

## WHEN IS ACCELERATION UNNECESSARY IN A DEGRADATION TEST?

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*Abstract:* Acceleration is widely used in reliability tests to yield sufficient reliability information within a short time frame. From the statistical point of view, the cost of acceleration is that additional parameters are needed to link the accelerating variables to the failure process. When the increase of statistical information is insufficient to compensate the introduction of additional parameters, acceleration is inefficient. This scenario may be rare in a life test, as acceleration yields more failures and failure is more informative than censoring. In a degradation test, however, information contained in a degradation measurement under high stress levels may not be much higher than that under normal use conditions; in this connection, acceleration may not be always necessary. This study identifies situations where acceleration is unnecessary when some common stochastic process models are used, including the Wiener, gamma, and inverse Gaussian (IG) processes. We assume that both the degradation rate and the volatility of degradation process are functions of the accelerating variable. An acceleration relation index is introduced to unify different kinds of acceleration relations seen in the literature. It is shown that when the acceleration relation index is at least one, acceleration is always inefficient. Otherwise, the necessity of acceleration depends on values of the model parameters as well as the acceleration relation index. These results are unified using a class of stochastic process models called the exponential dispersion (ED) class. A numerical example is given to illustrate the procedure.

*Key words and phrases:* Accelerated degradation test, exponential dispersion models, reliability, stochastic process models.

### 1. Introduction

Life tests are commonly used to obtain information about the time-to-failure distribution of materials and products. However, few failures might be observed in a life test of practical length under normal use conditions, and a large proportion of censoring makes estimation of the failure time distribution difficult. A common view is that the lifetime distribution can be estimated much more accurately using accelerated life tests (ALTs) (e.g., Tseng, Huang and Wu (1994); Tang and Liu (2010)): one tests units under several higher-than-normal levels of

stress to accelerate the failure process and induce more failures within the test duration. Information obtained under accelerated conditions is used in conjunction with a stress-acceleration lifetime model to estimate the reliability characteristics of interest under normal use conditions. Compared with the lifetime distribution under normal use conditions, the stress-acceleration lifetime model involves additional parameters to relate the life length to stress. To achieve the same statistical accuracy in estimating the reliability characteristics, more statistical information is needed to offset the increase in the number of parameters. In a life test, nevertheless, a failure provides much more information than censoring. The negative effect of introducing additional parameters is usually sufficiently compensated by the increase of statistical information.

The trade-off between information increase due to acceleration and information consumption caused by additional parameters is quite different in an accelerated degradation test (ADT). Degradation of a product, defined as the accumulation of irreversible damage over time, is the root cause of most aging failures. It is common that a product deteriorates over time and fails when the accumulated damage reaches a certain failure threshold. Based on this degradation-threshold failure mechanism, the failure time distribution of the product can be estimated through analysis of degradation data. The data are usually obtained from a degradation test. With appropriate degradation models, degradation tests achieve a good estimation precision by using a relatively small number of test units. They are widely employed in situations where product failure data are scarce, such as reliability tests of expensive devices or highly reliable products.

Similar to ALTs, ADTs apply accelerating variables to hasten the degradation process. Additional parameters used to link the accelerating variables to the degradation process require more statistical information to achieve the same level of estimation precision, and information obtained under accelerated conditions may not be sufficiently higher than that under normal use conditions. Considering the trade-off, there has been disagreement on whether to use acceleration in a degradation test. For example, many existing studies advocate acceleration in a degradation test (e.g., Yu and Tseng (1998); Liao and Elsayed (2006); Ye et al. (2014)), while some studies are devoted to degradation test planning under normal use conditions (e.g., Wu and Chang (2002); Tsai, Tseng and Balakrishnan (2012); Weaver et al. (2013); Kim and Bae (2013)). From an engineering point of view, ADTs usually require higher costs because extra equipment is needed to mimic the harsh conditions, and one wishes to ensure its necessity. The study is motivated by a current experiment to study degradation behaviors of emerging

contaminants. Emerging contaminants, recently shown to widely occur in water resources, have been identified as potential risks to the environment and public health. To reveal their degradation behaviors, we are currently conducting ADTs to estimate kinetic parameters such as the degradation rate in the wild. Considering the high costs of mimicking the harsh conditions, it is of great importance to justify the expense.

Generally speaking, there are two broad categories of degradation models: general path models (Meeker and Escobar (1998)) and stochastic process models (e.g., Si et al. (2011)). Commonly used stochastic process models are the Wiener process, gamma process, and the inverse Gaussian (IG) process. Under the Wiener process assumption, Doksum and Hbyland (1992), Lim and Yum (2011), and Hu, Lee and Tang (2015) investigated ADTs designs under different acceleration schemes, including constant-stress loading, step-stress loading and progressive-stress loading. Lee and Tang (2007) and Ye et al. (2013) considered degradation modelling and data analysis under normal use conditions in a Wiener process. When the monotonicity of the degradation path is required, the gamma and IG processes are good alternatives. Tseng, Balakrishnan and Tsai (2009) studied optimum step-stress ADT planning of the gamma process, while Wang (2008) and Tsai, Tseng and Balakrishnan (2012) investigated degradation data analysis and test planning for the gamma process under normal use conditions. Additionally, Ye et al. (2014) developed an optimum ADT plan for the IG process.

The necessity of acceleration depends also on the acceleration relations. Stress-degradation acceleration relations link the accelerating variables to such degradation characteristics as the degradation rate and the volatility of the degradation path. The degradation rate is usually assumed to be an increasing function of the stress levels. This kind of functional relation is called the link function. Commonly used link functions include the Arrhenius relation, power law relation, and the exponential relation (Ye and Xie (2015)). In applications, appropriate link functions can be specified based on the degradation physics, engineering experiences, or data analysis. For example, the Arrhenius function is widely used when the accelerating variable is temperature, while the power law is often used to characterize the effect of voltage (Park and Padgett (2005)). Regarding the volatility parameter that reflects the variation of a degradation path, many studies assume it to be a constant (e.g., Tang, Yang and Xie (2004); Liao and Tseng (2006); Lim and Yum (2011)). Some studies consider the positive correlation between the degradation variation and the stress levels, and assume that

the volatility parameter increases with the stress levels (e.g., Liao and Elsayed (2006); Tseng, Balakrishnan and Tsai (2009); Ye, Chen and Shen (2015)). Additionally, Tseng and Wen (2000) applied the cumulative exposure principle to link accelerating variables to a degradation model.

This paper aims to identify situations where acceleration is unnecessary in a degradation test. The results can be used as a guideline for ADT planning. We consider commonly used stochastic process models including the Wiener, gamma, and IG processes. Different kinds of acceleration relations are investigated in these models. The remainder of this paper is organized as follows. Section 2 gives model assumptions of the Wiener, gamma and IG processes. An acceleration relation index is introduced to unify different kinds of acceleration relations. Section 3 proposes a generic approximation of the first-hitting-time distribution for these three processes. Asymptotic variances of the estimated lifetime quantiles are derived. Section 4 compares the estimation precision between an optimum ADT plan and the corresponding nonaccelerated test. Situations where acceleration of the degradation process is unnecessary are identified. Section 5 applies a class of exponential dispersion (ED) models to unify the discussion of acceleration in the Wiener, gamma and IG processes. A numerical example is given in Section 6 to illustrate the procedure to determine the necessity of acceleration. Section 7 gives conclusions and describes possible areas for future research. Proofs and additional discussion are provided in the Supplementary Materials.

