\sigma_{i,s_4,s_4+q} - \hat{\sigma}_{i,s_4,s_4+q})\mathrm{E}\left(\mathrm{Y}_{i,j}^{s_1,s_1+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\right),\\
\mathrm{C}_{i,q,c6} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} (\sigma_{i,s_3,s_3+q} - \hat{\sigma}_{i,s_3,s_3+q})(\sigma_{i,s_4,s_4+q} - \hat{\sigma}_{i,s_4,s_4+q})\mathrm{E}\left(\mathrm{Y}_{i,j}^{s_1,s_1+q}\mathrm{Y}_{i,j}^{s_2,s_2+q}\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{C}_{h,q,d} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q} (\sigma_{i,s_1,s_1+q} - \hat{\sigma}_{i,s_1,s_1+q})(\sigma_{i,s_2,s_2+q} - \hat{\sigma}_{i,s_2,s_2+q})\\
&\qquad \times(\sigma_{i,s_3,s_3+q} - \hat{\sigma}_{i,s_3,s_3+q})(\sigma_{i,s_4,s_4+q} - \hat{\sigma}_{i,s_4,s_4+q}).
\end{aligned}$$

It has been shown that $\mathrm{C}_{i,q,a} = O\left((p-q)^2\right)$ is $O\left((p-q)^2\right)$ in Lemma 3. According to the definition of $\hat{\sigma}_{h,s,s+q}$ in (S2.8), $\mathrm{C}_{i,q,b1}$ can be rewritten as

$$\begin{aligned}
\mathrm{C}_{i,q,b1} &= \frac{1}{n_i}\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\sigma_{i,s_1,s_1+q}\mathrm{E}\left(\mathrm{Y}_{i,j}^{s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right)\\
&\quad -\frac{1}{n_i}\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\mathrm{E}\left(\mathrm{X}_{i,j,s_1}\mathrm{X}_{i,j,s_1+q}\mathrm{Y}_{i,j}^{s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right)\\
&= \frac{1}{n_i}\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\Big[\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&\quad +\sigma_{i,s_1s_1+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&\quad +\sigma_{i,s_2s_2+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&\quad +\sigma_{i,s_3s_3+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&\quad +\sigma_{i,s_4s_4+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\right)\\
&\quad -\sigma_{i,s_1s_1+q}\sigma_{i,s_2,s_2+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&\quad -\sigma_{i,s_1s_1+q}\sigma_{i,s_3,s_3+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&\quad -\sigma_{i,s_1s_1+q}\sigma_{i,s_4,s_4+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\right)\\
&\quad -\sigma_{i,s_2s_2+q}\sigma_{i,s_3,s_3+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&\quad -\sigma_{i,s_2s_2+q}\sigma_{i,s_4,s_4+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\right)\\
&\quad -\sigma_{i,s_3s_3+q}\sigma_{i,s_4,s_4+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\right)\\
&\quad -\sigma_{i,s_1s_1+q}\sigma_{i,s_3,s_3+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&\quad +3\sigma_{i,s_1s_1+q}\sigma_{i,s_2,s_2+q}\sigma_{i,s_3,s_3+q}\sigma_{i,s_4,s_4+q}\Big].
\end{aligned}$$

We derive each term above as we did in the proof of Lemma 3 and thus $\mathrm{C}_{i,q,b1} = O\left((p-q)^2\right)$. Analogous derivation shows that $\mathrm{C}_{i,q,bj}$, $j = 2, 3, 4$, is exactly the same with $\mathrm{C}_{i,q,b1}$. And $\mathrm{C}_{i,q,c1}$ can be derived in the same way because

$$\begin{aligned}
\mathrm{C}_{i,q,c1} &= \sum_{s_1,s_2,s_3,s_4=1}^{p-q}\Big[\sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}\mathrm{E}\left(\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right)\\
&\quad -\sigma_{i,s_1,s_1+q}\mathrm{E}\left(\hat{\sigma}_{i,s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right)\\
&\quad -\sigma_{i,s_2,s_2+q}\mathrm{E}\left(\hat{\sigma}_{i,s_1,s_1+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right)\\
&\quad +\mathrm{E}\left(\hat{\sigma}_{i,s_1,s_1+q}\hat{\sigma}_{i,s_2,s_2+q}\mathrm{Y}_{i,j}^{i,s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right)\Big]. \qquad \text{(S2.14)}
\end{aligned}$$

We expand each term in (S2.14) and write the explicit form of $\mathrm{C}_{i,q,c1}$ as follows:

$$
\begin{aligned}
&\sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1s_1+q}\mathrm{E}\left(\hat{\sigma}_{i,s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right)\\
= \;& \frac{1}{n_i}\sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_1+q}\mathrm{E}\left(\mathrm{X}_{i,j,s_2}\mathrm{X}_{i,j,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right)\\
&+\frac{n_i-1}{n_i}\sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}\mathrm{E}\left(\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right)\\
= \;& \sum_{s_1,s_2,s_3,s_4=1}^{p-q}\Bigg[\frac{1}{n_i}\sigma_{i,s_1,s_1+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&\qquad -\frac{1}{n_i}\sigma_{i,s_1,s_1+q}\sigma_{i,s_3,s_3+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&\qquad -\frac{1}{n_i}\sigma_{i,s_1,s_1+q}\sigma_{i,s_4,s_4+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\right)\\
&\qquad +\frac{n_i-1}{n_i}\sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&\qquad -\frac{n_i-2}{n_i}\sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}\sigma_{i,s_3,s_3+q}\sigma_{i,s_4,s_4+q}\Bigg],
\end{aligned}
$$

and $\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\sigma_{i,s_2,s_2+q}\mathrm{E}\left(\hat{\sigma}_{i,s_1,s_1+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right)$ can be derived in the same way. And

$$
\begin{aligned}
&\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\mathrm{E}\left(\hat{\sigma}_{i,s_1,s_1+q}\hat{\sigma}_{i,s_2,s_2+q}\mathrm{Y}_{i,j}^{s_3,s_3+q}\mathrm{Y}_{i,j}^{s_4,s_4+q}\right) \qquad \text{(S2.15)}\\
= \;& \sum_{s_1,s_2,s_3,s_4=1}^{p-q}\Big[\mathrm{E}\left(\hat{\sigma}_{i,s_1,s_1+q}\hat{\sigma}_{i,s_2,s_2+q}\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}\right)\\
&\qquad +\sigma_{i,s_3,s_3+q}\sigma_{i,s_4,s_4+q}\mathrm{E}\left(\hat{\sigma}_{i,s_1,s_1+q}\hat{\sigma}_{i,s_2,s_2+q}\right)\\
&\qquad -\sigma_{i,s_3,s_3+q}\mathrm{E}\left(\hat{\sigma}_{i,s_1,s_1+q}\hat{\sigma}_{i,s_2,s_2+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}\right)\\
&\qquad -\sigma_{i,s_4,s_4+q}\mathrm{E}\left(\hat{\sigma}_{i,s_1,s_1+q}\hat{\sigma}_{i,s_2,s_2+q}\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\right)\Big]
\end{aligned}
$$

where the first term in (S2.15) is

$$
\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\mathrm{E}(\hat{\sigma}_{i,s_1,s_1+q}\hat{\sigma}_{i,s_2,s_2+q}\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}) = (\mathrm{H}_{1,q}+\mathrm{H}_{2,q}+\mathrm{H}_{3,q}+\mathrm{H}_{4,q}),
$$

with

$$\begin{aligned}
\mathrm{H}_{1,q} &= \frac{1}{n_i^2}\sum_{s_1,\ldots,s_4=1}^{p-q}\sum_{j_1=1}^{n_i}\sum_{j_2=1}^{n_i}\mathrm{E}(\mathrm{X}_{i,j_1,s_1}\mathrm{X}_{i,j_1,s_1+q}\mathrm{X}_{i,j_2,s_2}\mathrm{X}_{i,j_2,s_2+q}\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}),\\
\mathrm{H}_{2,q} &= \frac{1}{n_iP_{n_i}^2}\sum_{s_1,\ldots,s_4=1}^{p-q}\sum_{j_1,k_1}^{*}\sum_{j_2=1}^{n_i}\mathrm{E}(\mathrm{X}_{i,j_1,s_1}\mathrm{X}_{i,k_1,s_1+q}\mathrm{X}_{i,j_2,s_2}\mathrm{X}_{i,j_2,s_2+q}\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}),\\
\mathrm{H}_{3,q} &= \frac{1}{n_iP_{n_i}^2}\sum_{s_1,\ldots,s_4=1}^{p-q}\sum_{j_1=1}^{n_i}\sum_{j_2,k_2}^{*}\mathrm{E}(\mathrm{X}_{i,j_1,s_1}\mathrm{X}_{i,j_1,s_1+q}\mathrm{X}_{i,j_2,s_2}\mathrm{X}_{i,k_2,s_2+q}\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}),
\end{aligned}$$

and

$$\mathrm{H}_{4,q} = \frac{1}{(P_{n_i}^2)^2}\sum_{s_1,\ldots,s_4=1}^{p-q}\sum_{j_1,k_1}^{*}\sum_{j_2,k_2}^{*}\mathrm{E}(\mathrm{X}_{i,j_1,s_1}\mathrm{X}_{i,k_1,s_1+q}\mathrm{X}_{i,j_2,s_2}\mathrm{X}_{i,k_2,s_2+q}\mathrm{X}_{i,j,s_3}\mathrm{X}_{i,j,s_3+q}\mathrm{X}_{i,j,s_4}\mathrm{X}_{i,j,s_4+q}).$$

Derivation shows that

$$\begin{aligned}
\mathrm{H}_{1,q} = \sum_{s_1,\ldots,s_4=1}^{p-q}\Bigg[&\frac{1}{n_i^2}\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&+\frac{n_i-1}{n_i^2}\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\right)\mathrm{E}\left(\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&+\frac{n_i-1}{n_i^2}\sigma_{i,s_1,s_1+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&+\frac{n_i-1}{n_i^2}\sigma_{i,s_2,s_2+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&+\frac{P_{n_i-1}^2}{n_i^2}\sigma_{i,s_1,s_1+q}\sigma_{i,s_2,s_2+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,ks_4+q}\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
\mathrm{H}_{2,q} = \frac{1}{n_i^2}\sum_{s_1,\ldots,s_4=1}^{p-q}\Big[&\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\right)\mathrm{E}\left(\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&+\mathrm{E}\left(\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_2+q}\right)\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\Big],
\end{aligned}$$

$$\begin{aligned}
\mathrm{H}_{3,q} = \frac{1}{n_i^2}\sum_{s_1,\ldots,s_4=1}^{p-q}\Big[&\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_2}\right)\mathrm{E}\left(\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\\
&+\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_2+q}\right)\mathrm{E}\left(\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\Big],
\end{aligned}$$

and

$$
\begin{aligned}
\mathrm{H}_{4,q} &= \frac{1}{n_i^2} \sum_{s_1,\ldots,s_4=1}^{p-q} \Big[(n_i-1)\sigma_{i,s_1+q,s_2+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right) \\
&\quad +(n_i-1)\sigma_{i,s_1,s_2}\mathrm{E}\left(\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right) \\
&\quad +(n_i-1)\sigma_{i,s_1,s_2+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_1+q}\mathrm{X}_{i,k,s_2}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right) \\
&\quad +(n_i-1)\sigma_{i,s_1+q,s_2}\mathrm{E}\left(\mathrm{X}_{i,k,s_1}\mathrm{X}_{i,k,s_2+q}\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right) \\
&\quad +P_{n_i-1}^2\sigma_{i,s_1,s_2}\sigma_{i,s_1+q,s_2+q}\mathrm{E}\left(\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right) \\
&\quad +P_{n_i-1}^2\sigma_{i,s_1,s_2+q}\sigma_{i,s_1+q,s_2}\mathrm{E}\left(\mathrm{X}_{i,k,s_3}\mathrm{X}_{i,k,s_3+q}\mathrm{X}_{i,k,s_4}\mathrm{X}_{i,k,s_4+q}\right)\Big]
\end{aligned}
$$

