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CONSTRUCTION OF TREND-RESISTANT FACTORIAL DESIGNS

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Abstract: The problem of constructing trend-resistant factorial designs is discussed. Suppose a factorial experiment is to be run in a time sequence with one observation taken at a time. Then the experimenter has to decide in which order to observe the treatment combinations. A common practice is to randomize. However, sometimes randomization may lead to an undesirable ordering, and a systematic run order may be preferred. Attention is focused on the construction of systematic run orders of factorial designs in which the estimates of important factorial effects are orthogonal to some polynomial trends. Some recent work on this subject is unified and extended.

Key words and phrases: Design key, generalized foldover method, polynomial trend, run order.

1. Introduction

An ordered application of treatments to experimental units over time is called a *run order*. The purpose of this paper is to unify and extend some recent work on the construction of systematic run orders of factorial designs in which the ordinary estimates of factorial effects are orthogonal to certain polynomial time trend effects. In a factorial experiment carried out in a time sequence, the observations may be affected by uncontrollable variables highly correlated with the time in which they occur. Then the experimenter may prefer systematic designs in which the usual estimates of the factorial effects of interest are not affected by the time trend effect. A survey of earlier work on this subject can be found, for example, in Cheng and Jacroux (1988). We shall mention only the important work of Daniel and Wilcoxon (1966), who recognized that certain contrasts in the standard order of the complete 2^n design are orthogonal to linear and quadratic trends, and suggested the important idea of using these trend-resistant contrasts to construct a run order so that they represent the desired main effects and interactions in the new order. This idea was utilized by Cheng and Jacroux for general 2^n designs. One drawback of this method is that its implementation can be quite cumbersome. Coster and Cheng (1988) presented a *generalized foldover* method for constructing a run order from a sequence of *generators* in a simple manner, while one still needs to find suitable generators which will produce desired trend-resistant properties. In a review of these two methods, Cheng (1990) discussed their relationship and pointed out that they are actually equivalent. Therefore the two methods complement each other nicely: trend-resistant contrasts are selected, the corresponding generators can be determined easily from the established equivalence, and a run order is then constructed by the generalized foldover method.

In this paper, we shall extend and unify the above-mentioned work of Cheng and Jacroux (1988), Coster and Cheng (1988), and Cheng (1990) to general symmetric and asymmetrical factorial designs. Many results in Coster and Cheng (1988) can be reproduced and strengthened by a different and simpler approach.

Definitions and some preliminary material are reviewed in Section 2. It is shown in Section 3 that, in a standard order of the complete $s_1 \times s_2 \times \cdots \times s_n$ design, the trend-resistant properties of main-effect and interaction contrasts can readily be determined. Examples are given to show how one can use higher-order interaction contrasts to construct trend-resistant run orders. Section 4 applies this method, in conjunction with the classical breakdown of treatment degrees of freedom via finite geometries, to the case where the numbers of factor levels are prime powers. The connection of this method to the generalized foldover method is also discussed. Section 5 is devoted to the generalized foldover method for general symmetric and asymmetrical factorial designs. Section 6 deals with the case where the factors are quantitative. The paper concludes with some remarks in Section 7.

2. Preliminaries

Consider an $s_1 \times s_2 \times \cdots \times s_n$ experiment with n factors $A_1, A_2, \ldots, A_n$, where A_i has s_i levels. The s_i levels can be represented by integers $0, 1, \ldots, s_i - 1$ and each treatment combination is denoted by an n-vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$, where x_i is the level of the ith factor. Let S be the set of all the $s_1 s_2 \cdots s_n$ treatment combinations. Then S is an abelian group under the following operation:

$$(x_1, x_2, \ldots, x_n)' + (y_1, y_2, \ldots, y_n)' = (z_1, z_2, \ldots, z_n)', \text{ where } z_i = x_i + y_i \bmod s_i. \tag{2.1}$$

The treatment combination $\mathbf{0} = (0, 0, \ldots, 0)'$ with all the factors at level 0 is the identity element. The *order* of any treatment combination $\mathbf{x}$ is the smallest positive integer l such that $l\,\mathbf{x} = \mathbf{0}$. Fractional factorial designs are often chosen to be subgroups of S. Each subgroup G of S contains a set of *independent generators*