## 2. Stochastic Process Models and Acceleration

Consider a product whose degradation is measurable. Let  $\{X(t), t \geq 0\}$  with  $X(0) = 0$  be the degradation path of a randomly selected unit, where  $X(t)$  is the degradation level measured at time  $t$ . In this paper, we consider a linear degradation path. In some cases, the degradation path of products is a linear process (Lu, Park and Yang (1997)). When linearity does not hold, the shape of the degradation path, exponential, power law, or others, is usually determined by the degradation physics or empirical experience of the product. For example, in a problem where degradation tests are needed, we may already have degradation data on similar products, e.g., products of previous vintages, from which the degradation pattern can be empirically determined. Time-scale transformation is commonly adopted in the degradation literature, e.g., see Whitmore and Schenkelberg (1997); Lee and Tang (2007); Weaver et al. (2013); Tseng and Lee

Table 1. Original/unified parameters in the Wiener, gamma, and IG processes.

| Stochastic Process | Distribution of $X(t)$              | Drift $\mu$ | Volatility $\sigma$                |
|--------------------|-------------------------------------|-------------|------------------------------------|
| Wiener             | $\mathcal{N}(\nu t, \varsigma^2 t)$ | $\nu$       | $\varsigma$                        |
| gamma              | $Ga(kt, \theta)$                    | $k\theta$   | $k^{1/2}\theta$                    |
| IG                 | $\mathcal{IG}(\alpha t, \beta t^2)$ | $\alpha$    | $\alpha^{\frac{3}{2}}\beta^{-1/2}$ |

(2016). Lifetime of a unit, denoted by  $T$ , is defined as the time until the product deteriorates to a specified failure threshold  $\mathcal{D}_f$ .

### 2.1. Stochastic process models

We first consider a stationary Wiener process under normal use conditions. A basic Wiener process model  $\{X(t), t \geq 0\}$  is often expressed as  $X(t) = \nu t + \varsigma \mathcal{B}(t)$ , where  $\nu$  is the drift parameter,  $\varsigma$  is the volatility parameter, and  $\mathcal{B}(\cdot)$  is the standard Brownian motion. Here, the degradation increment  $X(t)$  is  $\mathcal{N}(\nu t, \varsigma^2 t)$  with mean  $\nu t$  and variance  $\varsigma^2 t$ .

Consider then a stationary gamma process  $\{X(t), t \geq 0\}$ , where the degradation increment  $X(t)$  follows a gamma distribution  $Ga(kt, \theta)$  with shape  $kt$  and scale  $\theta$ . The probability density function (PDF) of  $X(t)$  is

$$f_{Ga}(x; k, \theta, t) = \frac{1}{\Gamma(kt)\theta^{kt}} x^{kt-1} \exp\left(-\frac{x}{\theta}\right), \quad x > 0,$$

for  $k > 0$  and  $\theta > 0$ , with mean and variance  $k\theta t$  and  $k\theta^2 t$ , respectively. A stationary IG process  $\{X(t), t \geq 0\}$  has the degradation increment  $X(t)$  as  $\mathcal{IG}(\alpha t, \beta t^2)$  with PDF

$$f_{\mathcal{IG}}(x; \alpha, \beta, t) = \left(\frac{\beta t^2}{2\pi x^3}\right)^{1/2} \exp\left[-\frac{\beta(x - \alpha t)^2}{2\alpha^2 x}\right], \quad x > 0,$$

for  $\alpha > 0$  and  $\beta > 0$ , with mean and variance of  $X(t)$   $\alpha t$  and  $\alpha^3 t/\beta$ , respectively.

The mean path functions in the Wiener, gamma, and IG processes have similar expressions. We take the mean of  $X(t)$  as  $\mu t$ , and the variance of  $X(t)$  as  $\sigma^2 t$  in all three models through reparameterization. After the reparameterization,  $\mu$  is the drift parameter that captures the mean degradation rate, while  $\sigma$  is the volatility parameter that reflects the variation of the degradation path. The relationship between the original parameters and the unified parameters  $\mu$  and  $\sigma$  are given in Table 1.

### 2.2. ADT settings

We consider a single stress with constant-stress loading in an ADT plan. Suppose  $r$  factor levels of the stress are employed. Let  $\tilde{s}_0$  and  $\tilde{s}_H$  be the normal

use level and the highest allowable level of the accelerating variable, respectively. Without loss of generality, we normalize the stress level  $\tilde{s}_i$  ( $i = 1, \dots, r$ ) as follows (Lim and Yum (2011)):

$$s_i = \frac{\psi(\tilde{s}_i) - \psi(\tilde{s}_0)}{\psi(\tilde{s}_H) - \psi(\tilde{s}_0)}, \quad (2.1)$$

where  $\psi(\cdot)$  is a monotone transformation of the stress level whose form depends on the acceleration relations. For example,  $\psi(\tilde{s}_i) = 1/\tilde{s}_i$  for the Arrhenius law, while  $\psi(\tilde{s}_i) = \ln \tilde{s}_i$  for the power law. After the normalization,  $s_0 = 0$ ,  $s_H = 1$ , and  $0 \leq s_i \leq 1$  for  $i = 1, \dots, r$ .

Let  $\{X(t|s_i), t \geq 0\}$  be the degradation path of a randomly selected unit under the standardized stress level  $s_i$ . A common practice is to assume that the degradation rate increases with the stress levels (e.g., Tang, Yang and Xie (2004); Tseng, Balakrishnan and Tsai (2009); Ye et al. (2014)). Let  $\mu_i$  be the unified drift parameter under the standardized stress level  $s_i$ . We normalize the functional relation between  $\mu_i$  and  $s_i$  as

$$\mu_i = \exp(\delta_1 + \delta_2 s_i) \text{ for } i = 1, \dots, r, \quad (2.2)$$

where  $\delta_1$  is a scaling factor, and  $\delta_2 > 0$  is the accelerating parameter. With the standardization of the stress levels, the acceleration relation (2.2) is a normalized form of common link functions including the Arrhenius relation, power law relation, and the exponential relation (Lim and Yum (2011)).

Regarding the volatility of the degradation path, some studies have assumed that the variation of  $X(t|s_i)$  is invariant of the stress levels  $s_i$  (e.g., Doksum and Normand (1995); Tang, Yang and Xie (2004); Lim and Yum (2011)), while others assumed it to increase with the stress levels (e.g., Liao and Elsayed (2006); Ye et al. (2014)). We adopt a more general acceleration relation that considers the constant, increasing, and even decreasing volatility parameters under accelerated conditions. The functional relation between the unified volatility parameter  $\sigma_i$  and the stress level  $s_i$  is modeled as

$$\sigma_i = \exp(\rho_1 + \rho_2 s_i) \text{ for } i = 1, \dots, r, \quad (2.3)$$

where  $\rho_1$  and  $\rho_2$  are model parameters. When  $\rho_2 = 0$ , the volatility parameter is a constant. When  $\rho_2 > 0$ , it increases with the stress levels; otherwise, it decreases with the stress levels.

Based on the acceleration relations (2.2) and (2.3), an acceleration relation index  $b$  is introduced to represent different kinds of acceleration relations used in the literature. We reexpress the unified volatility parameter  $\sigma_i$  as

$$\sigma_i = a\mu_i^b \text{ for } i = 1, \dots, r, \quad (2.4)$$

with the transformed parameters  $a = \exp(\rho_1 - \rho_2\delta_1/\delta_2) > 0$ , and  $b = \rho_2/\delta_2$ . After the reparameterization, the exponent  $b$  serves as an acceleration relation index that distinguishes different kinds of acceleration relations. We give some examples.

When the acceleration relation index  $b = 0$ , the drift parameter increases with the stress levels while the volatility parameter is a constant. This kind of acceleration relation is adopted in Tang, Yang and Xie (2004) and Lim and Yum (2011) under the Wiener process assumption. When the index  $b = 1/2$ , the acceleration model is a cumulative-exposure model (Ye and Xie (2015)). Adopting  $b = 1/2$  in a gamma process where  $X(t|s_i) \sim Ga(k_i t, \theta_i)$  is equivalent to assuming that the shape parameter  $k_i = \mu_i/a^2$  increases with the stress levels, while the scale parameter  $\theta_i = a^2$  is a constant (see Table 1). This kind of acceleration relation is adopted in Tseng, Balakrishnan and Tsai (2009). When the index  $b = 1$ , the volatility parameter  $\sigma_i$  can be expressed as a linear function of the drift parameter  $\mu_i$ . This kind of acceleration relation is adopted in Ye, Chen and Shen (2015) under the Wiener process assumption. When the index  $b = 3/2$  in an IG process where  $X(t|s_i) \sim \mathcal{IG}(\alpha_i t, \beta_i t^2)$ , the mean parameter  $\alpha_i = \mu_i$  is an increasing function of the stress levels, while the shape parameter  $\beta_i = a^{-2}$  is a constant (see Table 1). This kind of acceleration relation is adopted in Ye et al. (2014).