The other three terms in (S2.15) can be derived in the same way. Taking all the terms back into (S2.14), we have the exact form of $\mathrm{C}_{i,q,c1}$. Similarly, we can derive the explicit form of other terms $\mathrm{C}_{i,q,cj}$, $j=1,\ldots,6$, and $\mathrm{C}_{i,q,d}$. It is shown that all the terms involved with $\sigma_{i,s,s+q}$ can be canceled out, as shown in the proof of Lemma 3 and the rest of the terms have been shown to be $O\left((p-q)^2\right)$. Hence, $\mathrm{T}_{i,2}$ is $O\left((p-q)^2\right)$. The ratio consistency of $\mathrm{R}_{c,nq}$ can also be shown in the same way. This completes the proof of Proposition 2.

□

Proof of Theorem 1. It has been shown that the leading order term of $\hat{\mathrm{S}}_{nq}$ is $\hat{\mathrm{S}}_{nq}^{(1)} = \mathrm{A}_{1,nq}^{(1)} + \mathrm{A}_{2,nq}^{(1)} - 2\mathrm{A}_{c,nq}^{(1)}$ in the proof of Proposition 1 in the supplementary document. Meanwhile, we have shown that $\mathrm{E}\left(\hat{\mathrm{S}}_{nq}^{(1)}\right) = \mathrm{S}_q$ and $\mathrm{Var}\left(\hat{\mathrm{S}}_{nq}^{(1)}\right) = V_{nq}^2$. For the asymptotic normality of $\hat{\mathrm{S}}_{nq}$, it is sufficient to prove that,

$$V_{nq}^{-1}(\hat{\mathrm{S}}_{nq}^{(1)} - \mathrm{S}_q) \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as} \quad n,p\to\infty. \tag{S2.16}$$

We follow the martingale central limit theorem (Hall and Heyde, 1980), to establish (S2.16). Define a sequence of new random vectors $\boldsymbol{\xi}_i$ such that

$$
\begin{aligned}
\boldsymbol{\xi}_j &= \mathbf{X}_{1,j}, \quad \text{for} \quad j=1,2,\ldots,n_1; \\
\boldsymbol{\xi}_{n_1+k} &= \mathbf{X}_{2,k}, \quad \text{for} \quad k=1,2,\ldots,n_2.
\end{aligned}
$$

Let $\mathcal{F}_0 = \{\emptyset,\Omega\}$ and $\mathcal{F}_t = \sigma\{\boldsymbol{\xi}_1,\cdots,\boldsymbol{\xi}_t\}$ be the σ-field generated by $\{\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_t\}$ for $t=1,\ldots,n_1+n_2$. Denote the conditional expectation with respective to $\mathcal{F}_t$ by $\mathrm{E}_t(\cdot)$ and $\mathrm{E}_0(\cdot)=\mathrm{E}(\cdot)$. Let $\mathrm{G}_{q,t} = \mathrm{E}_t\left(\hat{\mathrm{S}}_{nq}^{(1)}\right) - \mathrm{E}_{t-1}\left(\hat{\mathrm{S}}_{nq}^{(1)}\right)$ and $\nu_{q,t}^2 = \mathrm{E}_{t-1}\left(\mathrm{G}_{q,t}^2\right)$, $t=1,\ldots,n_1+n_2$. Then for every q, $\{(\mathrm{G}_{q,t},\mathcal{F}_t) : t=1,\cdots,n_1+n_2\}$ forms a martingale difference array and $\hat{\mathrm{S}}_{nq}^{(1)} - \mathrm{E}(\hat{\mathrm{S}}_{nq}^{(1)}) = \sum_{t=1}^{n_1+n_2}\mathrm{G}_{q,t}$.

It suffices to show that for any $\varepsilon > 0$, as n_1, n_2 and $p \to \infty$, (see page 58, Theorem 3.2, Hall and Heyde (1980)),

$$V_{nq}^{-2} \sum_{t=1}^{n_1+n_2} \nu_{q,t}^2 \xrightarrow{p} 1, \quad \text{and} \tag{S2.17}$$

$$V_{nq}^{-4} \sum_{t=1}^{n_1+n_2} \mathrm{E}\left(\mathrm{G}_{q,t}^4\right) \to 0. \tag{S2.18}$$

For $t = 1, \ldots, n_1$, $\mathrm{G}_{q,t}$ can be written in the following explicit form:

$$\begin{aligned}
\mathrm{G}_{q,t} &= (\mathrm{E}_t - \mathrm{E}_{t-1})\, \mathrm{A}_{1,nq}^{(1)} - 2\,(\mathrm{E}_t - \mathrm{E}_{t-1})\, \mathrm{A}_{c,nq}^{(1)} \\
&= \frac{2}{n_1(n_1-1)} \sum_{j=1, j\neq t}^{n_1} \sum_{s=1}^{p-q} (\mathrm{E}_t - \mathrm{E}_{t-1})\, (\xi_{i,s}\xi_{i,s+q})(\xi_{t,s}\xi_{t,s+q}) \qquad \text{(S2.19)} \\
&\quad - \frac{2}{n_1 n_2} \sum_{k=1}^{n_2} \sum_{s=1}^{p-q} (\mathrm{E}_t - \mathrm{E}_{t-1})\, (\xi_{t,s}\xi_{t,s+q})(\xi_{n_1+k,s}\xi_{n_1+k,s+q}) \\
&= \frac{2}{n_1} \sum_{s=1}^{p-q} (\sigma_{1,s,s+q} - \sigma_{2,s,s+q}) \mathrm{Y}_{1,t}^{s,s+q} + \frac{2}{n_1(n_1-1)} \sum_{s=1}^{p-q} \mathrm{Y}_{1,t}^{s,s+q} \mathrm{Q}_{1,t-1}^{s,s+q} \qquad \text{(S2.20)}
\end{aligned}$$

where $\mathrm{Q}_{1,t}^{s_1,s_2} = \sum_{j=1}^{t} \mathrm{Y}_{1,j}^{s_1,s_2}$ for any integer $t \in \{1, \ldots, n_1\}$ and (S2.19) is valid since for $j \neq \ell$ except when $j = t$ or $\ell = t$, $(\mathrm{E}_t - \mathrm{E}_{t-1})\, (\mathrm{X}_{1,j,s}\mathrm{X}_{1,j,s+q})(\mathrm{X}_{1,\ell,s}\mathrm{X}_{1,\ell,s+q}) = 0$.

For $t = n_1 + 1, \ldots, n_1 + n_2$, we have

$$\begin{aligned}
\mathrm{G}_{q,t} &= (\mathrm{E}_t - \mathrm{E}_{t-1})\, \mathrm{A}_{2,nq}^{(1)} - 2\,(\mathrm{E}_t - \mathrm{E}_{t-1})\, \mathrm{A}_{c,nq}^{(1)} \\
&= \frac{2}{n_2(n_2-1)} \sum_{k=n_1+1, k\neq t}^{n_1+n_2} \sum_{s=1}^{p-q} (\mathrm{E}_t - \mathrm{E}_{t-1})\, (\xi_{k,s}\xi_{t,s+q})(\xi_{t,s}\xi_{t,s+q}) \\
&\quad - \frac{2}{n_1 n_2} \sum_{j=1}^{n_1} \sum_{s=1}^{p-q} (\mathrm{E}_t - \mathrm{E}_{t-1})\, (\xi_{j,s}\xi_{j,s+q})(\xi_{t,s}\xi_{t,s+q}) \\
&= \frac{2}{n_2} \sum_{s=1}^{p-q} (\sigma_{2,s,s+q} - \sigma_{1,s,s+q}) \mathrm{Y}_{2,t-n_1}^{s,s+q} + \frac{2}{n_2(n_2-1)} \sum_{s=1}^{p-q} \mathrm{Y}_{2,t-n_1}^{s,s+q} \mathrm{Q}_{2,t-n_1-1}^{s,s+q} \\
&\quad - \frac{2}{n_1 n_2} \sum_{s=1}^{p-q} \mathrm{Y}_{2,t-n_1}^{s,s+q} \mathrm{Q}_{1,n_1}^{s,s+q}, \qquad \text{(S2.21)}
\end{aligned}$$

where $\mathrm{Q}_{2,t}^{s_1,s_2} = \sum_{k=1}^{t} \mathrm{Y}_{2,k}^{s_1,s_2}$ for any integer $t \in \{1, \ldots, n_2\}$.