$\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ in the sense that G is the direct sum $< \mathbf{x}_1 > \oplus \cdots \oplus < \mathbf{x}_m >$, where $< \mathbf{x}_i >$ is the cyclic group generated by $\mathbf{x}_i$. Given a sequence of generators $\mathbf{x}_1, \ldots, \mathbf{x}_m$ with orders $l_1, \ldots, l_m$, respectively, a run order of G can be obtained in the following way. The first run is $\mathbf{0}$, which is followed by $\mathbf{x}_1, 2\mathbf{x}_1, \ldots, (l_1-1)\mathbf{x}_1$. Suppose a sequence U_j of $\prod_{i=1}^{j} l_i$ treatment combinations has been generated. Then U_j is followed by $U_j + \mathbf{x}_{j+1}, U_j + 2\mathbf{x}_{j+1}, \ldots, U_j + (l_{j+1} - 1)\mathbf{x}_{j+1}$, where $U_j + t\mathbf{x}_{j+1}$ is the sum of $t\mathbf{x}_{j+1}$ with the treatment combinations in U_j in the same order as in U_j. Once a sequence of generators is chosen, a run order can be constructed systematically. Of course, the experimenter needs to solve the design problem of choosing an appropriate generator sequence. The run order of a complete $s_1 \times s_2 \times \cdots \times s_n$ design obtained by applying the above method to the generator sequence $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, where $\mathbf{e}_i$ has the ith component equal to 1 and all the other components equal to 0, is called a *standard order.*

The above method is essentially the *generalized foldover method* defined in Coster and Cheng (1988), except that they consider symmetric factorial designs with prime-power number of levels, in which field operations, instead of (2.1), are used, and the $t\mathbf{x}_{j+1}$'s in the construction are the products of $\mathbf{x}_{j+1}$ with all the nonzero elements in the field. We continue to call the method described above a generalized foldover method. Coster (1988) also gives a version of generalized foldover method for asymmetrical factorial designs. He only considers the case where the numbers of levels are prime numbers and uses pseudo factors for the general case. In his construction, each l_j is one of the s_i's, which needs to be specified and is not necessarily the order of $\mathbf{x}_j$. We will comment more on this work in Section 5.

Another convenient notation for the treatment combination $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$ is to denote it by $a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n}$, with $a_i^{x_i}$ omitted if $x_i = 0$. If the exponent of a letter is 1, then that exponent is also omitted. Then (2.1) becomes

$$a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n} \cdot a_1^{y_1} a_2^{y_2} \cdots a_n^{y_n} = a_1^{z_1} a_2^{z_2} \cdots a_n^{z_n}.$$

All the definitions in the previous paragraph are modified accordingly. The treatment combination in which all the factors are at level 0 is then denoted as (1), and the generators $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ become $a_1, a_2, \ldots, a_n$. Throughout this paper, we shall use both notations freely; from the context, there should be no danger of confusion.

In an $s_1 \times s_2 \times \cdots \times s_n$ experiment, $s_1 s_2 \cdots s_n - 1$ mutually orthogonal contrasts of the treatment effects can be constructed to represent the main effects and interactions. A convenient choice is based on orthogonal polynomials. Let $\{P_s^k, k = 0, 1, \ldots, s-1\}$ be a system of orthogonal polynomials on s equally spaced points $u = 0, 1, \ldots, s-1$, i.e.,

$$P_s^0(u) \equiv 1,$$

$$\sum_{u=0}^{s-1} P_s^k(u)P_s^{k'}(u) = 0, \text{ for all } 0 \le k < k' \le s-1,$$

where P_s^k is a polynomial of degree k. These polynomials are unique up to constant multiples and can be scaled so that the values $P_s^k(u)$ are all integers. Then the kth order main-effect contrast of factor A_i, denoted by A_i^k, is the contrast of the $s_1 s_2 \cdots s_n$ treatment effects in which the coefficient of a treatment effect is equal to $P_{s_i}^k(u)$ if factor A_i appears at level u. The first- (second, third, $\cdots$) order main-effect contrast is usually referred to as the linear (quadratic, cubic, $\cdots$) main-effect contrast and is denoted by lin A_i (quad A_i, cubic $A_i, \ldots$, respectively). For any r distinct factors $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ and integers $1 \le k_j \le s_{i_j} - 1$, $j = 1, 2, \ldots, r$, $A_{i_1}^{k_1} \times A_{i_2}^{k_2} \times \cdots \times A_{i_r}^{k_r}$ denotes the contrast of the $s_1 s_2 \cdots s_n$ treatment effects in which the coefficient of a treatment effect is equal to $\prod_{j=1}^{r} P_{s_{i_j}}^{k_j}(u_j)$ if factor A_{i_j} appears at level u_j, $j = 1, 2, \ldots, r$. This contrast represents a component of the interaction of factors $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$. For example, $A_1^1 \times A_2^1$ and $A_1^1 \times A_2^2$ are usually referred to as the linear $\times$ linear and linear $\times$ quadratic components of the interaction of factors A_1 and A_2.