Suppose a total number of  $n$  units are used for test. Let  $n_i$  be the number of units allocated to the standardized stress level  $s_i$ ,  $i = 1, \dots, r$ . The proportion of test units allocated to  $s_i$  is denoted as  $\pi_i$ ,  $\pi_i = n_i/n$ . We take test duration and number of measurements as predetermined, and adopt an equally-spaced inspection policy. Such a schedule has been widely used in existing literature (e.g., Lim and Yum (2011); Ye et al. (2014); Tseng and Lee (2016)). Based on it, let  $t_M$  and  $m$  be the test duration and the number of measurements for each unit, respectively. The time interval  $\Delta t$  between two adjacent measurements is then  $t_M/m$ .

### 3. Asymptotic Variance of the Estimated Lifetime Quantile

We compare the estimation precision of an optimum ADT plan and the corresponding degradation test without acceleration. Usually, the objective of a degradation test is to estimate a lifetime quantile under normal use conditions. Let  $t_q$  be the  $q$ th lifetime quantile and  $\hat{t}_q$  be the corresponding maximum like-

likelihood (ML) estimator. The asymptotic variance of  $\hat{t}_q$ , denoted as  $\text{AVar}(\hat{t}_q)$ , is widely adopted to represent the estimation precision of an ADT plan (e.g., Wu and Shao (1999)). Minimization of  $\text{AVar}(\hat{t}_q)$  is adopted as the planning criterion here (e.g., Lu, Meeker and Escobar (1996)).

### 3.1. A generic form of $t_q$

Although there might be closed-form CDFs of the lifetime  $T$  for the Wiener, gamma and IG processes (Chhikara (1988); Lawless and Crowder (2004); Ye and Chen (2014)), the associated PDFs are too complex to derive closed-form life quantiles. Following Onar and Padgett (2000), a generic approximation of  $t_q$  can be derived as follows.

Consider a discrete degradation path  $\{X_n; n \in \mathbb{N}\}$  with drift parameter  $\mu > 0$  and volatility parameter  $\sigma > 0$ , where  $X_n$  is the degradation level after  $n$  time units and  $X_0 = 0$ . Let  $N$  denote the discrete first hitting time of the process  $\{X_n; n \in \mathbb{N}\}$  to the pre-specified threshold  $\mathcal{D}_f$ . For a stochastically increasing case, the failure probability after  $n$  time units is  $\Pr[N > n] = \Pr[X_n < \mathcal{D}_f]$ . Let  $Y_n = X_n - X_{n-1}$  for  $n \geq 1$ . The increments  $Y_n$  are i.i.d. with common mean  $\mu$  and variance  $\sigma^2$ . Since  $X_n = Y_n + Y_{n-1} + \cdots + Y_1$ ,

$$\Pr[N > n] = \Pr\left[\sum_{i=1}^n Y_i < \mathcal{D}_f\right].$$

It follows from the Central Limit Theorem that

$$\Pr[N \leq n] \cong \Phi\left(\frac{\mu n - \mathcal{D}_f}{\sqrt{n}\sigma}\right), \quad (3.1)$$

where  $\Phi(\cdot)$  is the standard normal CDF.

Here  $N$  is a discretization of the first hitting time  $t$ . Based on (3.1), a continuous version of the first hitting time  $t$  can be represented by the Birnbaum–Saunders distribution (Birnbaum and Saunders (1969)),

$$F_T(t; \mathcal{D}_f) \cong \Phi\left(\frac{\mu\sqrt{t}}{\sigma} - \frac{\mathcal{D}_f}{\sqrt{t}\sigma}\right). \quad (3.2)$$

Denote the standard normal quantile  $\Phi^{-1}(q)$  as  $z_q$ . Inverting (3.2), an approximation of the  $q$ th lifetime quantile is

$$t_q \approx \frac{\left[z_q\sigma + \sqrt{z_q^2\sigma^2 + 4\mu\mathcal{D}_f}\right]^2}{4\mu^2}.$$

According to our simulation experience, the approximation is accurate, especially when  $\mu$  is large compared with  $\sigma^2$ , or when  $\mathcal{D}_f$  is large compared with  $\mu$ . Under

normal use conditions,

$$t_q \approx \frac{\left[ z_q a \exp(b\delta_1) + \sqrt{z_q^2 a^2 \exp(2b\delta_1) + 4 \exp(\delta_1) \mathcal{D}_f} \right]^2}{4 \exp(2\delta_1)}, \tag{3.3}$$

since  $\mu_0 = \exp(\delta_1)$  and  $\sigma_0 = a \exp(b\delta_1)$  with the normal use stress level  $s_0 = 0$ .

**3.2. Asymptotic variance of  $\hat{t}_q$**

Let  $\boldsymbol{\theta} = (\delta_1, \delta_2, a)'$  be the model parameter in an ADT plan, and  $\hat{\boldsymbol{\theta}} = (\hat{\delta}_1, \hat{\delta}_2, \hat{a})'$  be the ML estimators of  $\boldsymbol{\theta}$ . Based on the invariance property of the ML estimation, the ML estimator of  $t_q$  under normal use conditions can be obtained by evaluating (3.3) at  $\hat{\boldsymbol{\theta}}$ . By the delta method (e.g., Lawless (2011, Appendix B)), the asymptotic variance of  $\hat{t}_q$  is

$$\text{AVar}_a(\hat{t}_q) = \mathbf{h}'[\mathcal{I}(\boldsymbol{\theta})]^{-1}\mathbf{h}, \tag{3.4}$$

where  $\mathcal{I}(\boldsymbol{\theta})$  is the Fisher information matrix of  $\boldsymbol{\theta}$ ,  $\mathbf{h} = (h_1, h_2, h_3)'$  is the gradient of the quantile  $t_q$  with respect to  $\boldsymbol{\theta}$ , and

$$\begin{aligned} h_1 &= \frac{\partial t_q}{\partial \delta_1} = \frac{(w_1 + w_2^{1/2})(bw_1 + bw_1^2 w_2^{-1/2} + 2 \exp(\delta_1) \mathcal{D}_f w_2^{-1/2}) - (w_1 + w_2^{1/2})^2}{2 \exp(2\delta_1)}, \\ h_2 &= \frac{\partial t_q}{\partial \delta_2} = 0, \\ h_3 &= \frac{\partial t_q}{\partial a} = \frac{(w_1 + w_2^{1/2})(w_1 + w_1^2 w_2^{-1/2})}{2a \exp(2\delta_1)}, \end{aligned} \tag{3.5}$$

where  $w_1 = z_q a \exp(b\delta_1)$ , and  $w_2 = z_q^2 a^2 \exp(2b\delta_1) + 4 \exp(\delta_1) \mathcal{D}_f$ .

There are many possible planning variables for an optimum ADT design, such as the stress levels, the allocation of test units, and the number of measurements. We simultaneously determine the optimum setting of the stress levels and the optimum allocation of test units. The optimum problem is formulated as:

$$\begin{aligned} &\text{minimize} && \text{AVar}_a(\hat{t}_q) \\ &\text{subject to} && 0 \leq s_i \leq 1, \quad i = 1, \dots, r, \\ &&& 0 \leq \pi_i \leq 1, \quad i = 1, \dots, r, \\ &&& \sum_{i=1}^r \pi_i = 1, \quad i = 1, \dots, r. \end{aligned}$$

In the corresponding degradation test without acceleration, the model parameter  $\boldsymbol{\theta}$  degenerates as  $\boldsymbol{\theta}_0 = (\delta_1, a)'$ . Let  $\mathcal{I}(\boldsymbol{\theta}_0)$  denote the Fisher information matrix of  $\boldsymbol{\theta}_0$ . Using the delta method, the asymptotic variance of  $\hat{t}_q$  can be

obtained as

$$A\text{Var}_n(\hat{t}_q) = \mathbf{h}'_0[\mathcal{I}(\boldsymbol{\theta}_0)]^{-1}\mathbf{h}_0, \tag{3.6}$$

with  $\mathbf{h}_0 = (h_1, h_3)'$  where  $h_1$  and  $h_3$  are given in (3.5).