Under these new notations, invoking the definition that $\nu_{q,t}^2 = \mathrm{E}\left(\mathrm{G}_{q,t}^2 | \mathcal{F}_{t-1}\right)$, we have, for $t = 1, \ldots, n_1$,

$$\begin{aligned}
\nu_{q,t}^2 &= \frac{4}{n_1^2} \sum_{s_1,s_2=1}^{p-q} (\sigma_{1,s_1,s_1+q} - \sigma_{2,s_1,s_1+q})(\sigma_{1,s_2,s_2+q} - \sigma_{2,s_2,s_2+q})\omega_{1,q}^{s_1,s_2} \\
&\quad + \frac{4}{n_1^2(n_1-1)^2} \sum_{s_1,s_2=1}^{p-q} \omega_{1,q}^{s_1,s_2} \mathrm{Q}_{1,t-1}^{s_1,s_1+q} \mathrm{Q}_{1,t-1}^{s_2,s_2+q} \\
&\quad + \frac{4}{n_1^2(n_1-1)} \sum_{s_1,s_2=1}^{p-q} (\sigma_{1,s_1,s_1+q} - \sigma_{2,s_1,s_1+q})\omega_{1,q}^{s_1,s_2} \mathrm{Q}_{1,t-1}^{s_2,s_2+q}, \qquad \text{(S2.22)}
\end{aligned}$$

and for $t = n_1 + 1, \ldots, n_1 + n_2$,

$$\begin{aligned}
\nu_{q,t}^2 &= \frac{4}{n_2^2} \sum_{s_1,s_2=1}^{p-q} (\sigma_{2,s_1,s_1+q} - \sigma_{1,s_1,s_1+q})(\sigma_{2,s_2,s_2+q} - \sigma_{1,s_2,s_2+q})\omega_{2,q}^{s_1,s_2} \\
&\quad + \frac{4}{n_2^2(n_2-1)^2} \sum_{s_1,s_2=1}^{p-q} \omega_{2,q}^{s_1,s_2} \mathrm{Q}_{2,(t-n_1)-1}^{s_1,s_1+q} \mathrm{Q}_{2,(t-n_1)-1}^{s_2,s_2+q} \\
&\quad + \frac{4}{n_2^2(n_2-1)} \sum_{s_1,s_2=1}^{p-q} (\sigma_{2,s_1,s_1+q} - \sigma_{1,s_1,s_1+q})\omega_{2,q}^{s_1,s_2} \mathrm{Q}_{2,(t-n_1)-1}^{s_2,s_2+q} \\
&\quad + \frac{4}{n_1^2 n_2^2} \sum_{s_1,s_2=1}^{p-q} \omega_{2,q}^{s_1,s_2} \mathrm{Q}_{1,n_1}^{s_1,s_1+q} \mathrm{Q}_{1,n_1}^{s_2,s_2+q} \\
&\quad - \frac{4}{n_1 n_2^2} \sum_{s_1,s_2=1}^{p-q} (\sigma_{2,s_1,s_1+q} - \sigma_{1,s_1,s_1+q})\omega_{2,q}^{s_1,s_2} \mathrm{Q}_{1,n_1}^{s_2,s_2+q} \\
&\quad - \frac{4}{n_1 n_2^2(n_2-1)} \sum_{s_1,s_2=1}^{p-q} \omega_{2,q}^{s_1,s_2} \mathrm{Q}_{2,(t-n_1)-1}^{s_1,s_1+q} \mathrm{Q}_{1,n_1}^{s_2,s_2+q}. \qquad \text{(S2.23)}
\end{aligned}$$

Using the explicit forms of $\mathrm{G}_{q,t}$ and $\nu_{q,t}^2$, we now derive (S2.17) and (S2.18). According to (S2.22) and (S2.23), we have

$$\sum_{t=1}^{n_1+n_2} \nu_{q,t}^2 = \sum_{i=1}^{9} I_i,$$

where

$$\begin{aligned}
I_1 &= \frac{4}{n_1} \sum_{s_1,s_2=1}^{p-q} (\sigma_{1,s_1,s_1+q} - \sigma_{2,s_1,s_1+q})(\sigma_{1,s_2,s_2+q} - \sigma_{2,s_2,s_2+q})\omega_{1,q}^{s_1,s_2}, \\
I_2 &= \frac{4}{n_2} \sum_{s_1,s_2=1}^{p-q} (\sigma_{2,s_1,s_1+q} - \sigma_{1,s_1,s_1+q})(\sigma_{2,s_2,s_2+q} - \sigma_{1,s_2,s_2+q})\omega_{2,q}^{s_1,s_2},
\end{aligned}$$

$$
\begin{aligned}
I_3 &= \frac{4}{n_1^2(n_1-1)^2}\sum_{s_1,s_2=1}^{p-q}\omega_{1,q}^{s_1,s_2}\left[\sum_{j=1}^{n_1}\mathrm{Q}_{1,j-1}^{s_1,s_1+q}\mathrm{Q}_{1,j-1}^{s_2,s_2+q}\right],\\
I_4 &= \frac{4}{n_2^2(n_2-1)^2}\sum_{s_1,s_2=1}^{p-q}\omega_{2,q}^{s_1,s_2}\left[\sum_{k=1}^{n_2}\mathrm{Q}_{2,k-1}^{s_1,s_1+q}\mathrm{Q}_{2,k-1}^{s_2,s_2+q}\right],\\
I_5 &= \frac{4}{n_1^2(n_1-1)}\sum_{s_1,s_2=1}^{p-q}(\sigma_{1,s_1,s_1+q}-\sigma_{2,s_1,s_1+q})\omega_{1,q}^{s_1,s_2}\left[\sum_{j=1}^{n_1}\mathrm{Q}_{1,j-1}^{s_2,s_2+q}\right],\\
I_6 &= \frac{4}{n_2^2(n_2-1)}\sum_{s_1,s_2=1}^{p-q}(\sigma_{2,s_1,s_1+q}-\sigma_{1,s_1,s_1+q})\omega_{2,q}^{s_1,s_2}\left[\sum_{k=1}^{n_2}\mathrm{Q}_{2,k-1}^{s_2,s_2+q}\right],\\
I_7 &= \frac{4}{n_1^2 n_2}\sum_{s_1,s_2=1}^{p-q}\omega_{2,q}^{s_1,s_2}\mathrm{Q}_{1,n_1}^{s_1,s_1+q}\mathrm{Q}_{1,n_1}^{s_2,s_2+q},\\
I_8 &= -\frac{4}{n_1 n_2}\sum_{s_1,s_2=1}^{p-q}(\sigma_{2,s_1,s_1+q}-\sigma_{1,s_1,s_1+q})\omega_{2,q}^{s_1,s_2}\mathrm{Q}_{1,n_1}^{s_2,s_2+q},\\
I_9 &= -\frac{4}{n_1 n_2^2(n_2-1)}\sum_{s_1,s_2=1}^{p-q}\omega_{2,q}^{s_1,s_2}\mathrm{Q}_{1,n_1}^{s_2,s_2+q}\left[\sum_{k=1}^{n_2}\mathrm{Q}_{2,k-1}^{s_1,s_1+q}\right].
\end{aligned}
$$

By taking expectation of each I_i, $i=1,\ldots,9$, it can be readily shown that $\mathrm{E}\left(\sum_{t=1}^{n_1+n_2}\nu_{q,t}^2\right)=V_{nq}^2$. It suffices to show that the variance of $\sum_{t=1}^{n_1+n_2}\nu_{q,t}^2$ is at a smaller order of V_{nq}^4. The order of V_{nq}^2 is specified under different regimes of q as follows (see details in the proof of Proposition 1 in the supplementary material):

$$
V_{nq}^2 \asymp \begin{cases} \frac{(p-q)q^{-\beta}}{n}, & \text{for } q=o(n^{1/\beta}),\\ \frac{(p-q)q^{-\beta}}{n}\asymp\frac{p-q}{n^2} & \text{for } q \text{ such that } q^{-\beta}\asymp 1/n,\\ \frac{p-q}{n^2}, & \text{for } q \text{ such that } n^{1/\beta}=o(q) \text{ and } q=o(p).\end{cases} \tag{S2.24}
$$

We first calculate the variances of I_i, $i=3,\ldots,9$, since I_1 and I_2 are deterministic. The variances of I_3 and I_4 can be derived in the same way and we take I_3 as an example. Let

$$
V_{j,k}=\sum_{s_1,s_2=1}^{p-q}\omega_{1,q}^{s_1,s_2}\left(\mathrm{Y}_{1,j}^{s_1,s_1+q}\mathrm{Y}_{1,j}^{s_2,s_2+q}-\omega_{1,q}^{s_1,s_2}\mathrm{I}(i=j)\right).
$$

Notice that $\mathrm{E}(I_3)=\frac{2}{n_1(n_1-1)}\sum_{s_1,s_2=1}^{p-q}(\omega_{1,q}^{s_1,s_2})^2$ and we have

$$
I_3-\mathrm{E}(I_3)=\sum_{j,k=1}^{n_1-1}\frac{4(n_1-\max\{j,k\})}{n_1^2(n_1-1)^2}V_{j,k}.
$$

Thus,

$$\begin{aligned}\mathrm{Var}\,(I_3) &= \sum_{j=1}^{n_1}\frac{16(n_1-j)^2}{n_1^4(n_1-1)^4}\mathrm{Var}(V_{j,j})+\sum_{j,k}^{*}\frac{16(n_1-\max\{j,k\})^2}{n_1^4(n_1-1)^4}\mathrm{Var}(V_{j,k})\\&\quad+\sum_{j,k}^{*}\frac{16(n_1-\max\{j,k\})^2}{n_1^4(n_1-1)^4}\mathrm{Cov}(V_{j,k},V_{j,k}).\end{aligned}\tag{S2.25}$$

According to Lemma 2, for $j=1,\ldots,n_1$, we have

$$\begin{aligned}\mathrm{Var}(V_{j,j}) &= \sum_{s_1,s_2=1}^{p-q}\sum_{s_3,s_4=1}^{p-q}\omega_{1,q}^{s_1,s_2}\omega_{1,q}^{s_3,s_4}\mathrm{E}\left(\mathrm{Y}_{1,j}^{s_1,s_1+q}\mathrm{Y}_{1,j}^{s_2,s_2+q}\mathrm{Y}_{1,j}^{s_3,s_3+q}\mathrm{Y}_{1,j}^{s_4,s_4+q}\right)\\&\quad-\sum_{s_1,s_2=1}^{p-q}\sum_{s_3,s_4=1}^{p-q}(\omega_{1,q}^{s_1,s_2})^2(\omega_{1,q}^{s_3,s_4})^2\\&= O\left((p-q)^2\right).\end{aligned}$$

For $j\neq k$, we have $\mathrm{Var}(V_{j,k})=\mathrm{tr}(W_{1,q}^4)$ and $\mathrm{Cov}(V_{j,k},V_{k,j})=\mathrm{tr}(W_{1,q}^4)$, which are all $O(p-q)$. Thus, by (S2.25), we have

$$\mathrm{Var}(I_3)=O\left(\frac{(p-q)^2}{n_1^5}\right)+O\left(\frac{(p-q)}{n_1^4}\right).$$

Comparing with the leading order of V_{nq}^2 in (S2.24), we have $\mathrm{Var}(I_3)=o(V_{nq}^4)$ for any $q=o(p)$.

Similarly, the variances of I_5 and I_6 can be shown to be $o(V_{nq}^4)$. Specifically, I_5 can be written as

$$I_5=\sum_{k=1}^{n_1-1}\frac{8(n_1-k)}{n_1^2(n_1-1)}\mathrm{T}_k,$$

where $\mathrm{T}_k=\sum_{s_1,s_2=1}^{p-q}(\sigma_{1,s_1,s_1+q}-\sigma_{2,s_1,s_1+q})\omega_{1,q}^{s_1,s_2}\mathrm{Y}_{1,k}^{s_2,s_2+q}$. Likewise, for $k=1,\ldots,n_1$,

$$\mathrm{Var}(\mathrm{T}_k)=\frac{1}{(p-q)^2}J_q^TW_{1,q}^3J_q.$$

Thus, according to Lemma 2, it can be shown that $\mathrm{Var}(I_5)=o(V_{nq}^4)$ for any $q=o(p)$ in the three regimes in (S2.24).

