Sensitivity Analysis of Nongaussianity by Projection Pursuit

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Supplementary Material

1 Appendix

This technical section gives the details of derivation of the linear system (14) and the exact expressions for the λ functions and the π functions used thereof. Let us start with reminding the readers of the defining system of the statistic $\mathbf{T}$. For every $\varepsilon \geq 0$, $\mathbf{T}^*_\varepsilon$ is a solution to

$$\mathbf{H}(\mathbf{T}) = (H_1(\mathbf{T}), \dots, H_{n-1}(\mathbf{T})) = \mathbf{0},$$

i.e.,

$$\mathbf{H}(\mathbf{T}^*_\varepsilon; F_\varepsilon) = (H_1(\mathbf{T}^*_\varepsilon; F_\varepsilon), \dots, H_{n-1}(\mathbf{T}^*_\varepsilon; F_\varepsilon)) = \mathbf{0},$$

where $H_k(\mathbf{T}) = h^k_1(\mathbf{T}) - 3h^k_2(\mathbf{T}) - 6h^k_3(\mathbf{T}) + 12h^k_4(\mathbf{T}) - 4h^k_5(\mathbf{T})$ and

$$\begin{aligned}
h^k_1(\mathbf{T}) &= \phi^k_1(\mathbf{T}, 4),\\
h^k_2(\mathbf{T}) &= 2\phi_2(\mathbf{T}, 2)\phi^k_1(\mathbf{T}, 2),\\
h^k_3(\mathbf{T}) &= 4\rho^3_1(\mathbf{T})\rho^k_2(\mathbf{T}),\\
h^k_4(\mathbf{T}) &= 2\rho_1(\mathbf{T})\rho^k_2(\mathbf{T})\phi_2(\mathbf{T}, 2) + \rho^2_1(\mathbf{T})\phi^k_1(\mathbf{T}, 2),\\
h^k_5(\mathbf{T}) &= \rho^k_2(\mathbf{T})\phi_2(\mathbf{T}, 3) + \rho_1(\mathbf{T})\phi^k_1(\mathbf{T}, 3).
\end{aligned}$$

The expressions for the ϕ functions and the ρ functions are given in (4):(7). For fixed $k \in \{1, \dots, n-1\}$, implicitly differentiating with respect to ε the equation $H_k(\mathbf{T}^*_\varepsilon; F_\varepsilon) = 0$ yields

$$\frac{d}{d\varepsilon}\left[h^k_1(\mathbf{T}_\varepsilon; F_\varepsilon) - 3h^k_2(\mathbf{T}_\varepsilon; F_\varepsilon) - 6h^k_3(\mathbf{T}_\varepsilon; F_\varepsilon) + 12h^k_4(\mathbf{T}_\varepsilon; F_\varepsilon) - 4h^k_5(\mathbf{T}_\varepsilon; F_\varepsilon)\right] = 0,$$

which in turn boils down to determine the derivatives with respect to ε of the functions $\phi^k_1(\mathbf{T}_\varepsilon, p; F_\varepsilon)$, $\phi_2(\mathbf{T}_\varepsilon, p; F_\varepsilon)$, $\rho_1(\mathbf{T}_\varepsilon; F_\varepsilon)$, and $\rho^k_2(\mathbf{T}_\varepsilon; F_\varepsilon)$ for p being an integer. Let us work out the derivative of $\phi_2(\mathbf{T}_\varepsilon, p; F_\varepsilon)$ as an example; the other derivatives can be derived similarly but sometimes more tedious. Recall that

$$\phi_2(\mathbf{T}_\varepsilon, p; F_\varepsilon) = \sum_{|\mathbf{q}|=p}\left\{c^p_{\mathbf{q}}\prod_{i=1}^{n-1}\xi_{i,\varepsilon}^{p-|\mathbf{q}^{i+1}|}\left(1+\xi^2_{i,\varepsilon}\right)^{\frac{|\mathbf{q}^i|}{2}}\mathbb{E}[\mathbf{X}^{\mathbf{q}}; F_\varepsilon]\right\}.$$