#### 4. Necessity of Acceleration

We identify the necessity of acceleration in the Wiener, gamma, and IG processes. In an ADT plan, let  $X_{ijl}$  be the  $l$ th degradation measurement on unit  $j$  under the standardized stress  $s_i$  for  $i = 1, \dots, r, j = 1, \dots, n_i$ , and  $l = 1, \dots, m$ . Denote the degradation data as  $\mathbf{D} = \{X_{ijl}; i = 1, \dots, r, j = 1, \dots, n_i, l = 1, \dots, m\}$ . The time interval between two adjacent measurements is  $\Delta t = t_M/m$  for all test units. Let  $\Delta X_{ijl} = X_{ijl} - X_{ij(l-1)}$  be the degradation increment within the measurement time interval  $\Delta t$ . Then  $\Delta X_{ijl}$  has mean  $\mu_i \Delta t$  and variance  $\sigma_i^2 \Delta t$  with the unified drift parameter  $\mu_i = \exp(\delta_1 + \delta_2 s_i)$  and volatility parameter  $\sigma_i = a\mu_i^b$  for  $i = 1, \dots, r$ . In the corresponding nonaccelerated test, all units are tested under the normal use stress level  $s_0 = 0$ . The drift parameter is  $\mu_0 = \exp(\delta_1)$ , and the volatility parameter is  $\sigma_0 = a\mu_0^b$ .

##### 4.1. Wiener process

Under the Wiener process assumption,  $\Delta X_{ijl}$  is  $\mathcal{N}(\mu_i \Delta t, \sigma_i^2 \Delta t)$ . Based on the degradation data  $\mathbf{D}$ , the log-likelihood function of  $n$  test units (up to a constant) is

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^r \sum_{j=1}^{n_i} \sum_{l=1}^m \left[ -\ln a - b \ln \mu_i - \frac{\mu_i^{-2b}}{2a^2 \Delta t} (\Delta x_{ijl} - \mu_i \Delta t)^2 \right].$$

The Fisher information matrix of  $\boldsymbol{\theta} = (\delta_1, \delta_2, a)'$ , denoted as  $\mathcal{I}_W(\boldsymbol{\theta})$ , is (see Supplement, Section S.2)

$$\mathcal{I}_W(\boldsymbol{\theta}) = \frac{nt_M}{a^2} \begin{bmatrix} \sum_{i=1}^r \pi_i \mu_i^{-2b+2} + \frac{2ma^2b^2}{t_M} & \sum_{i=1}^r s_i \pi_i \left( \mu_i^{-2b+2} + \frac{2ma^2b^2}{t_M} \right) & \frac{2mab}{t_M} \\ & \sum_{i=1}^r s_i^2 \pi_i \left( \mu_i^{-2b+2} + \frac{2ma^2b^2}{t_M} \right) & \frac{2mab}{t_M} \sum_{i=1}^r s_i \pi_i \\ \text{symmetric} & & \frac{2m}{t_M} \end{bmatrix}. \tag{4.1}$$

The Fisher information matrix of  $\boldsymbol{\theta}_0$  in the corresponding nonaccelerated test, denoted as  $\mathcal{I}_W(\boldsymbol{\theta}_0)$ , is

$$\mathcal{I}_W(\boldsymbol{\theta}_0) = \frac{nt_M}{a^2} \begin{bmatrix} \mu_0^{-2b+2} + \frac{2ma^2b^2}{t_M} & \frac{2mab}{t_M} \\ \frac{2mab}{t_M} & \frac{2m}{t_M} \end{bmatrix}. \quad (4.2)$$

Substituting the Fisher information matrix  $\mathcal{I}_W(\boldsymbol{\theta})$  into (3.4) yields  $\text{AVar}_a(\hat{t}_q)$ , the asymptotic variance of the estimated lifetime quantile in an ADT plan. Similarly, substituting  $\mathcal{I}_W(\boldsymbol{\theta}_0)$  into (3.6) yields  $\text{AVar}_n(\hat{t}_q)$ , the asymptotic variance of  $\hat{t}_q$  in the corresponding nonaccelerated test. Since the test duration  $t_M$ , the number of measurements  $m$ , and the failure threshold  $\mathcal{D}_f$  are assumed to be predetermined, the asymptotic variances of the estimated lifetime quantiles depend on the model parameters as well as the acceleration relation index  $b$ . When the acceleration relation index  $b = 1$ , one finds that

$$\text{AVar}_a(\hat{t}_q) > \text{AVar}_n(\hat{t}_q), \quad (4.3)$$

for all  $\boldsymbol{\theta}$  (see Supplement, Section S.3). Acceleration is unnecessary in this case. When  $b = 1$ , the volatility parameter  $\sigma_i$  can be expressed as a linear function of the drift parameter  $\mu_i$ ,  $\sigma_i = a\mu_i$  for  $i = 1, \dots, r$ . Therefore, the ratio of mean to standard deviation of a degradation signal, the signal-to-noise ratio (SNR), remains constant for all stress levels. Information contained in a degradation measurement under high stress levels is as large as that under normal use conditions, but the introduction of additional parameters in an ADT model requires more information to achieve the same level of precision. Therefore, acceleration lowers the estimation precision and thus is unnecessary. When the acceleration relation index  $b > 1$ , the volatility parameter  $\sigma_i$  can be expressed as a power function of  $\mu_i$  with a greater-than-one exponent. The SNR of the degradation data decreases with the drift parameter  $\mu_i$  and thus decreases with the stress levels. A high stress will cause a decrease of statistical information contained a degradation measurement, not to mention the information consumption caused by additional parameters. Therefore, when  $b \geq 1$ , acceleration is always unnecessary in a Wiener process. Epistemic uncertainty may be introduced into the ADTs due to the possible deviance between the assumed stress-acceleration relation and the true one. In its presence, it is even less likely that acceleration is necessary or desirable.

When  $b < 1$ , increase of the drift parameter  $\mu_i$  improves the SNR in the degradation data, and additional information provided by acceleration should be

sufficiently large to compensate the information consumption due to the increase in the number of parameters. Considering this trade-off, there exists a region of  $\boldsymbol{\theta}$  where acceleration of the degradation process is unnecessary. For each  $\boldsymbol{\theta}_0 = (\delta_1, a)'$ ,  $\text{AVar}_n(\hat{t}_q)$  in the corresponding nonaccelerated test is fixed, while the value of  $\text{AVar}_a(\hat{t}_q)$  in an optimum ADT plan decreases with the accelerating parameter  $\delta_2$ , as the SNR increases with  $\delta_2$  when  $b < 1$ . With the increase of  $\delta_2$ , the value of  $\text{AVar}_a(\hat{t}_q)$  in the optimal ADT plan is greater than  $\text{AVar}_n(\hat{t}_q)$  at the beginning, and then smaller than  $\text{AVar}_n(\hat{t}_q)$ , hence there is a break-even point for  $\delta_2$ . Denote this break-even point of the accelerating parameter  $\delta_2$  as  $\delta_2^*$ . When  $\delta_2 < \delta_2^*$ , acceleration is unnecessary.