Furthermore, according to Lemma 3, we have

$$\begin{aligned}\mathrm{Var}(I_7) &= \frac{16}{n_1^4n_2^2}\sum_{j=1}^{n_1}\sum_{s_1,\ldots,s_4=1}^{p-q}\omega_{2,q}^{s_1,s_2}\omega_{2,q}^{s_3,s_4}\mathrm{E}(\mathrm{Y}_{1,j}^{s_1,s_1+q}\mathrm{Y}_{1,j}^{s_2,s_2+q}\mathrm{Y}_{1,j}^{s_3,s_3+q}\mathrm{Y}_{1,j}^{s_4,s_4+q})\\&\quad+\frac{32P_{n_1}^2}{n_1^4n_2^2}\sum_{j,k}^{*}\sum_{s_1,\ldots,s_4=1}^{p-q}\omega_{2,q}^{s_1,s_2}\omega_{2,q}^{s_3,s_4}\omega_{1,q}^{s_1,s_3}\omega_{1,q}^{s_2,s_4}\end{aligned}$$

$$
\begin{aligned}
&\le \frac{16M_2^2}{n_1^3n_2^2}\sum_{s_1,\ldots,s_4=1}^{p-q}\mathrm{E}(\mathrm{Y}_{1,j}^{s_1,s_1+q}\mathrm{Y}_{1,j}^{s_2,s_2+q}\mathrm{Y}_{1,j}^{s_3,s_3+q}\mathrm{Y}_{1,j}^{s_4,s_4+q})\\
&\quad+\frac{32M_2^2(n_1-1)}{n_1^3n_2^2}\mathrm{tr}\left((W_{1,q}W_{2,q})^2\right).
\end{aligned}
$$

Comparing $\mathrm{Var}(I_7)$ with the order of V_{nq}^2 in (S2.24), we have $\mathrm{Var}(I_7)=o(V_{nq}^4)$ for any $q=o(p)$.

We also have

$$
\begin{aligned}
\mathrm{Var}(I_8) &= \frac{16}{n_1^2n_2^2}\sum_{i=1}^{n_1}\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\omega_{2,q}^{s_1,s_2}\omega_{2,q}^{s_3,s_4}\omega_{1,q}^{s_2,s_4}\\
&\qquad\times(\sigma_{2,s_1,s_1+q}-\sigma_{1,s_1,s_1+q})(\sigma_{2,s_3,s_3+q}-\sigma_{1,s_3,s_3+q})\\
&= \frac{16}{n_1n_2^2}J_q^\top(W_{2,q}W_{1,q}W_{2,q})J_q. \qquad \text{(S2.26)}
\end{aligned}
$$

Similar to the derivation for I_5, according to Lemma 2, $\mathrm{Var}(I_8)$ is shown to be at a smaller order of V_{nq}^4 for any $q=o(p)$ in the three regimes in (S2.24).

The last term in (S2.23) equals to

$$
\begin{aligned}
\mathrm{Var}(I_9) &= \frac{16}{n_1^2n_2^4(n_2-1)^2}\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\omega_{2,q}^{s_1,s_2}\omega_{2,q}^{s_3,s_4}\mathrm{E}(\mathrm{Q}_{1,n_1}^{s_2,s_2+q}\mathrm{Q}_{1,n_1}^{s_4,s_4+q})\\
&\qquad\times\sum_{k_1,k_2=1}^{n_2}\mathrm{E}(\mathrm{Q}_{2,k_1-1}^{s_1,s_1+q}\mathrm{Q}_{2,k_2-1}^{s_3,s_3+q})\\
&= \frac{16(2n_2-1)}{6n_1n_2^3(n_2-1)}\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\omega_{2,q}^{s_1,s_2}\omega_{2,q}^{s_3,s_4}\omega_{2,q}^{s_1,s_3}\omega_{1,q}^{s_2,s_4},
\end{aligned}
$$

which is also at a smaller order of V_{nq}^4 for any $q=o(p)$.

From the above derivations, the variances of I_i are all at a smaller order of V_{nq}^2. By the Cauchy-Schwartz inequality, the covariances between I_i are also at a smaller order of V_{nq}^4. Until now, we have proved (S2.17).

In order to establish (S2.18), according to the explicit form of $\mathrm{G}_{q,t}$ in (S2.20) and (S2.21), for some constant $\tilde{\gamma}$, we have

$$
\sum_{t=1}^{n_1+n_2}\mathrm{E}(G_{q,t}^4)\le\tilde{\gamma}\left[\sum_{t=1}^{n_1}\mathrm{E}(\mathrm{B}_{t1}^4)+\sum_{t=1}^{n_1}\mathrm{E}(\mathrm{B}_{t2}^4)+\sum_{t=1}^{n_2}\mathrm{E}(\mathrm{B}_{t3}^4)+\sum_{t=1}^{n_2}\mathrm{E}(\mathrm{B}_{t4}^4)+\sum_{t=1}^{n_2}\mathrm{E}(\mathrm{B}_{t5}^4)\right],
$$

where

$$
\mathrm{B}_{t1} = \frac{2}{n_1}\sum_{s=1}^{p-q}(\sigma_{1,s,s+q}-\sigma_{2,s,s+q})\mathrm{Y}_{1,t}^{s,s+q};
$$

$$\begin{aligned}
\mathrm{B}_{t2} &= \frac{2}{n_1(n_1-1)}\sum_{s=1}^{p-q}\mathrm{Y}_{1,t}^{s,s+q}\mathrm{Q}_{1,t-1}^{s,s+q};\\
\mathrm{B}_{t3} &= \frac{2}{n_2}\sum_{s=1}^{p-q}(\sigma_{2,s,s+q}-\sigma_{1,s,s+q})\mathrm{Y}_{2,t}^{s,s+q};\\
\mathrm{B}_{t4} &= \frac{2}{n_2(n_2-1)}\sum_{s=1}^{p-q}\mathrm{Y}_{2,t}^{s,s+q}\mathrm{Q}_{2,t-1}^{s,s+q};\\
\mathrm{B}_{t5} &= \frac{2}{n_1n_2}\sum_{s=1}^{p-q}\mathrm{Y}_{2,t}^{s,s+q}\mathrm{Q}_{1,n_1}^{s,s+q}.
\end{aligned}$$

In the following, we prove that the above five terms are of smaller orders of V_{nq}^4. Under Assumption (C3), we have

$$\sum_{t=1}^{n_1}\mathrm{E}(\mathrm{B}_{t1}^4)\le\frac{16cM}{n_1^3}(p-q)^2q^{-2\beta}.$$

In Regime (i) where $q=o(n^{1/\beta})$ and Regime (ii) where q satisfies that $q^{-\beta}\asymp 1/n$ in (S2.24), $V_{nq}^2\asymp(p-q)q^{-\beta}/n$, thus we have $\sum_{t=1}^{n_1}\mathrm{E}(\mathrm{B}_{t1}^4)=o(V_{nq}^4)$ in these two regimes. In Regime (iii) where q satisfies that $n^{1/\beta}=o(q)$ and $q=o(p)$, $V_{nq}^2\asymp(p-q)/n^2$ and we also have $\sum_{t=1}^{n_1}\mathrm{E}(\mathrm{B}_{t1}^4)=o(V_{nq}^4)$.

Similarly, according to Lemma 3,

$$\begin{aligned}
\sum_{t=1}^{n_1}\mathrm{E}(\mathrm{B}_{t2}^4) &\le \frac{16M}{(P_{n_1}^2)^4}\sum_{t=1}^{n_1}\sum_{s_1,\ldots,s_4=1}^{p-q}\mathrm{E}(\mathrm{Q}_{1,t-1}^{s_1,s_1+q}\mathrm{Q}_{1,t-1}^{s_2,s_2+q}\mathrm{Q}_{1,t-1}^{s_3,s_3+q}\mathrm{Q}_{1,t-1}^{s_4,s_4+q})\\
&= \frac{16M}{(P_{n_1}^2)^4}\sum_{s_1,\ldots,s_4=1}^{p-q}\sum_{k=1}^{n_1}(k-1)\mathrm{E}(\mathrm{Y}_{1,k}^{s_1,s_1+q}\mathrm{Y}_{1,k}^{s_2,s_2+q}\mathrm{Y}_{1,k}^{s_3,s_3+q}\mathrm{Y}_{1,k}^{s_4,s_4+q})\\
&\quad+\frac{48M}{(P_{n_1}^2)^4}\sum_{s_1,\ldots,s_4=1}^{p-q}\sum_{k=1}^{n_1}(k-1)(k-2)\omega_{1,q}^{s_1,s_2}\omega_{1,q}^{s_3,s_4}\\
&= O\left(\frac{(p-q)^2}{n_1^6}\right)+O\left(\frac{(p-q)^2}{n_1^5}\right)=o\left(V_{nq}^4\right).
\end{aligned}$$

In the same way, $\sum_{k=1}^{n_2}\mathrm{E}(\mathrm{B}_{k3}^4)$ and $\sum_{k=1}^{n_2}\mathrm{E}(\mathrm{B}_{k4}^4)$ can also be shown to be of smaller order of V_{nq}^4. Then, the last term

$$\sum_{t=1}^{n_2}\mathrm{E}(\mathrm{B}_{t5}^4)\le\frac{16M}{n_1^4n_2^3}\sum_{s_1,s_2,s_3,s_4=1}^{p-q}\mathrm{E}(\mathrm{Q}_{1,n_1}^{s_1,s_1+q}\mathrm{Q}_{1,n_1}^{s_2,s_2+q}\mathrm{Q}_{1,n_1}^{s_3,s_3+q}\mathrm{Q}_{1,n_1}^{s_4,s_4+q})$$

$$
\begin{aligned}
&= \frac{16M}{n_1^4 n_2^3} \sum_{s_1,s_2,s_3,s_4=1}^{p-q} \sum_{k=1}^{n_1} \mathrm{E}(\mathrm{Y}_{1,k}^{s_1,s_1+q}\mathrm{Y}_{1,k}^{s_2,s_2+q}\mathrm{Y}_{1,k}^{s_3,s_3+q}\mathrm{Y}_{1,k}^{s_4,s_4+q}) \\
&\quad + \frac{48M}{n_1^4 n_2^3} \sum_{s_1,s_2,s_3,s_4=1}^{p-q} (n_1-1)(n_1-2)\omega_{1,q}^{s_1,s_2}\omega_{1,q}^{s_3,s_4} \\
&= O\left(\frac{(p-q)^2}{n_1^3 n_2^3}\right) + O\left(\frac{(p-q)^2}{n_1^2 n_2^3}\right) = o\left(V_{nq}^4\right).
\end{aligned}
$$