Therefore, by the product rule (Leibnitz rule) of differentiation, we have

$$\begin{aligned}
&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\phi_2(\mathbf{T}_\varepsilon,p;F_\varepsilon)\\
&=\sum_{|\mathbf{q}|=p}\left\{c^p_{\mathbf{q}}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\prod_{i=1}^{n-1}\xi_{i,\varepsilon}^{p-|\mathbf{q}^{i+1}|}\right)\prod_{i=1}^{n-1}\left(1+\xi_i^2\right)^{\frac{|\mathbf{q}^i|}{2}}\mathbb{E}[\mathbf{X}^{\mathbf{q}}]\right.\\
&\quad+c^p_{\mathbf{q}}\prod_{i=1}^{n-1}\xi_i^{p-|\mathbf{q}^{i+1}|}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\prod_{i=1}^{n-1}\left(1+\xi_{i,\varepsilon}^2\right)^{\frac{|\mathbf{q}^i|}{2}}\right)\mathbb{E}[\mathbf{X}^{\mathbf{q}}]\\
&\quad\left.+c^p_{\mathbf{q}}\prod_{i=1}^{n-1}\xi_i^{p-|\mathbf{q}^{i+1}|}\left(1+\xi_i^2\right)^{\frac{|\mathbf{q}^i|}{2}}\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathbb{E}[\mathbf{X}^{\mathbf{q}};F_\varepsilon]\right)\right\}. \qquad (1.1)
\end{aligned}$$

which in turn amounts to differentiating $\prod_{i=1}^{n-1}\xi_{i,\varepsilon}^{p-|\mathbf{q}^{i+1}|}$, $\prod_{i=1}^{n-1}\left(1+\xi_{i,\varepsilon}^2\right)^{\frac{|\mathbf{q}^i|}{2}}$, and $\mathbb{E}[\mathbf{X}^{\mathbf{q}};F_\varepsilon]$. Note that in (1.1) we have evaluated the derivative of $\phi_2(\mathbf{T}_\varepsilon,p;F_\varepsilon)$ at $\varepsilon=0$ and have applied the convention that $\xi_{i,0}=\xi_i$, $\mathbb{E}[\cdot;F_0]=\mathbb{E}[\cdot]$. We have seen that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathbb{E}[\mathbf{X}^{\mathbf{q}};F_\varepsilon]=-\mathbb{E}[\mathbf{X}^{\mathbf{q}}]+\mathbf{x}^{\mathbf{q}}.$$

Consider the following terms,

$$\begin{aligned}
&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\prod_{i=1}^{n-1}\xi_{i,\varepsilon}^{p-|\mathbf{q}^{i+1}|}=\sum_{j=1}^{n-1}I_j(\mathbf{T},\mathbf{x})\left\{(p-|\mathbf{q}^{i+1}|)\xi_j^{p-|\mathbf{q}^{i+1}|-1}\prod_{i\neq j}\xi_i^{p-|\mathbf{q}^{i+1}|}\right\}\\
&\equiv\sum_{j=1}^{n-1}I_j(\mathbf{T},\mathbf{x})\psi_{3,j}(\mathbf{T},p,\mathbf{q})
\end{aligned}$$

$$\begin{aligned}
&\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\prod_{i=1}^{n-1}\left(1+\xi_{i,\varepsilon}^2\right)^{\frac{|\mathbf{q}^i|}{2}}=\sum_{j=1}^{n-1}I_j(\mathbf{T},\mathbf{x})\left\{|\mathbf{q}^j|\xi_j\left(1+\xi_j^2\right)^{\frac{|\mathbf{q}^j|}{2}-1}\prod_{i\neq j}\left(1+\xi_{i,\varepsilon}^2\right)^{\frac{|\mathbf{q}^i|}{2}}\right\}\\
&\equiv\sum_{j=1}^{n-1}I_j(\mathbf{T},\mathbf{x})\psi_{2,j}(\mathbf{T},p,\mathbf{q}).
\end{aligned}$$