The value of the break-even point  $\delta_2^*$  depends on the model parameters  $\delta_1$  and  $a$ , as well as the acceleration relation index  $b$ , but it is difficult to obtain an analytical form for  $\delta_2^*$ . To visualize the effects of  $\delta_1$ ,  $a$ , and  $b$  on the necessity of acceleration, we calculated  $\delta_2^*$  under different settings of the three parameters. Without loss of generality, we considered ADTs with two stress levels. The number of test units was set as  $n = 100$ , with  $\mathcal{D}_f = 100$ ,  $m = 10$ ,  $t_M = 10$ , and  $q = 0.1$ . Our simulations showed that these parameters have negligible impacts on the value of  $\delta_2^*$  (see Supplement, Section S.9). Settings of  $\delta_1$ ,  $a$ , and  $b$  were chosen based on the estimations from existing degradation datasets. Figure 1(a) plots  $\delta_2^*$  for  $b \in [0, 1)$  with  $\delta_1 = -2, -1, 0.1, \text{ and } 1$  when  $a = 0.5$ . Similarly, Figure 1(b) gives  $\delta_2^*$  for  $b \in [0, 1)$  with  $a = 0.1, 0.5, 1, \text{ and } 1.5$  while  $\delta_1$  is fixed at 1. As can be seen,  $\delta_2^*$  is not sensitive to the model parameter  $a$  when  $0 < a \leq 1$ . When  $a > 1$ ,  $\delta_2^*$  decreases with  $a$ , especially when  $b$  is close to one. Given  $a$  and  $\delta_1$ ,  $\delta_2^*$  increases with  $b$ . When  $b$  is close to one,  $\delta_2^*$  goes to infinity, suggesting that acceleration is unnecessary for almost all  $\delta_2$  in this case. Additionally, when  $b = 0$ ,  $\delta_2^*$  does not depend on the model parameters  $\delta_1$  and  $a$  in a Wiener process model. In this case,  $\delta_2^*$  is around 1.28 for all combinations of  $(\delta_1, a)'$  (see Supplement, Section S.4).

#### 4.2. Gamma process and IG process

Under the gamma process assumption,  $\Delta X_{ijl} \sim Ga(k_i \Delta t, \theta_i)$  where  $k_i = \mu_i^2 / \sigma_i^2$ , and  $\theta_i = \sigma_i^2 / \mu_i$  (Table 1). Based on the data  $\mathbf{D}$ , the Fisher information matrix of  $\boldsymbol{\theta}$ , denoted as  $\mathcal{I}_{Ga}(\boldsymbol{\theta})$ , is (see Supplement, Section S.5)

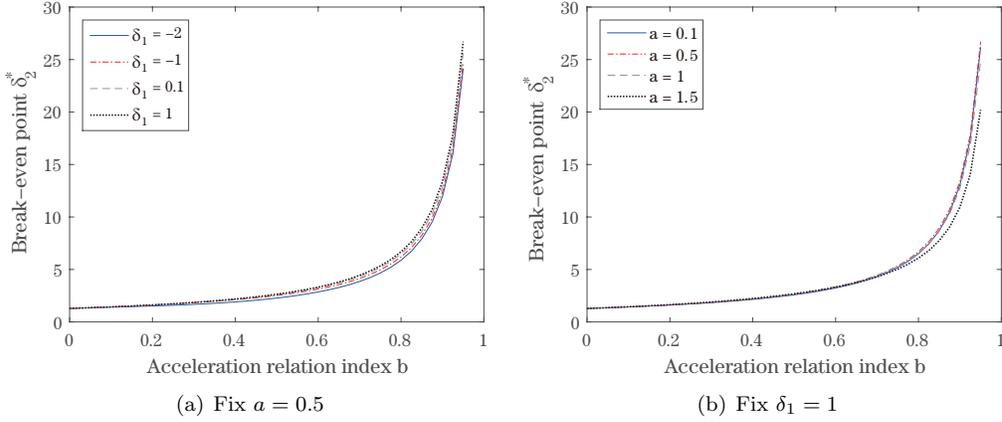


Figure 1. Values of the break-even point  $\delta_2^*$  under different settings of the model parameters  $\delta_1$  and  $a$  when the acceleration relation index  $b \in [0, 1)$  in a Wiener process.

$$\mathcal{I}_{Ga}(\boldsymbol{\theta}) = \frac{4nt_M^2}{a^2m} \begin{bmatrix} \frac{(1-b)^2}{a^2}A + \frac{(3-2b)(2b-1)m}{4t_M}B & \frac{(1-b)^2}{a^2}C + \frac{(3-2b)(2b-1)m}{4t_M}D & \frac{-(1-b)}{a^3}A + \frac{(1-b)m}{at_M}B \\ \text{symmetric} & \frac{(1-b)^2}{a^2}E + \frac{(3-2b)(2b-1)m}{4t_M}F & \frac{-(1-b)}{a^3}C + \frac{(1-b)m}{at_M}D \\ & & \frac{1}{a^4}A - \frac{m}{a^2t_M}B \end{bmatrix}, \tag{4.4}$$

where

$$A = \sum_{i=1}^r \pi_i \psi_{1i} \mu_i^{4(1-b)}, \quad B = \sum_{i=1}^r \pi_i \mu_i^{2(1-b)}, \quad C = \sum_{i=1}^r s_i \pi_i \psi_{1i} \mu_i^{4(1-b)}, \\
 D = \sum_{i=1}^r s_i \pi_i \mu_i^{2(1-b)}, \quad E = \sum_{i=1}^r s_i^2 \pi_i \psi_{1i} \mu_i^{4(1-b)}, \quad F = \sum_{i=1}^r s_i^2 \pi_i \mu_i^{2(1-b)},$$

and  $\psi_{1i} = \psi_1(\mu_i^{2(1-b)} \Delta t / a^2)$  with the trigamma function  $\psi_1(\cdot)$ .

Substituting the Fisher information matrix  $\mathcal{I}_{Ga}(\boldsymbol{\theta})$  into (3.4) yields  $\text{AVar}_a(\hat{t}_q)$  in an ADT. The asymptotic variance  $\text{AVar}_n(\hat{t}_q)$  in the corresponding nonaccelerated test can also be obtained using (3.6). When the acceleration relation index  $b = 1$ , the asymptotic variance  $\text{AVar}_a(\hat{t}_q)$  in an ADT plan is always larger than  $\text{AVar}_n(\hat{t}_q)$  in the corresponding nonaccelerated test (see Supplement, Section S.6). Acceleration is thus unnecessary in this case. When  $b > 1$ , applying acceleration in a degradation test increases the number of parameters, and lowers the SNR in the degradation data. Acceleration is unnecessary when  $b \geq 1$ .

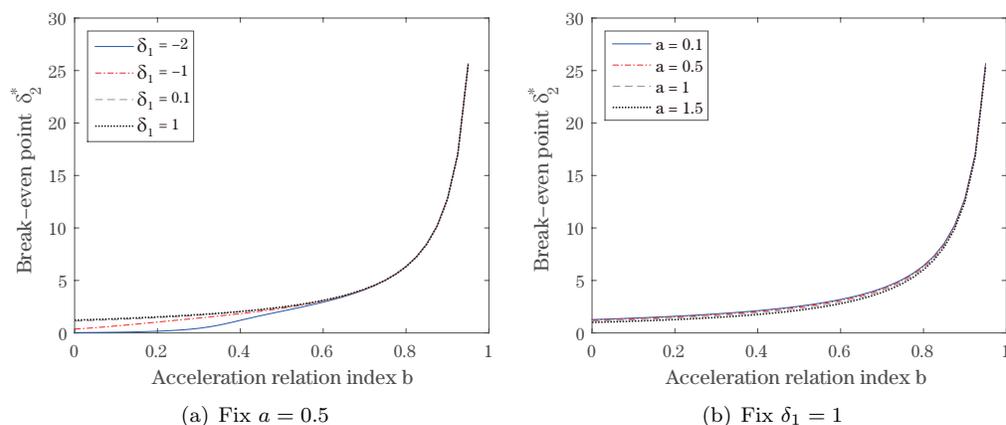


Figure 2. Values of  $\delta_2^*$  under different settings of the model parameters  $\delta_1$  and  $a$  when the acceleration relation index  $b \in [0, 1)$  in a gamma process.