Until now, we have proved (S2.18). By verifying the two conditions (S2.17) and (S2.18) of the Martingale Central Limit Theorem, we assure the asymptotic normality in (S2.16). The asymptotic normality of $\hat{\mathrm{S}}_{nq}$ in Theorem 1 is established by the Slutsky Theorem. This completes the proof for Theorem 1. □

Proof of Theorem 2. For any $q = o(p)$, since we have proved the ratio consistency of $\hat{V}_{0,nq}^2$ in Proposition 2, for any $\xi > 0$ and the set $B_\xi = \{\hat{V}_{0,nq}^2 < V_{0,nq}^2(1+\eta)\}$, $\Pr(B_\xi) \to 1$ as $\min\{n_1,n_2\}, p \to \infty$. That is, for any $\varepsilon > 0$, there exists positive integers N_1 and P_1, such that for all n_1, n_2 that satisfy $\min\{n_1,n_2\} > N_1$ and $p > P_1$, $\Pr(B_\xi) > 1-\varepsilon$. Thus, according to (3.4), $\beta_{np,q}(\alpha)$ satisfies that

$$
\begin{aligned}
\beta_{np,q}(\alpha) &\geq \Pr\left(\frac{\hat{\mathrm{S}}_{nq} - \mathrm{S}_q}{V_{nq}} > z_{1-\alpha}\frac{\hat{V}_{0,nq}}{V_{0,nq}} - \delta_{np,q}, B_\xi\right) \\
&\geq \Pr\left(\frac{\hat{\mathrm{S}}_{nq} - \mathrm{S}_q}{V_{nq}} > z_{1-\alpha}(1+\eta) - \delta_{np,q}\right) - \Pr(B_\xi^c),
\end{aligned}
$$

where B_ξ^c is the complementary set of B_ξ. According to Theorem 1, $V_{nq}^{-1}(\hat{\mathrm{S}}_{nq} - \mathrm{S}_q) \xrightarrow{d} \mathcal{N}(0,1)$ as $\min\{n_1,n_2\}, p \to \infty$. Hence, we have

$$
\liminf_{\min\{n_1,n_2\},p\to\infty} \beta_{np,q}(\alpha) \geq 1 - \Phi\left(z_{1-\alpha}(1+\eta) - \liminf_{\min\{n_1,n_2\},p\to\infty} \delta_{np,q}\right) - \varepsilon.
$$

Let $\varepsilon \to 0$ and $\eta \to 0$, and the following inequality holds:

$$
\liminf_{\min\{n_1,n_2\},p\to\infty} \beta_{np,q}(\alpha) \geq 1 - \Phi\left(z_{1-\alpha} - \liminf_{\min\{n_1,n_2\},p\to\infty} \delta_{np,q}\right).
$$

This completes the proof of Theorem 2. □

S3. Simulation results

S3.1 Simulation results for Gaussian Data with Benjamini and Hochberg (1995) procedure

In this section, we report simulation results for Gaussian distributed data using the Benjamini and Hochberg (1995) multiple test procedure by controlling the FDR at 5%. The multiple testing was implemented for $\mathrm{H}_{0,q} : \mathrm{S}_q = 0$, $q = 0, 1, \ldots, N$ where $N = \lfloor p^{0.7} \rfloor$ and the joint hypothesis $\mathrm{H}_0 : \Sigma_1 = \Sigma_2$ is rejected if there exists any individual test $\mathrm{H}_{0,q}$ rejected in the multiple test procedure.

Tables 1 and 2 provide the empirical sizes and powers for the joint test $\mathrm{H}_0 : \Sigma_1 = \Sigma_2$. And Tables 3 and 4 reporte the empirical False Discovery Rates (FDRs) and the Correct Rejection Rates (CRRs) of the Benjamini and Hochberg (1995) procedure, which is equivalent to the Storey et al. (2004) procedure with $\lambda = 0$. The Benjamini and Hochberg (1995) procedure obtains very similar results with those using the Storey et al. (2004) procedure in Tables 1 to 4 in the paper. This is due to the positive correlation between S_q under Models (4.1) - (4.4) in our simulation study. This result indicates that we can still use the Benjamini and Hochberg (1995) method if the positive correlation assumption can be verified by some inference methods or induced by prior knowledge. Otherwise, we recommend using the Storey et al. (2004) method to accommodate more general types of dependence.

S3.2 Simulation results for non-Gaussian Data

In this section, we present additional simulation results for the Gamma distributed data to evaluate its performance for non-Gaussian distributions.

We generated iid random variables $\{\mathrm{Z}_{i,j,k}\}_{k=1}^{p}$, $i = 1, 2$, $j = 1, \ldots, n_i$, from the standardized Gamma(1,0.5) distribution with zero mean and unit variance. Random vectors $\mathbf{X}_{i,j} = \Gamma_i Z_{i,j}$, $i = 1, 2$, $j = 1, \ldots, n_i$, were generated from models (4.1) and (4.3). To evaluate the power, the first sample was generated from models (4.1) and (4.3) and the second sample was generated from models (4.2) and (4.4) correspondingly. The multiple testing was implemented for $\mathrm{H}_{0,q} : \mathrm{S}_q = 0$, $q = 0, 1, \ldots, N$ where $N = \lfloor p^{0.7} \rfloor$ with the Storey et al. (2004) procedure controlling the FDR at 5%. The numerical performance of the proposed test was compared with the tests proposed by SY, LC and CLX. The simulation results are summarized in Tables 5 to 8. Figures 1 and 2 plot the empirical sizes of the individual tests $\mathrm{H}_{0,q} : \mathrm{S}_q = 0$ from $q = 0$ to $\lfloor p^{0.7} \rfloor$ for the Gamma distributed data $\mathbf{X}_{i,j}$ generated from models (4.1) and (4.3), respectively. And the individual empirical powers are shown in Figures 3 and 4 for models (4.2) and (4.4) respectively. These results are largely similar with the Gaussian case in the paper. It indicates that the proposed test is robust under Gaussian and non-Gaussian distributions.

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Table 1: *Empirical sizes and powers of the proposed test in conjunction with the Benjamini and Hochberg (1995) procedure and the tests of Srivastava and Yanagihara (2010) (SY), Li and Chen (2012) (LC) and Cai et al. (2013) (CLX) for the Gaussian distributed data generated from model* (4.1) *for size and* (4.2) *for power. The multiple test procedure is conducted by controlling the FDR at* $\alpha = 0.05$ *and* $N = \lfloor p^{0.7} \rfloor$. *The figures in the parentheses are the adjusted empirical sizes and powers so that the empirical sizes are smaller than the other three tests.*

		Empirical Size				Empirical Power			
p	n	SY	LC	CLX	Proposed	SY	LC	CLX	Proposed
50	30	0.068	0.072	0.053	0.108(0.051)	0.250	0.158	0.271	0.660(0.585)
	50	0.059	0.060	0.041	0.100(0.040)	0.423	0.219	0.288	0.900(0.869)
	80	0.046	0.057	0.040	0.090(0.042)	0.608	0.484	0.443	0.992(0.984)
100	50	0.058	0.053	0.049	0.068(0.052)	0.311	0.126	0.267	0.990(0.986)
	80	0.053	0.053	0.056	0.061(0.043)	0.422	0.240	0.269	1.000(1.000)
	100	0.049	0.059	0.047	0.068(0.046)	0.602	0.520	0.409	1.000(1.000)
200	80	0.045	0.039	0.057	0.068(0.040)	0.325	0.126	0.255	1.000(1.000)
	100	0.040	0.046	0.045	0.063(0.045)	0.618	0.211	0.265	1.000(1.000)
	120	0.046	0.050	0.057	0.067(0.045)	0.706	0.531	0.381	1.000(1.000)
400	100	0.038	0.040	0.046	0.043(0.041)	0.339	0.109	0.260	1.000(1.000)
	120	0.046	0.049	0.060	0.047(0.034)	0.638	0.210	0.268	1.000(1.000)
	150	0.048	0.052	0.041	0.047(0.038)	0.757	0.555	0.330	1.000(1.000)
600	120	0.040	0.043	0.047	0.045(0.043)	0.347	0.106	0.245	1.000(1.000)
	150	0.047	0.049	0.069	0.042(0.041)	0.795	0.209	0.274	1.000(1.000)
	180	0.051	0.059	0.046	0.048(0.039)	1.000	0.594	0.322	1.000(1.000)
1000	150	0.046	0.043	0.047	0.040(0.040)	0.341	0.131	0.242	1.000(1.000)
	180	0.049	0.041	0.047	0.043(0.040)	0.801	0.233	0.246	1.000(1.000)
	200	0.043	0.043	0.042	0.045(0.039)	1.000	0.558	0.283	1.000(1.000)

Table 2: *Empirical sizes and powers of the proposed test in conjunction with the Benjamini and Hochberg (1995) procedure and the tests of SY, LC and CLX for the Gaussian distributed data generated from model* (4.3) *for size and* (4.4) *for power. The multiple test procedure is conducted by controlling the FDR at* $\alpha = 0.05$ *and* $N = \lfloor p^{0.7} \rfloor$. *The figures in the parentheses are the adjusted empirical sizes and powers so that the empirical sizes are smaller than the other three tests.*

		Empirical Size				Empirical Power			
p	n	SY	LC	CLX	Proposed	SY	LC	CLX	Proposed
50	30	0.047	0.057	0.043	0.108(0.047)	0.356	0.372	0.295	0.566(0.496)
	50	0.063	0.069	0.042	0.099(0.045)	0.604	0.589	0.243	0.779(0.718)
	80	0.050	0.069	0.065	0.092(0.041)	0.935	0.943	0.527	0.936(0.911)
100	50	0.050	0.063	0.051	0.061(0.039)	0.518	0.401	0.282	0.916(0.885)
	80	0.058	0.059	0.031	0.057(0.050)	0.785	0.637	0.309	0.995(0.994)
	100	0.031	0.042	0.042	0.058(0.041)	0.990	0.966	0.469	0.999(0.998)
200	80	0.039	0.047	0.057	0.065(0.041)	0.754	0.403	0.263	1.000(1.000)
	100	0.043	0.053	0.049	0.071(0.041)	0.968	0.672	0.292	1.000(1.000)
	120	0.043	0.054	0.042	0.068(0.045)	1.000	0.990	0.400	1.000(1.000)
400	100	0.052	0.044	0.051	0.049(0.038)	0.896	0.421	0.251	1.000(1.000)
	120	0.051	0.046	0.039	0.048(0.038)	0.994	0.706	0.270	1.000(1.000)
	150	0.061	0.039	0.061	0.045(0.038)	1.000	0.992	0.353	1.000(1.000)
600	120	0.044	0.032	0.039	0.046(0.045)	0.943	0.377	0.256	1.000(1.000)
	150	0.048	0.042	0.055	0.048(0.038)	1.000	0.733	0.258	1.000(1.000)
	180	0.049	0.056	0.045	0.043(0.043)	1.000	0.996	0.321	1.000(1.000)
1000	150	0.057	0.041	0.041	0.048(0.048)	0.986	0.361	0.245	1.000(1.000)
	180	0.056	0.057	0.040	0.049(0.043)	1.000	0.758	0.260	1.000(1.000)
	200	0.066	0.042	0.052	0.044(0.044)	1.000	0.996	0.280	1.000(1.000)