Now plug the above two expressions back into (1.1) we have

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\phi_2(\mathbf{T}_\varepsilon,p;F_\varepsilon)=\sum_{j=1}^{n-1}I_j(\mathbf{T},\mathbf{x})\xi_{2,j}(\mathbf{T},p)-\phi_2(\mathbf{T},p)+\phi_2(\mathbf{T},p,\mathbf{x}), \qquad (1.2)$$

where

$$\begin{aligned}
\xi_{2,j}(\mathbf{T},p)&=\sum_{|\mathbf{q}|=p}c^p_{\mathbf{q}}\left(\psi_{3,j}(\mathbf{T},p,\mathbf{q})\prod_{i=1}^{n-1}\left(1+\xi_i^2\right)^{\frac{|\mathbf{q}^i|}{2}}+\psi_{2,j}(\mathbf{T},p,\mathbf{q})\prod_{i=1}^{n-1}\xi_i^{p-|\mathbf{q}^{i+1}|}\right)\mathbb{E}[\mathbf{X}^{\mathbf{q}}],\\
\phi_2(\mathbf{T},p,\mathbf{x})&=\sum_{|\mathbf{q}|=p}c^p_{\mathbf{q}}\left(\prod_{i=1}^{n-1}\xi_i^{p-|\mathbf{q}^{i+1}|}\left(1+\xi^2\right)^{\frac{|\mathbf{q}^i|}{2}}\right)\mathbf{x}^{\mathbf{q}}. \qquad (1.3)
\end{aligned}$$

Here we remark that the function $\phi_2(\mathbf{T}, p, \mathbf{x})$ above is exactly the function obtained by replacing the expectation term $\mathbb{E}[\mathbf{X}^{\mathbf{q}}]$ in (5) by $\mathbf{x}^{\mathbf{q}}$.

The derivatives of $\phi_1^k(\mathbf{T}_\varepsilon, p; F_\varepsilon)$, $\rho_1(\mathbf{T}_\varepsilon; F_\varepsilon)$, and $\rho_2^k(\mathbf{T}_\varepsilon; F_\varepsilon)$ can be calculated similarly. We list the formulas in the following without detailed derivations.

$$\begin{aligned}
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi_1^k(\mathbf{T}_\varepsilon, p; F_\varepsilon) &= \sum_{j=1}^{n-1} I_j(\mathbf{T}, \mathbf{x})\xi_{1,j}^k(\mathbf{T}, p) - \phi_1^k(\mathbf{T}, p) + \phi_1^k(\mathbf{T}, p, \mathbf{x}),\\
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \rho_1(\mathbf{T}_\varepsilon; F_\varepsilon) &= \sum_{j=1}^{n-1} I_j(\mathbf{T}, \mathbf{x})\varrho_{1,j}(\mathbf{T}) - \rho_1(\mathbf{T}) + \rho_1(\mathbf{T}, \mathbf{x}),\\
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \rho_2^k(\mathbf{T}_\varepsilon; F_\varepsilon) &= \sum_{j=1}^{n-1} I_j(\mathbf{T}, \mathbf{x})\varrho_{2,j}^k(\mathbf{T}) - \rho_2^k(\mathbf{T}) + \rho_2^k(\mathbf{T}, \mathbf{x}),
\end{aligned}$$

where

$$\begin{aligned}
\xi_{1,j}^k(\mathbf{T}, p) = \sum_{|\mathbf{q}|=p} c_{\mathbf{q}}^p \Bigg(& \psi_{1,j}^k(\mathbf{T}, p, \mathbf{q}) \prod_{i=1}^{n-1}\left(1+\xi_i^2\right)^{\frac{|\mathbf{q}^i|}{2}} + \psi_{2,j}(\mathbf{T}, p, \mathbf{q}) \prod_{i\neq k} \xi_i^{p-|\mathbf{q}^{i+1}|}\\
& \left((p - |\mathbf{q}^{k+1}|)\xi_k^{p-1-|\mathbf{q}^{k+1}|} - q_k \xi_k^{p+1-|\mathbf{q}^{k+1}|}\right)\Bigg)\mathbb{E}[\mathbf{X}^{\mathbf{q}}]
\end{aligned}$$