When  $b < 1$ , acceleration increases the SNR in the degradation data. However, because of the information consumption from the additional parameters, there exists a region of  $\theta$  where the estimation precision in an ADT is lower than that in a nonaccelerated test. As the value of  $\text{AVar}_n(\hat{t}_q)$  in the corresponding degradation test is determined by  $\theta_0 = (\delta_1, a)'$ , while the value of  $\text{AVar}_a(\hat{t}_q)$  in an optimal ADT plan decreases with the accelerating parameter  $\delta_2$ , for each  $\theta_0 = (\delta_1, a)'$ , there exists a break-even point of the accelerating parameter  $\delta_2$ ,  $\delta_2^*$ . When  $\delta_2 < \delta_2^*$ , the optimum value of  $\text{AVar}_a(\hat{t}_q)$  is greater than  $\text{AVar}_n(\hat{t}_q)$ , acceleration of the degradation process is unnecessary. The value of  $\delta_2^*$  can be numerically calculated under different settings of  $\delta_1$ ,  $a$ , and  $b$ . Figure 2(a) plots the value of  $\delta_2^*$  with different settings of  $\delta_1$  when  $b \in [0, 1)$ . When  $\delta_1 > 0$ ,  $\delta_2^*$  is not sensitive to  $\delta_1$ . When  $\delta_1 < 0$  and  $b \leq 1/2$ ,  $\delta_2^*$  increases with  $\delta_1$ . Another notable result is that when  $\delta_1 \leq -1$  and  $b$  is close to zero,  $\delta_2^*$  is around zero, suggesting that acceleration is almost always meaningful in this case. On the other hand, Figure 2(b) shows that  $\delta_2^*$  is not sensitive to the parameter  $a$  in a gamma process. As shown in Figure 2,  $\delta_2^*$  increases with  $b$  when  $\delta_1$  and  $a$  are fixed. Moreover,  $\delta_2^*$  goes to infinity as  $b$  goes to one, suggesting that acceleration is unnecessary for almost all  $\delta_2$ .

For the IG process model,  $\Delta X_{ijl} \sim \mathcal{IG}(\alpha_i \Delta t, \beta_i \Delta t^2)$ , where  $\alpha_i = \mu_i$  and  $\beta_i = \mu_i^3 / \sigma_i^2$  (see Table 1). Based on the data  $\mathbf{D}$ , the Fisher information matrix of  $\theta$  is (see Supplement, Section S.7)

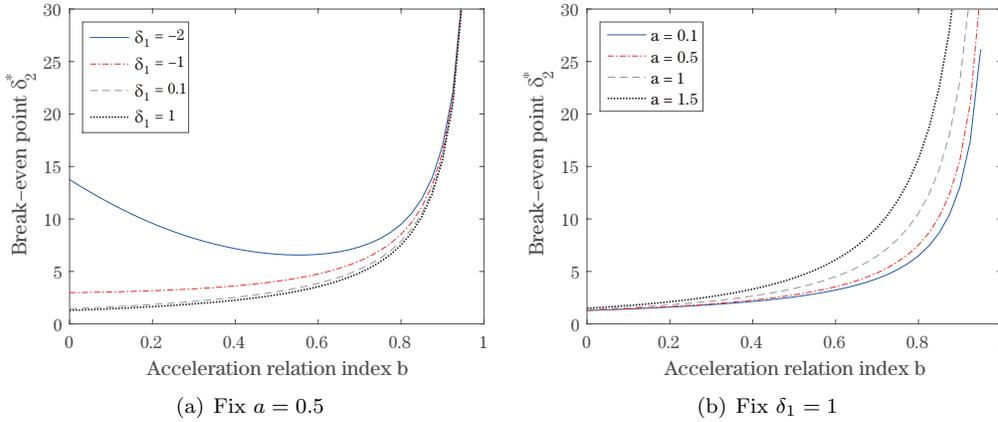


Figure 3. Values of  $\delta_2^*$  under different settings of the model parameters  $\delta_1$  and  $a$  when the acceleration relation index  $b \in [0, 1)$  in an IG process.

$$\mathcal{I}_{IG}(\boldsymbol{\theta}) = \frac{nt_M}{a^2} \begin{bmatrix} p + \sum_{i=1}^r \pi_i \mu^{2-2b} & \sum_{i=1}^r s_i \pi_i \mu^{2-2b} + p \sum_{i=1}^r s_i \pi_i & \frac{ma(2b-1)}{t_M} \\ \sum_{i=1}^r s_i^2 \pi_i \mu^{2-2b} + p \sum_{i=1}^r s_i^2 \pi_i & \frac{ma(2b-1)}{t_M} \sum_{i=1}^r s_i \pi_i & \\ \text{symmetric} & & \frac{2m}{t_M} \end{bmatrix}, \quad (4.5)$$

where  $p = (a^2 m) / (2t_M)(3 - 4b + 4b^2)$ .

Again, acceleration of the degradation process is unnecessary when the acceleration relation index  $b \geq 1$ . When  $b < 1$ , for each  $\boldsymbol{\theta}_0 = (\delta_1, a)'$  with a certain acceleration relation index  $b$ , there exists a break-even point  $\delta_2^*$  of the accelerating parameter  $\delta_2$ . When  $\delta_2 < \delta_2^*$ , acceleration is unnecessary. The value of  $\delta_2^*$  is numerically calculated in a two-point plan. Figure 3(a) plots  $\delta_2^*$  under different values of  $\delta_1$  when  $b \in [0, 1)$ . When  $\delta_1 > 0$ ,  $\delta_2^*$  is not sensitive to  $\delta_1$ . When  $\delta_1 < 0$ ,  $\delta_2^*$  increases with  $\delta_1$ . Figure 3(b) shows that  $\delta_2^*$  increases with  $a$  when  $b \in [0, 1)$ . As shown in Figure 3,  $\delta_2^*$  goes to infinity when  $b$  goes to one. This means that acceleration is unnecessary for almost all  $\delta_2$  in this scenario.

### 4.3. Some further remarks

In practical applications, estimates of the model parameter  $\boldsymbol{\theta}$  and an appropriate acceleration relation index  $b$  can be roughly obtained by conducting a

preliminary ADT. When the acceleration relation index  $b \geq 1$ , acceleration of the degradation process is unnecessary in the Wiener, gamma, and IG processes. Subsequent degradation tests can be conducted under normal use conditions. When  $b < 1$ , for each  $\theta_0 = (\delta_1, a)'$ , there exists a break-even point  $\delta_2^*$  of the accelerating parameter  $\delta_2$ . Figures 1-3 gives the value of  $\delta_2^*$  under different settings of  $\delta_1$ ,  $a$ , and  $b$  in these three models. One can identify the necessity of acceleration by simply comparing the estimate of  $\delta_2$  with the break-even point  $\delta_2^*$ . For example, while assuming a constant volatility parameter (i.e.,  $b = 0$ ) in a Wiener process, acceleration is unnecessary if the estimate of  $\delta_2$  is less than  $\delta_2^* = 1.28$ . A numerical example is given in Section 6 to further demonstrate the use of the proposed procedure.

## 5. Exponential Dispersion Model

We investigate a class of exponential dispersion (ED) degradation models that unifies the three models (Tseng and Lee (2016)). A stationary ED degradation model has independent increments, and  $X(t)$  is  $ED(\mu t, \lambda)$  with PDF

$$f_{ED}(x; \varpi, \lambda) = c(x; \lambda, t) \exp\{\lambda[x\varpi - t\kappa(\varpi)]\}, x > 0,$$

where the normalization term  $c(\cdot)$  is a function of the dispersion parameter  $\lambda$ , while  $\varpi$  and  $\kappa(\varpi)$  are suitable functions of the mean parameter  $\mu$  such that  $\mu = \kappa'(\varpi)$ . Tseng and Lee (2016) assumed that the mean parameter  $\mu$  is an increasing function of the stress levels, while the dispersion parameter  $\lambda$  is a constant. This study extends their work and considers a generic form of the acceleration relations.