Table 3: *False discovery rate and correct rejection rate of the proposed test in conjunction with the Benjamini and Hochberg (1995) procedure for the Gaussian distributed data where the first sample is generated from model* (4.1) *and the second generated from model* (4.2). *The multiple test procedure is performed by controlling the FDR at* $\alpha = 0.05$ *and* $N = \lfloor Cp^{0.7} \rfloor$. *Three dimensions are considered for each dimension, that is* $n = 30, 50, 80$ *for* $p = 50$*;* $n = 50, 80, 100$ *for* $p = 100$*;* $n = 80, 100, 120$ *for* $p = 200$*;* $n = 100, 120, 150$ *for* $p = 400$*;* $n = 120, 150, 180$ *for* $p = 600$*; and* $n = 150, 180, 200$ *for* $p = 1000$*, respectively.*

False Discovery Rate						Correct Rejection Rate					
p						p					
50	100	200	400	600	1000	50	100	200	400	600	1000
$C = 1$						$C = 1$					
0.076	0.067	0.061	0.055	0.049	0.047	0.193	0.269	0.439	0.748	0.933	0.997
0.044	0.045	0.053	0.052	0.054	0.045	0.246	0.343	0.552	0.837	0.974	1.000
0.041	0.056	0.053	0.054	0.048	0.046	0.309	0.421	0.657	0.928	0.993	1.000
$C = 1.5$						$C = 1.5$					
0.062	0.052	0.063	0.052	0.051	0.044	0.149	0.205	0.399	0.688	0.920	0.996
0.051	0.060	0.050	0.052	0.056	0.043	0.205	0.321	0.484	0.824	0.971	1.000
0.054	0.046	0.058	0.059	0.051	0.042	0.287	0.368	0.652	0.921	0.993	1.000
$C = 2$						$C = 2$					
0.071	0.057	0.059	0.050	0.049	0.052	0.154	0.162	0.407	0.644	0.927	0.996
0.046	0.056	0.049	0.053	0.051	0.050	0.210	0.304	0.448	0.827	0.961	1.000
0.060	0.052	0.059	0.059	0.051	0.053	0.289	0.350	0.616	0.918	0.991	1.000
$C = 2.5$						$C = 2.5$					
0.061	0.049	0.050	0.046	0.051	0.053	0.143	0.167	0.382	0.592	0.922	0.996
0.042	0.067	0.046	0.057	0.055	0.052	0.202	0.303	0.454	0.795	0.969	1.000
0.065	0.038	0.059	0.052	0.049	0.052	0.272	0.353	0.623	0.905	0.991	1.000
$C = 3$						$C = 3$					
0.066	0.042	0.052	0.043	0.053	0.042	0.144	0.197	0.391	0.524	0.922	0.996
0.059	0.048	0.042	0.052	0.055	0.049	0.201	0.305	0.485	0.805	0.971	1.000
0.075	0.046	0.046	0.049	0.057	0.041	0.270	0.366	0.620	0.920	0.991	1.000

Table 4: *False discovery rate and correct rejection rate of the proposed test in conjunction with the Benjamini and Hochberg (1995) procedure for the Gaussian distributed data where the first sample is generated from model (4.3) and the second generated from model (4.4). The multiple test procedure is performed by controlling the FDR at $\alpha = 0.05$ and $N = \lfloor Cp^{0.7} \rfloor$. Three dimensions are considered for each dimension, that is $n = 30, 50, 80$ for $p = 50$; $n = 50, 80, 100$ for $p = 100$; $n = 80, 100, 120$ for $p = 200$; $n = 100, 120, 150$ for $p = 400$; $n = 120, 150, 180$ for $p = 600$; and $n = 150, 180, 200$ for $p = 1000$, respectively.*

False Discovery Rate						Correct Rejection Rate					
p						p					
50	100	200	400	600	1000	50	100	200	400	600	1000
$C = 1$						$C = 1$					
0.033	0.062	0.049	0.048	0.053	0.052	0.324	0.652	0.983	1.000	1.000	1.000
0.048	0.048	0.051	0.053	0.049	0.054	0.544	0.874	0.998	1.000	1.000	1.000
0.037	0.057	0.055	0.052	0.054	0.053	0.684	0.955	0.999	1.000	1.000	1.000
$C = 1.5$						$C = 1.5$					
0.067	0.053	0.057	0.048	0.053	0.043	0.253	0.601	0.980	1.000	1.000	1.000
0.051	0.049	0.052	0.046	0.051	0.044	0.464	0.843	0.996	1.000	1.000	1.000
0.046	0.052	0.058	0.053	0.056	0.044	0.655	0.943	0.999	1.000	1.000	1.000
$C = 2$						$C = 2$					
0.065	0.050	0.050	0.048	0.053	0.055	0.214	0.572	0.980	1.000	1.000	1.000
0.050	0.057	0.052	0.047	0.052	0.057	0.425	0.858	0.995	1.000	1.000	1.000
0.050	0.051	0.058	0.055	0.057	0.057	0.623	0.935	0.999	1.000	1.000	1.000
$C = 2.5$						$C = 2.5$					
0.061	0.059	0.052	0.048	0.050	0.053	0.185	0.532	0.976	1.000	1.000	1.000
0.068	0.061	0.052	0.049	0.050	0.056	0.362	0.843	0.995	1.000	1.000	1.000
0.049	0.053	0.059	0.055	0.051	0.053	0.641	0.929	0.999	1.000	1.000	1.000
$C = 3$						$C = 3$					
0.056	0.047	0.056	0.050	0.060	0.047	0.169	0.513	0.981	1.000	1.000	1.000
0.044	0.056	0.052	0.050	0.060	0.046	0.373	0.850	0.993	1.000	1.000	1.000
0.061	0.056	0.053	0.054	0.060	0.044	0.609	0.924	0.998	1.000	1.000	1.000

Table 5: *Empirical sizes and powers of the proposed test in conjunction with the Storey et al. (2004) procedure and the tests of SY, LC and CLX for the Gamma distributed data generated from model* (4.1) *for size and* (4.2) *for power. The multiple test procedure is conducted by controlling the FDR at* $\alpha = 0.05$ *and* $N = \lfloor p^{0.7} \rfloor$. *The figures in the parentheses are the adjusted empirical sizes and powers so that the empirical sizes are smaller than the SY's and LC's tests.*

		Empirical Size				Empirical Power			
p	$n_1 = n_2$	SY	LC	CLX	Proposed	SY	LC	CLX	Proposed
50	30	0.078	0.060	0.041	0.102(0.060)	0.237	0.177	0.255	0.599(0.422)
	50	0.075	0.071	0.045	0.096(0.057)	0.409	0.265	0.250	0.877(0.720)
	80	0.055	0.065	0.046	0.096(0.047)	0.644	0.402	0.416	0.990(0.860)
100	50	0.069	0.058	0.041	0.061(0.058)	0.537	0.258	0.260	0.986(0.820)
	80	0.067	0.067	0.040	0.063(0.063)	0.831	0.397	0.378	1.000(0.996)
	100	0.058	0.065	0.044	0.066(0.055)	0.927	0.521	0.405	1.000(0.994)
200	80	0.063	0.053	0.047	0.069(0.051)	0.944	0.401	0.364	1.000(0.999)
	100	0.056	0.068	0.042	0.067(0.051)	0.988	0.556	0.383	1.000(1.000)
	120	0.054	0.046	0.042	0.067(0.046)	0.997	0.661	0.433	1.000(1.000)
400	100	0.060	0.062	0.044	0.047(0.047)	0.999	0.540	0.349	1.000(1.000)
	120	0.054	0.063	0.039	0.047(0.047)	1.000	0.678	0.380	1.000(1.000)
	150	0.044	0.044	0.050	0.051(0.042)	1.000	0.830	0.355	1.000(1.000)
600	120	0.056	0.048	0.049	0.044(0.044)	1.000	0.693	0.380	1.000(1.000)
	150	0.075	0.055	0.041	0.036(0.036)	1.000	0.850	0.415	1.000(1.000)
	180	0.048	0.048	0.042	0.043(0.043)	1.000	0.932	0.530	1.000(1.000)
1000	150	0.061	0.059	0.039	0.041(0.041)	1.000	0.867	0.386	1.000(1.000)
	180	0.050	0.063	0.042	0.052(0.050)	1.000	0.929	0.450	1.000(1.000)
	200	0.055	0.048	0.044	0.047(0.047)	1.000	0.980	0.562	1.000(1.000)

Table 6: *Empirical sizes and powers of the proposed test in conjunction with the Storey et al. (2004) procedure and the tests of SY, LC and CLX for the Gamma distributed data generated from model* (4.3) *for size and* (4.4) *for power. The multiple test procedure is conducted by controlling the FDR at* $\alpha = 0.05$ *and* $N = \lfloor p^{0.7} \rfloor$. *The figures in the parentheses are the adjusted empirical sizes and powers so that the empirical sizes are smaller than the SY's and LC's tests.*

		Empirical Size				Empirical Power			
p	$n_1 = n_2$	SY	LC	CLX	Proposed	SY	LC	CLX	Proposed
50	30	0.066	0.063	0.043	0.102(0.046)	0.555	0.373	0.293	0.604(0.577)
	50	0.062	0.069	0.035	0.092(0.060)	0.787	0.618	0.242	0.812(0.793)
	80	0.046	0.061	0.042	0.098(0.042)	0.952	0.867	0.350	0.964(0.954)
100	50	0.063	0.065	0.043	0.069(0.053)	0.905	0.663	0.292	0.925(0.914)
	80	0.062	0.065	0.039	0.058(0.058)	0.991	0.902	0.289	0.994(0.992)
	100	0.051	0.068	0.043	0.063(0.051)	0.998	0.969	0.346	1.000(0.999)
200	80	0.046	0.057	0.041	0.065(0.046)	1.000	0.939	0.223	1.000(1.000)
	100	0.057	0.073	0.042	0.060(0.056)	1.000	0.987	0.359	1.000(1.000)
	120	0.043	0.068	0.043	0.061(0.041)	1.000	0.998	0.363	1.000(1.000)
400	100	0.047	0.061	0.044	0.042(0.042)	1.000	0.992	0.229	1.000(1.000)
	120	0.059	0.054	0.043	0.050(0.050)	1.000	1.000	0.282	1.000(1.000)
	150	0.056	0.058	0.044	0.047(0.047)	1.000	1.000	0.413	1.000(1.000)
600	120	0.050	0.042	0.045	0.046(0.036)	1.000	1.000	0.246	1.000(1.000)
	150	0.058	0.053	0.042	0.043(0.043)	1.000	1.000	0.376	1.000(1.000)
	180	0.056	0.055	0.043	0.037(0.037)	1.000	1.000	0.511	1.000(1.000)
1000	150	0.049	0.065	0.036	0.057(0.043)	1.000	1.000	0.310	1.000(1.000)
	180	0.050	0.052	0.039	0.049(0.049)	1.000	1.000	0.407	1.000(1.000)
	200	0.033	0.063	0.037	0.039(0.029)	1.000	1.000	0.533	1.000(1.000)