$$\phi_1^k(\mathbf{T}, p, \mathbf{x}) = \sum_{|\mathbf{q}|=p} c_{\mathbf{q}}^p \left(\prod_{i\neq k} \xi_i^{p-|\mathbf{q}^{i+1}|} \prod_{i=1}^{n-1}\left(1+\xi_i^2\right)^{\frac{|\mathbf{q}^i|}{2}} \left[(p - |\mathbf{q}^{k+1}|)\xi_k^{p-1-|\mathbf{q}^{k+1}|} - q_k \xi_k^{p+1-|\mathbf{q}^{k+1}|}\right]\right)\mathbf{x}^{\mathbf{q}}, \tag{1.4}$$

$$\begin{aligned}
\varrho_{1,j}(\mathbf{T}) = \sum_{i=1}^{n} \Bigg(& \prod_{l=1, l\neq j}^{i-1} \xi_l \prod_{l=i+1}^{n-1} (1+\xi_l^2)^{\frac{1-\delta_{in}}{2}} \mathbb{E}[X_i] 1_{\{i>j\}}\\
& + \prod_{l=1}^{i-1} \xi_l \prod_{l=1, l\neq j}^{n-1} (1+\xi_l^2)^{\frac{1-\delta_{in}}{2}} (1-\delta_{in})\xi_j (1+\xi_j^2)^{\frac{-1-\delta_{in}}{2}} \mathbb{E}[X_i] 1_{\{i<j\}}\Bigg)
\end{aligned}$$

$$\begin{aligned}
\varrho_{2,j}^k(\mathbf{T}) = \sum_{i=1}^{n} \Bigg\{ & \prod_{l\neq j, l\neq k}^{i-1} \xi_l \prod_{l=i+1}^{n-1} (1+\xi_l^2)^{\frac{1-\delta_{in}}{2}} \left(\xi_k^2(1-\delta_{ik}) - (1-\delta_{in})\delta_{ik}\right)(1-\delta_{kj})\mathbb{E}[X_i] 1_{\{i>j\}}\\
+ \prod_{l\neq k}^{i-1} \xi_l \prod_{l\neq j}^{n-1} & (1+\xi_l^2)^{\frac{1-\delta_{in}}{2}} (1-\delta_{in})\xi_j (1+\xi_j^2)^{\frac{-1-\delta_{in}}{2}} \left(\xi_k^2(1-\delta_{ik}) - (1-\delta_{in})\delta_{ik}\right)(1-\delta_{kj})\mathbb{E}[X_i] 1_{\{i<j\}}\\
+ \prod_{l\neq k}^{i-1} \xi_l \prod_{l=i+1}^{n-1} & (1+\xi_l^2)^{\frac{1-\delta_{in}}{2}} \left(2\xi_k(1-\delta_{ik})\right)\delta_{kj}\Bigg\},
\end{aligned}$$

$$\rho_1(\mathbf{T}, \mathbf{x}) = \sum_{i=1}^{n} \left(\prod_{l=1}^{i-1} \xi_l \prod_{l=i+1}^{n-1} \left(1+\xi_l^2\right)^{\frac{1-\delta_{in}}{2}} x_i \right), \tag{1.5}$$

$$\rho_2^k(\mathbf{T}, \mathbf{x}) = \sum_{i=1}^{n} \left(\prod_{l=1, l\neq k}^{i-1} \xi_l \prod_{l=i+1}^{n-1} \left(1+\xi_l^2\right)^{\frac{1-\delta_{in}}{2}} \left[\xi_k^2(1-\delta_{ik}) - (1-\delta_{in})\delta_{ik}\right] x_i \right). \tag{1.6}$$