Denote the mapping from  $\mu$  to the second derivative of  $\kappa$  with respect to  $\varpi$  as  $V(\cdot)$ ,  $V(\mu) = \kappa''(\varpi)$ . Based on the properties of the natural exponential families, the mean and variance of  $X(t)$  are  $\mu t$  and  $V(\mu)t/\lambda$ , respectively. An important class of the ED models is known as the Tweedie models (Tweedie (1984)), where  $V(\mu)$  is

$$V(\mu) = \mu^d, \quad d \in (-\infty, 0] \cup [1, \infty).$$

As shown in Table 2, the Wiener, gamma, and IG processes are special cases of the Tweedie models with  $d = 0, 2$ , and  $3$ , respectively. We unify the mean and variance of  $X(t)$  in the class of ED models as  $\mu t$  and  $\sigma^2 t$ , respectively. Then, the dispersion parameter is  $\lambda = \mu^d/\sigma^2$ .

When a product is subject to higher stress levels, we adopt the acceleration relations given in Section 2. Under the standardized stress level  $s_i$  ( $i = 1, \dots, r$ ),

Table 2. Wiener, gamma and IG process as Tweedie models.

|                    | Wiener   | gamma   | IG   |
|--------------------|--|---|--|
| $c(x; \lambda, t)$ | $\sqrt{\frac{\lambda}{2\pi t}} \exp\left(-\frac{\lambda x^2}{2t}\right)$ | $\frac{\lambda^{\lambda t}}{\Gamma(\lambda t)} x^{\lambda t-1}$ | $\sqrt{\frac{\lambda t^2}{2\pi x^3}} \exp\left(-\frac{\lambda t^2}{2x}\right)$ |
| $\kappa(\varpi)$   | $\varpi^2/2$   | $-\ln(-\varpi)$   | $-(-2\varpi)^{1/2}$  |
| $V(\mu)$           | 1  | $\mu^2$   | $\mu^3$  |
| $d$                | 0  | 2   | 3  |

$u_i = \exp(\delta_1 + \delta_2 s_i)$ , and  $\sigma_i = a\mu_i^b$ , where  $\delta_1, \delta_2 > 0$  and  $a > 0$  are model parameters, and  $b$  is the acceleration relation index.

Following Jorgensen (1997), we define the mapping from  $\varpi$  to  $\mu$  as the mean value mapping  $\eta(\cdot)$ ,  $\mu = \eta(\varpi)$  and  $\varpi = \eta^{-1}(\mu)$ . An approximation of  $c(x; \lambda, t)$  proposed by Jorgensen (1997, Chapter 3.5) is

$$c(x; \lambda, t) \sim \exp\left\{-\frac{\lambda}{t^{1-d}} [x\tilde{\varpi} - \kappa(\tilde{\varpi})]\right\} \left[\frac{\lambda}{2\pi V(x)t^{1-d}}\right]^{1/2},$$

where  $\tilde{\varpi} = \eta^{-1}(x)$ . The approximation is accurate especially when  $\lambda$  large.

Based on the degradation data  $\mathbf{D}$ , the log-likelihood function of  $n$  test units (up to a constant) can be expressed as:

$$\begin{aligned} \ell(\boldsymbol{\theta}) &= \sum_{i=1}^r \sum_{j=1}^{n_i} \sum_{l=1}^m \left[-C_{ijl}\lambda_i + \frac{1}{2} \ln \lambda_i + \lambda_i[\varpi_i \Delta x_{ijl} - \kappa(\varpi_i) \Delta t]\right] \\ &= \sum_{i=1}^r \sum_{j=1}^{n_i} \sum_{l=1}^m \left[-\frac{\mu_i^{d-2b}}{a^2} C_{ijl} + \frac{d-2b}{2} \ln \mu_i - \ln a + \frac{\mu_i^{d-2b}}{a^2} [\varpi_i \Delta x_{ijl} - \kappa(\varpi_i) \Delta t]\right], \end{aligned}$$

where  $C_{ijl} = [\Delta x_{ijl} \tilde{\varpi}_{ijl} - \kappa(\tilde{\varpi}_{ijl})] / \Delta t^{1-d}$  is not related to  $\lambda_i$ .

By the chain rule,

$$\frac{\partial \varpi_i}{\partial \delta_1} = \frac{1}{V(\mu_i)} \frac{\partial \mu_i}{\partial \delta_1}, \quad \frac{\partial \varpi_i}{\partial \delta_2} = \frac{1}{V(\mu_i)} \frac{\partial \mu_i}{\partial \delta_2}.$$

Therefore, elements of the Fisher information matrix of  $\boldsymbol{\theta}$  in an ED model are given as (see Supplement, Section S.8)

$$\begin{aligned} E\left[-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \delta_1^2}\right] &= \frac{nt_M}{a^2} \sum_{i=1}^r \pi_i \mu_i^{2-2b} + \frac{mn(d-2b)^2}{a^2} \sum_{i=1}^r \pi_i C_i, \\ E\left[-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \delta_1 \partial \delta_2}\right] &= \frac{nt_M}{a^2} \sum_{i=1}^r s_i \pi_i \mu_i^{2-2b} + \frac{mn(d-2b)^2}{a^2} \sum_{i=1}^r s_i \pi_i C_i, \\ E\left[-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \delta_1 \partial a}\right] &= -\frac{2mn(d-2b)}{a^3} \sum_{i=1}^r \pi_i C_i, \end{aligned}$$

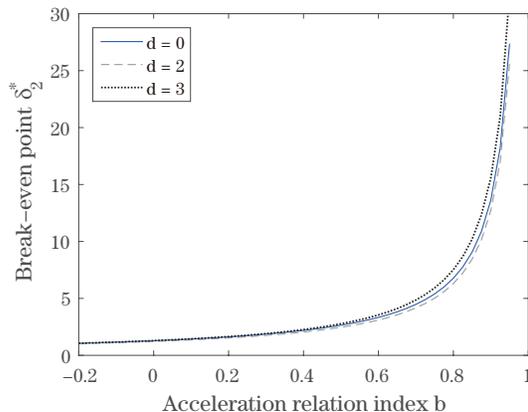


Figure 4. Relation between the acceleration relation index  $b$  and  $\delta_2^*$  under different settings of the model index  $d$  ( $\delta_1 = 1, a = 0.5$ ).

$$\begin{aligned}
 E \left[ -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \delta_2^2} \right] &= \frac{nt_M}{a^2} \sum_{i=1}^r s_i^2 \pi_i \mu_i^{2-2b} + \frac{mn(d-2b)^2}{a^2} \sum_i^r s_i^2 \pi_i C_i, \\
 E \left[ -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \delta_2 \partial a} \right] &= -\frac{2mn(d-2b)}{a^3} \sum_{i=1}^r \pi_i s_i C_i, \\
 E \left[ -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial a^2} \right] &= \frac{mn}{a^2} + \frac{6mn}{a^4} \sum_{i=1}^r \pi_i C_i,
 \end{aligned} \tag{5.1}$$

where

$$C_i = E \left\{ \sum_{j=1}^{n_i} \sum_{l=1}^m \mu_i^{d-2b} [\varpi_i \Delta x_{ijl} - \kappa(\varpi_i) \Delta t - C_{ijl}] \right\}.$$

The Fisher information matrices of  $\boldsymbol{\theta}$  in the class of ED models are only different in  $C_i (i = 1, \dots, r)$  and the model index  $d$ . When the elements containing  $\mu_i^{1-b}$  have dominant influence on the Fisher information matrices, the asymptotic variance of the estimated lifetime quantile in an ADT plan mainly depends on the acceleration relation index  $b$  rather than the model index  $d$ . Therefore, there exists a region of  $\boldsymbol{\theta}$  where the necessity of acceleration is similar in the class of ED models. Our simulation experience in a two-point plan suggests that, when the model parameters  $0 < a < 1$  and  $\delta_1 > 0$ , the value of the break-even point  $\delta_2^*$  is not sensitive to the model index  $d$ . For example, Figure 4 plots the value of  $\delta_2^*$  with  $d = 0, 2$ , and  $3$  when  $\delta_1 = 1, a = 0.5$  and  $b \in [-0.2, 1)$ . The patterns of  $\delta_2^*$  in these three models are similar. Therefore, when the elements containing  $\mu_i^{1-b}$  in