Table 7: *False discovery rate and correct rejection rate of the proposed test in conjunction with the Storey et al. (2004) procedure for the Gamma distributed data where the first sample is generated from model (4.1) and the second generated from model (4.2). The multiple test procedure is performed at level* $\alpha = 0.05$ *and* $N = \lfloor Cp^{0.7} \rfloor$. *Three dimensions are considered for each dimension, that is* $n = 30, 50, 80$ *for* $p = 50$; $n = 50, 80, 100$ *for* $p = 100$; $n = 80, 100, 120$ *for* $p = 200$; $n = 100, 120, 150$ *for* $p = 400$; $n = 120, 150, 180$ *for* $p = 600$; *and* $n = 150, 180, 200$ *for* $p = 1000$, *respectively.*

False Discovery Rate						Correct Rejection Rate					
p						p					
50	100	200	400	600	1000	50	100	200	400	600	1000
$C = 1$						$C = 1$					
0.064	0.050	0.055	0.045	0.054	0.045	0.163	0.226	0.321	0.517	0.791	0.907
0.060	0.049	0.047	0.047	0.051	0.046	0.245	0.273	0.401	0.639	0.831	0.937
0.041	0.055	0.048	0.050	0.048	0.049	0.257	0.334	0.481	0.754	0.891	0.960
$C = 1.5$						$C = 1.5$					
0.063	0.051	0.055	0.051	0.057	0.046	0.121	0.209	0.318	0.516	0.772	0.905
0.058	0.055	0.051	0.045	0.052	0.049	0.197	0.255	0.397	0.582	0.832	0.938
0.065	0.057	0.054	0.058	0.050	0.043	0.240	0.291	0.480	0.760	0.890	0.959
$C = 2$						$C = 2$					
0.052	0.055	0.041	0.043	0.052	0.050	0.123	0.206	0.319	0.508	0.732	0.897
0.054	0.055	0.046	0.058	0.051	0.055	0.196	0.261	0.375	0.573	0.840	0.946
0.047	0.050	0.058	0.054	0.049	0.051	0.234	0.322	0.495	0.739	0.880	0.959
$C = 2.5$						$C = 2.5$					
0.067	0.051	0.054	0.054	0.049	0.065	0.136	0.209	0.325	0.511	0.757	0.907
0.045	0.066	0.046	0.051	0.052	0.068	0.198	0.260	0.371	0.575	0.848	0.929
0.056	0.043	0.051	0.053	0.052	0.064	0.237	0.315	0.457	0.761	0.888	0.963
$C = 3$						$C = 3$					
0.069	0.045	0.061	0.055	0.047	0.060	0.125	0.212	0.300	0.515	0.724	0.909
0.043	0.043	0.042	0.055	0.054	0.061	0.186	0.265	0.380	0.591	0.840	0.931
0.064	0.038	0.058	0.059	0.060	0.054	0.241	0.313	0.441	0.707	0.892	0.961

Table 8: *False discovery rate and correct rejection rate of the proposed test in conjunction with the Storey et al. (2004) procedure for the Gamma distributed data where the first sample is generated from model* (4.3) *and the second generated from model* (4.4)*. The multiple test procedure is performed by controlling the FDR at* $\alpha = 0.05$ *and* $N = \lfloor Cp^{0.7} \rfloor$*. Three dimensions are considered for each dimension, that is* $n = 30, 50, 80$ *for* $p = 50$*;* $n = 50, 80, 100$ *for* $p = 100$*;* $n = 80, 100, 120$ *for* $p = 200$*;* $n = 100, 120, 150$ *for* $p = 400$*;* $n = 120, 150, 180$ *for* $p = 600$*; and* $n = 150, 180, 200$ *for* $p = 1000$*, respectively.*

False Discovery Rate						Correct Rejection Rate					
p						p					
50	100	200	400	600	1000	50	100	200	400	600	1000
$C = 1$						$C = 1$					
0.052	0.073	0.046	0.054	0.052	0.052	0.230	0.652	0.966	1.000	1.000	1.000
0.051	0.053	0.048	0.054	0.054	0.049	0.422	0.851	0.993	1.000	1.000	1.000
0.052	0.066	0.050	0.049	0.054	0.053	0.669	0.929	0.997	1.000	1.000	1.000
$C = 1.5$						$C = 1.5$					
0.044	0.055	0.053	0.057	0.058	0.044	0.225	0.635	0.957	1.000	1.000	1.000
0.046	0.045	0.053	0.058	0.055	0.044	0.427	0.861	0.991	1.000	1.000	1.000
0.055	0.038	0.055	0.055	0.057	0.048	0.661	0.938	0.997	1.000	1.000	1.000
$C = 2$						$C = 2$					
0.062	0.051	0.056	0.058	0.057	0.056	0.231	0.629	0.958	1.000	1.000	1.000
0.053	0.054	0.057	0.058	0.058	0.058	0.415	0.847	0.990	1.000	1.000	1.000
0.052	0.043	0.051	0.056	0.044	0.044	0.678	0.945	0.996	1.000	1.000	1.000
$C = 2.5$						$C = 2.5$					
0.071	0.057	0.050	0.056	0.054	0.054	0.238	0.643	0.961	1.000	1.000	1.000
0.067	0.053	0.060	0.059	0.053	0.055	0.400	0.862	0.991	1.000	1.000	1.000
0.049	0.046	0.053	0.057	0.054	0.059	0.669	0.940	0.996	1.000	1.000	1.000
$C = 3$						$C = 3$					
0.066	0.057	0.055	0.057	0.063	0.047	0.221	0.638	0.968	1.000	1.000	1.000
0.070	0.059	0.053	0.058	0.062	0.047	0.431	0.827	0.992	1.000	1.000	1.000
0.055	0.051	0.057	0.057	0.051	0.047	0.660	0.937	0.996	1.000	1.000	1.000

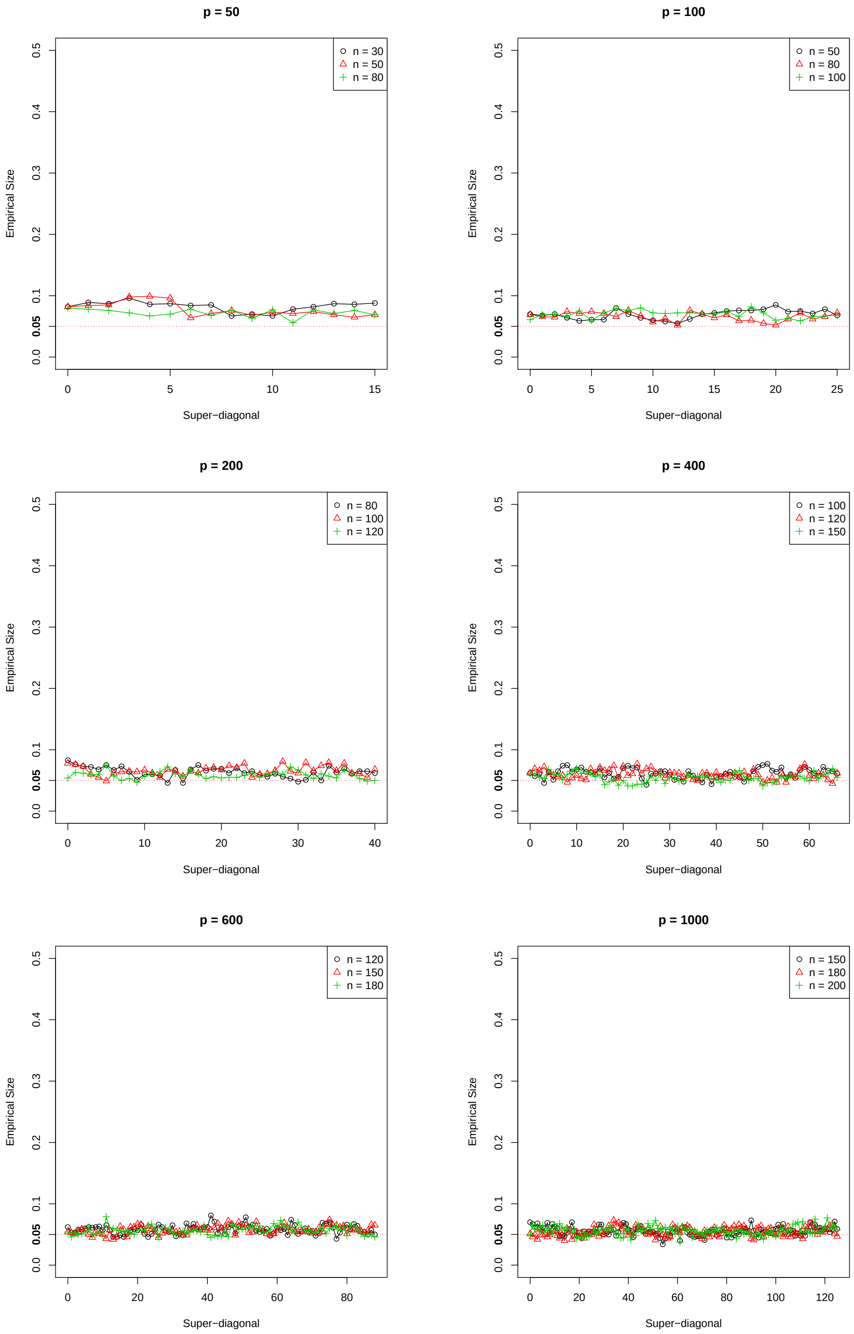

Figure 1: Empirical sizes of the individual tests $H_{0,q} : S_q = 0$ for the Gamma distributed data generated from model (4.1). The range of the horizontal axis is from $q = 0$ to $q = \lfloor p^{0.7} \rfloor$.