and

$$\begin{aligned}
&\psi_{1,j}^k(\mathbf{T},p,\mathbf{q})\\
&= (p-|\mathbf{q}^{j+1}|)\xi_j^{p-|\mathbf{q}^{j+1}|-1}\prod_{i\neq j, i\neq k}\xi_i^{p-|\mathbf{q}^{i+1}|}\left((p-|\mathbf{q}^{k+1}|)\xi_k^{p-|\mathbf{q}^{k+1}|-1}-q_k\xi_k^{p-|\mathbf{q}^{k+1}|+1}\right)(1-\delta_{jk})\\
&\quad+\prod_{i\neq k}\xi_i^{p-|\mathbf{q}^{i+1}|}\left[(p-|\mathbf{q}^{k+1}|)(p-|\mathbf{q}^{k+1}|-1)\xi_k^{p-|\mathbf{q}^{k+1}|-2}-q_k(p-|\mathbf{q}^{k+1}|+1)\xi_k^{p-|\mathbf{q}^{k+1}|}\right]\delta_{jk}.
\end{aligned}$$

Now we have the derivatives of the ϕ functions and the ρ functions, which are involved in the definition of h functions. Let us compute as an example the derivative of h_2^k, for fixed $k\in\{1,\ldots,n-1\}$, as follows.

$$\begin{aligned}
&\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} h_2^k(\mathbf{T}_\varepsilon,F_\varepsilon) = 2\,\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}[\phi_2(\mathbf{T}_\varepsilon,2;F_\varepsilon)\phi_1^k(\mathbf{T}_\varepsilon,2;F_\varepsilon)]\\
&= 2\phi_1^k(\mathbf{T},2)\,\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}[\phi_2(\mathbf{T}_\varepsilon,2;F_\varepsilon)] + 2\phi_2(\mathbf{T},2)\,\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}[\phi_1^k(\mathbf{T}_\varepsilon,2;F_\varepsilon)]\\
&= 2\phi_1^k(\mathbf{T},2)\left(\sum_{j=1}^{n-1}I_j(\mathbf{T},\mathbf{x})\xi_{2,j}(\mathbf{T},2)-\phi_2(\mathbf{T},2)+\phi_2(\mathbf{T},2,\mathbf{x})\right)\\
&\quad+2\phi_2(\mathbf{T},2)\left(\sum_{j=1}^{n-1}I_j(\mathbf{T},\mathbf{x})\xi_{1,j}^k(\mathbf{T},2)-\phi_1^k(\mathbf{T},2)+\phi_1^k(\mathbf{T},2,\mathbf{x})\right)\\
&\equiv \sum_{j=1}^{n-1}I_j(\mathbf{T},\mathbf{x})\lambda_{2,j}^k(\mathbf{T})-h_2^k(\mathbf{T})+\pi_2^k(\mathbf{T},\mathbf{x}),
\end{aligned}$$

where

$$\begin{aligned}
\lambda_{2,j}^k(\mathbf{T}) &= 2[\xi_{2,j}(\mathbf{T},2)\phi_1^k(\mathbf{T},2)+\xi_{1,j}^k(\mathbf{T},2)\phi_2(\mathbf{T},2)],\\
\pi_2^k(\mathbf{T},\mathbf{x}) &= 2[\phi_2(\mathbf{T},2,\mathbf{x})\phi_1^k(\mathbf{T},2)+\phi_2(\mathbf{T},2)\phi_1^k(\mathbf{T},2,\mathbf{x})-\phi_2(\mathbf{T},2)\phi_1^k(\mathbf{T},2)].
\end{aligned}$$