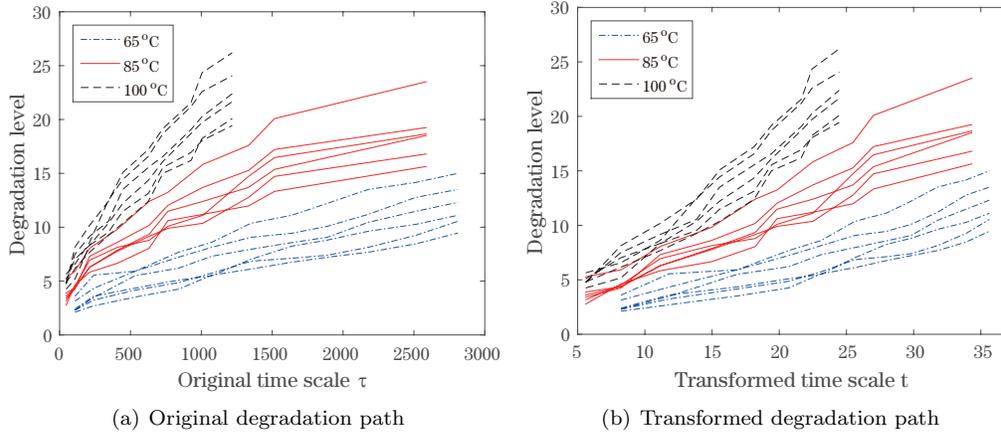


Figure 5. Original degradation path and transformed degradation path with  $t = \tau^{0.45}$  of the stress relaxation data.

the Fisher information matrix of  $\theta$  are dominating, the necessity of acceleration mainly depends on the acceleration relations, rather than the specific underlying degradation model. The results indicate that identifying the necessity of acceleration is robust to model misspecification, given that the degradation rate  $\mu_i$  and the degradation volatility  $\sigma_i$  can be accurately estimated. See Table 3 in the next section for example.

## 6. Illustrative Example

The stress relaxation data in Yang (2007) are used to illustrate the procedure of identifying the necessity of acceleration. Stress relaxation of an electrical connector (in percentage) is the observed decrease in stress in response to the same amount of strain over time. A connector is considered to have failed if the stress relaxation exceeds a specific failure threshold  $\mathcal{D}_f$  (e.g.,  $\mathcal{D}_f = 30\%$ ). It is of interest to estimate the lifetime quantile of the connector under normal use conditions ( $\tilde{s}_0 = 40^\circ\text{C}$ ). An ADT was thus conducted with temperature levels 65°C, 85°C, and 100°C. A total of 18 test units were randomly selected, divided into three equal groups, and allocated to these stress levels. The original degradation paths are displayed in Figure 5(a). An empirical power transformation  $t = \tau^{0.45}$  was applied on the chronological time to linearize the degradation paths (Tseng and Lee (2016)), where  $\tau$  is the chronological time and  $t$  is the transformed time scale. As shown in Figure 5(b), the transformed degradation paths is nearly linear. The Arrhenius relation is applied to characterize the effect of the test temperature

Table 3. ML estimates, maximum log-likelihoods, and AIC of different acceleration relations in Wiener, gamma and IG processes.

| Acceleration<br>relation index $b$ | Stochastic<br>process models | Model parameters |                  |           | Maximum<br>log-likelihoods | AIC    |
|------------------------------------|------------------------------|------------------|------------------|-----------|----------------------------|--------|
|                                    |                              | $\hat{\delta}_1$ | $\hat{\delta}_2$ | $\hat{a}$ |                            |        |
| 0                                  | Wiener                       | -2.13            | 2.02             | 0.52      | -202.79                    | 411.58 |
|                                    | gamma                        | -1.56            | 1.34             | 0.48      | -192.32                    | 390.64 |
|                                    | IG                           | -1.37            | 1.30             | 0.61      | -176.10                    | 358.21 |
| 1/2                                | Wiener                       | -2.16            | 2.06             | 0.65      | -187.64                    | 381.27 |
|                                    | gamma                        | -1.90            | 1.74             | 0.60      | -183.14                    | 372.29 |
|                                    | IG                           | -1.64            | 1.61             | 0.70      | -170.65                    | 347.31 |
| 1                                  | Wiener                       | -1.91            | 1.75             | 0.87      | -184.08                    | 374.15 |
|                                    | gamma                        | -1.93            | 1.77             | 0.82      | -181.83                    | 369.66 |
|                                    | IG                           | -1.75            | 1.74             | 0.86      | -169.21                    | 344.41 |
| 3/2                                | Wiener                       | -1.70            | 1.42             | 1.25      | -186.56                    | 379.13 |
|                                    | gamma                        | -1.72            | 1.45             | 1.18      | -190.49                    | 386.99 |
|                                    | IG                           | -1.65            | 1.60             | 1.13      | -175.53                    | 357.06 |

$\tilde{s}$ . Corresponding normalization of the stress levels is obtained using (2.1), with  $\psi(\tilde{s}) = 1/\tilde{s}$ .

We fit the data using Wiener, gamma, and IG processes by considering several commonly used acceleration relations:  $b = 0, 1/2, 1,$  and  $3/2$ . The Akaike information criterion (AIC) was used for model selection. Table 3 summarizes the ML estimates of  $\theta$ , the maximum log-likelihood, and the AIC of different models. The models were divided into four groups with  $b = 0, 1/2, 1,$  and  $3/2$ , respectively. As can be seen, the group with  $b = 1$  has a relatively smaller AIC compared with other groups. Within this group, the IG process model shows the best fit. According to our previous discussions, acceleration is unnecessary in this case, as  $b = 1$ .

To verify the necessity of acceleration, we compared the asymptotic variance of the estimated lifetime quantile between an optimum ADT plan and the corresponding nonaccelerated test. Consider a two-point plan with two stress levels. The number of test units  $n$ , number of measurements  $m$ , maximum test duration  $t_M$ , and the failure threshold  $\mathcal{D}_f$  were fixed at  $n = 30, m = 10, t_M = 10,$  and  $\mathcal{D}_f = 30$ . The 0.1th quantile of the lifetime distribution was estimated. The asymptotic variance of the estimated lifetime quantile in the optimum ADT plan was  $\text{AVar}_a(\hat{t}_q) = 9.28$ , with the optimal stress levels  $s_1 = 0, s_2 = 1,$  and the corresponding allocation  $n_1 = 29, n_2 = 1$ . In the corresponding nonaccelerated test, the asymptotic variance of the estimated lifetime quantile was obtained as  $\text{AVar}_n(\hat{t}_q) = 9.00$ , smaller than that using acceleration.

## 7. Conclusions

Our study may be extended in several directions. The first is to consider measurement errors in a degradation model. The variation of a degradation process usually has two major causes, internal degradation variation and external measurement errors. Therefore, it is meaningful to consider the impact of measurement errors on the necessity of acceleration. As the measurement error is usually assumed normally distributed, it is straightforward to incorporate the measurement error into a Wiener process. In addition, a random-effects model can be considered to capture the heterogeneity among test units, where some parameters in the Wiener, gamma, and IG processes are treated as random across the population. More flexible measurement schedules and more acceleration schemes can also be considered.

## Supplementary Materials

The supplementary materials provide proofs that are not included in the main paper. In addition, we investigate the impacts of the failure threshold  $\mathcal{D}_f$ , number of measurements  $m$ , test duration  $t_M$ , and the quantile  $q$  on the necessity of acceleration.

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