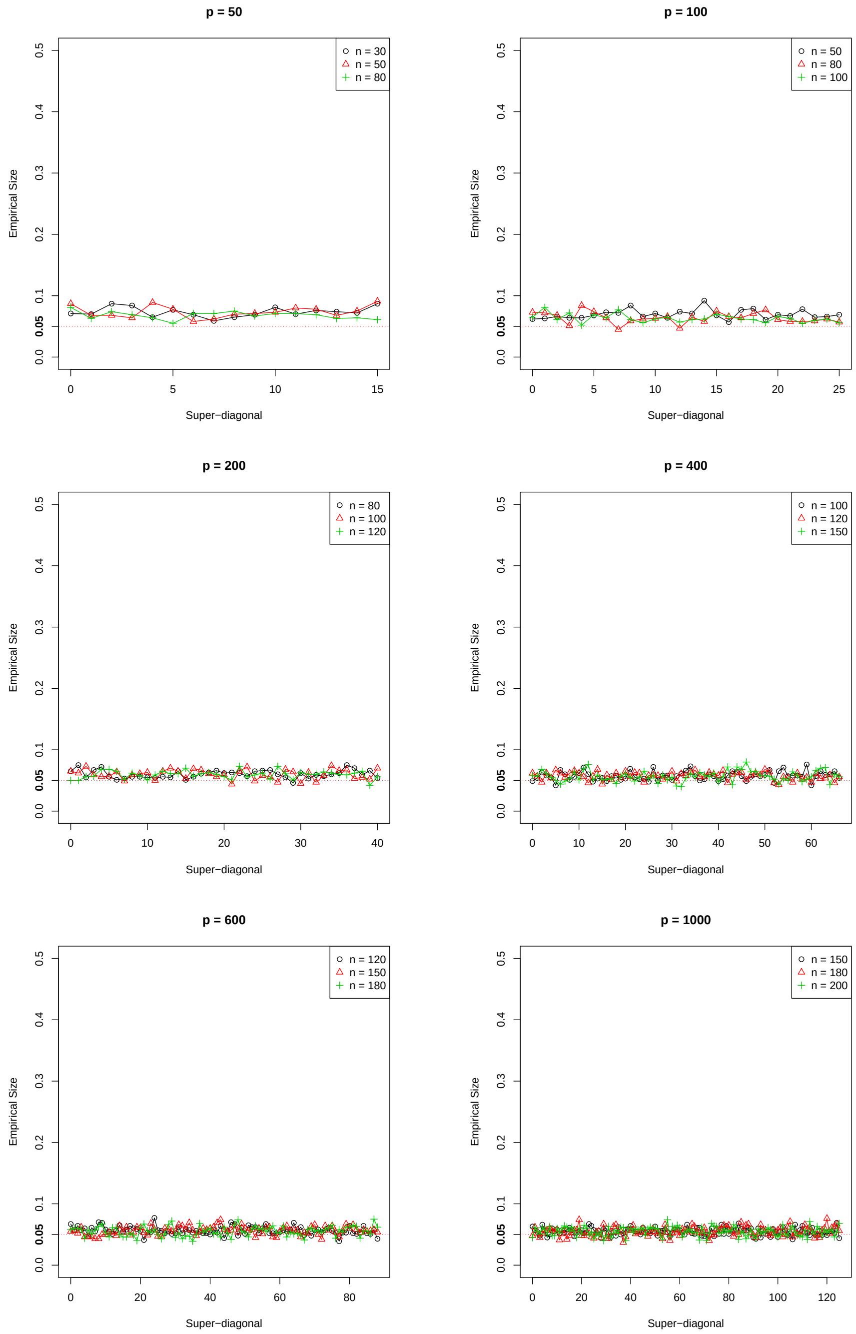

Figure 2: Empirical sizes of the individual tests $H_{0,q} : S_q = 0$ for the Gamma distributed data generated from model (4.3). The range of the horizontal axis is from $q = 0$ to $q = \lfloor p^{0.7} \rfloor$.

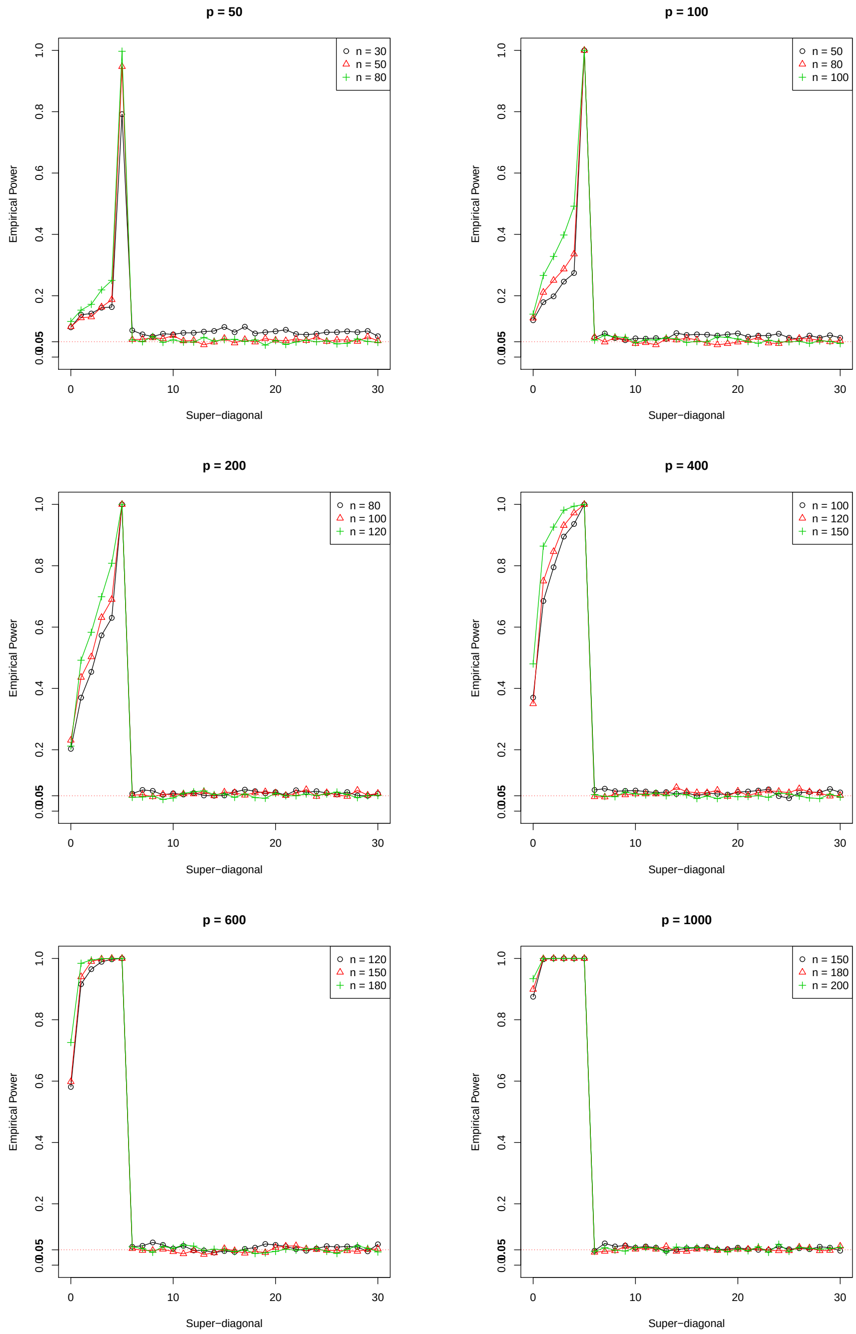

Figure 3: Empirical powers of the individual tests $H_{0,q} : S_q = 0$ for the Gamma distributed data with the first sample generated from model (4.1) while the second sample generated from model (4.2).

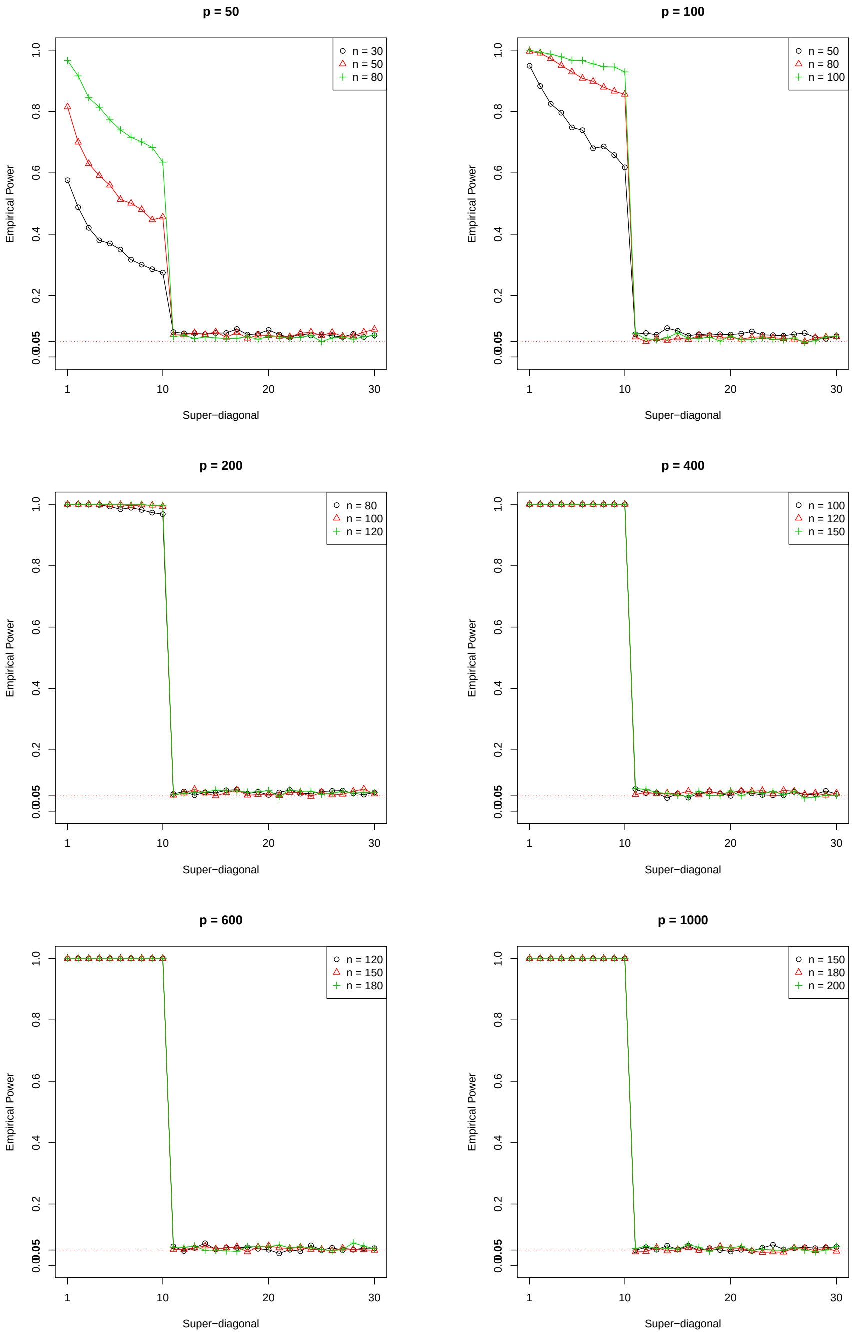

Figure 4: Empirical powers of the individual tests $H_{0,q} : S_q = 0$ for the Gamma distributed data with the first sample generated from model (4.3) while the second sample generated from model (4.4).