Similarly, the derivatives of the other h functions can be written as, for $a=1,3,4,5$,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} h_a^k(\mathbf{T}_\varepsilon,F_\varepsilon) = \sum_{j=1}^{n-1}I_j(\mathbf{T},\mathbf{x})\lambda_{a,j}^k(\mathbf{T})-h_a^k(\mathbf{T})+\pi_a^k(\mathbf{T},\mathbf{x}),$$

where

$$\lambda_{1,j}^k(\mathbf{T}) = \xi_{1,j}^k(\mathbf{T},4), \tag{1.7}$$

$$\lambda_{2,j}^k(\mathbf{T}) = 2[\xi_{2,j}(\mathbf{T},2)\phi_1^k(\mathbf{T},2)+\xi_{1,j}^k(\mathbf{T},2)\phi_2(\mathbf{T},2)], \tag{1.8}$$

$$\lambda_{3,j}^k(\mathbf{T}) = 12\rho_1^2(\mathbf{T})\rho_2^k(\mathbf{T})\varrho_{1,j}(\mathbf{T})+4\rho_1^3(\mathbf{T})\varrho_{2,j}^k(\mathbf{T}), \tag{1.9}$$

$$\begin{aligned}\lambda_{4,j}^k(\mathbf{T}) &= 2\varrho_{1,j}(\mathbf{T})\rho_2^k(\mathbf{T})\phi_2(\mathbf{T},2)+2\rho_1(\mathbf{T})\varrho_{2,j}^k(\mathbf{T})\phi_2(\mathbf{T},2)+2\rho_1(\mathbf{T})\rho_2^k(\mathbf{T})\xi_{2,j}(\mathbf{T},2)\\
&\quad+2\rho_1(\mathbf{T})\varrho_{1,j}(\mathbf{T})\phi_1^k(\mathbf{T},2)+\rho_1^2(\mathbf{T})\xi_{1,j}^k(\mathbf{T},2),\end{aligned} \tag{1.10}$$

$$\lambda_{5,j}^k(\mathbf{T}) = \varrho_{2,j}^k(\mathbf{T})\phi_2(\mathbf{T},3)+\rho_2^k(\mathbf{T})\xi_{2,j}(\mathbf{T},3)+\varrho_{1,j}(\mathbf{T})\phi_1^k(\mathbf{T},3)+\rho_1(\mathbf{T})\xi_{1,j}^k(\mathbf{T},3) \tag{1.11}$$

and

$$
\begin{aligned}
\pi_1^k(\mathbf{T},\mathbf{x}) &= \phi_1^k(\mathbf{T},4,\mathbf{x}), && (1.12)\\
\pi_2^k(\mathbf{T},\mathbf{x}) &= 2[\phi_2(\mathbf{T},2,\mathbf{x})\phi_1^k(\mathbf{T},2) + \phi_2(\mathbf{T},2)\phi_1^k(\mathbf{T},2,\mathbf{x}) - \phi_2(\mathbf{T},2)\phi_1^k(\mathbf{T},2)], && (1.13)\\
\pi_3^k(\mathbf{T},\mathbf{x}) &= 4[3\rho_1^2(\mathbf{T})\rho_2^k(\mathbf{T})(-\rho_1(\mathbf{T}) + \rho_1(\mathbf{T},\mathbf{x})) + \rho_1^3(\mathbf{T})\rho_2^k(\mathbf{T},\mathbf{x})], && (1.14)\\
\pi_4^k(\mathbf{T},\mathbf{x}) &= 2(-\rho_1(\mathbf{T}) + \rho_1(\mathbf{T},\mathbf{x}))(\rho_2^k(\mathbf{T})\phi_2(\mathbf{T},2) + \rho_1(\mathbf{T})\phi_1^k(\mathbf{T},2) + 2\rho_1(\mathbf{T})\rho_2^k(\mathbf{T})\phi_2(\mathbf{T},2,\mathbf{x})\\
&\quad + \rho_1^2(\mathbf{T})\phi_1^k(\mathbf{T},2,\mathbf{x}) + 2\rho_1(\mathbf{T})\phi_2(\mathbf{T},2)(-\rho_2^k(\mathbf{T}) + \rho_2^k(\mathbf{T},\mathbf{x})), && (1.15)\\
\pi_5^k(\mathbf{T},\mathbf{x}) &= \phi_2(\mathbf{T},3)(-\rho_2^k(\mathbf{T}) + \rho_2^k(\mathbf{T},\mathbf{x})) + \rho_2^k(\mathbf{T})\phi_2(\mathbf{T},3,\mathbf{x}) + \phi_1^k(\mathbf{T},3)(-\rho_1(\mathbf{T}) + \rho_1(\mathbf{T},\mathbf{x}))\\
&\quad + \rho_1(\mathbf{T})\phi_1^k(\mathbf{T},3,\mathbf{x}). && (1.16)
\end{aligned}
$$

Finally, recall that

$$H_k(\mathbf{T}_\varepsilon;F_\varepsilon) = h_1^k(\mathbf{T}_\varepsilon;F_\varepsilon) - 3h_2^k(\mathbf{T}_\varepsilon;F_\varepsilon) - 6h_3^k(\mathbf{T}_\varepsilon;F_\varepsilon) + 12h_4^k(\mathbf{T}_\varepsilon;F_\varepsilon) - 4h_5^k(\mathbf{T}_\varepsilon;F_\varepsilon)$$

and note that, for every $\varepsilon \geq 0$,

$$h_1^k(\mathbf{T}_\varepsilon^*;F_\varepsilon) - 3h_2^k(\mathbf{T}_\varepsilon^*;F_\varepsilon) - 6h_3^k(\mathbf{T}_\varepsilon^*;F_\varepsilon) + 12h_4^k(\mathbf{T}_\varepsilon^*;F_\varepsilon) - 4h_5^k(\mathbf{T}_\varepsilon^*;F_\varepsilon) = 0,$$

we obtain

$$\frac{d}{d\varepsilon}H_k(\mathbf{T}_\varepsilon^*,F_\varepsilon)\bigg|_{\varepsilon=0} = \sum_{j=1}^{n-1} I_j(\mathbf{T}^*,\mathbf{x})g_j^k(\mathbf{T}^*) + t_k(\mathbf{T}^*,\mathbf{x}),$$

where

$$
\begin{aligned}
g_j^k(\mathbf{T}^*) &= \lambda_{1,j}^k - 3\lambda_{2,j}^k - 6\lambda_{3,j}^k + 12\lambda_{4,j}^k - 4\lambda_{5,j}^k,\\
t_k(\mathbf{T}^*,\mathbf{x}) &= \pi_1^k - 3\pi_2^k - 6\pi_3^k + 12\pi_4^k - 4\pi_5^k.
\end{aligned}
$$

Note that $\lambda_{i,j}^k$, π_i^k denote $\lambda_{i,j}^k(\mathbf{T}^*)$ and $\pi_i^k(\mathbf{T}^*,\mathbf{x})$ $i = 1,\ldots,5$, respectively for short. Therefore, the system of equations can be written in matrix form as

$$
\begin{aligned}
&\frac{d}{d\varepsilon}\mathbf{H}(\mathbf{T}_\varepsilon^*,F_\varepsilon)\bigg|_{\varepsilon=0} = \mathbf{G}(\mathbf{T}^*)\cdot\mathbb{I}(\mathbf{T}^*,\mathbf{x}) + \mathbf{t}(\mathbf{T}^*,\mathbf{x})\\
&\equiv \begin{bmatrix} g_1^1 & \cdots & g_{n-1}^1 \\ \vdots & \ddots & \vdots \\ g_1^{n-1} & \cdots & g_{n-1}^{n-1} \end{bmatrix} \cdot \begin{bmatrix} I_1(\mathbf{T}^*,\mathbf{x}) \\ \vdots \\ I_{n-1}(\mathbf{T}^*,\mathbf{x}) \end{bmatrix} + \begin{bmatrix} t_1 \\ \vdots \\ t_{n-1} \end{bmatrix} = \mathbf{0}.
\end{aligned}
$$

This is to say, the influence function $\mathbb{I}(\mathbf{T}^*,\mathbf{x})$ solves the system of linear equations

$$\mathbf{G}(\mathbf{T}^*)\cdot\mathbb{I}(\mathbf{T}^*,\mathbf{x}) = -\mathbf{t}(\mathbf{T}^*,\mathbf{x}).$$