For a run order of a subgroup of S of size N, define $\mathbf{u}_i^k$ to be the $N \times 1$ vector with the lth component equal to $P_{s_i}^k(u)$ if factor A_i appears at level u on the lth run, where $l = 1, 2, \ldots, N$, $i = 1, \ldots, n$, $1 \le k \le s_i - 1$. For any two $N \times 1$ vectors $\mathbf{u} = (u_1, \ldots, u_N)'$ and $\mathbf{v} = (v_1, \ldots, v_N)'$. let $\mathbf{u} \circ \mathbf{v}$ be the vector $(u_1 v_1, \ldots, u_N v_N)'$. If $\mathbf{y}$ denotes $(y_1, y_2, \ldots, y_N)'$, where $y_1, y_2, \ldots, y_N$ are the N observations in the given run order, then under the usual homoscedastic linear model for factorial designs without trend effects, the least squares estimator of A_i^k is proportional to $(\mathbf{u}_i^k)'\mathbf{y}$, and the least squares estimator of $A_{i_1}^{k_1} \times A_{i_2}^{k_2} \times \cdots \times A_{i_r}^{k_r}$ is proportional to $(\mathbf{u}_{i_1}^{k_1} \circ \mathbf{u}_{i_2}^{k_2} \circ \cdots \circ \mathbf{u}_{i_r}^{k_r})'\mathbf{y}$, provided that they are estimable. For simplicity, we shall call $\mathbf{u}_i^k$ a main-effect contrast of order k, and $\mathbf{u}_{i_1}^{k_1} \circ \mathbf{u}_{i_2}^{k_2} \circ \cdots \circ \mathbf{u}_{i_r}^{k_r}$ an r-factor interaction contrast of order $\sum_{j=1}^{r} k_j$. Now suppose the observations are equally spaced in time and are affected by a trend effect which is a tth degree polynomial function of the time. If the least squares estimator of a factorial effect is not affected by the time trend, then we say that it is *orthogonal* to a time trend of degree t. If it is orthogonal to all polynomial trend effects of degrees up to and including t, then we say that it is *t-trend free.* It is easy to see that a necessary and sufficient condition for a main-effect or interaction contrast $\mathbf{u}$ to be t-trend free is that

$$\mathbf{u}'\mathbf{T}_i = 0 \text{ for all } i = 0, \ldots, t, \text{ where } \mathbf{T}_i = (1^i, 2^i, \ldots, N^i)'. \tag{2.2}$$

In general an $N \times 1$ vector $\mathbf{u}$ (not necessarily a main-effect or interaction contrast) is called t-trend free if (2.2) holds.

Bailey, Gilchrist and Patterson (1977) and Bailey (1985) describe a method of dividing the treatment degrees of freedom for any factorial design into homogeneous orthogonal subsets. Here homogeneity means that all the contrasts in the same set belong to the same main effect or interaction. For any subgroup $G = \langle \mathbf{x}_1 \rangle \oplus \cdots \oplus \langle \mathbf{x}_m \rangle$ of S, consider the vector space R^G in which the components of each vector are indexed by the elements of G. Let $I = \{1, \ldots, m\}$. For any subset J of I, we can define a partition of G by putting two elements in the same class if their projections onto $\oplus_{j \in J} \langle \mathbf{x}_j \rangle$ are the same. Let V_J be the subspace of R^G consisting of the vectors which are constant on each class of the partition defined above. Further, let W_J be the orthogonal complement in V_J of all those $V_{J'}$ for which $J' \subset J$ and $J' \neq J$. If the order of $\mathbf{x}_i$ is l_i, then the dimension of W_J is $\prod_{j \in J}(l_j - 1)$. Notice that if $G = S = \langle \mathbf{e}_1 \rangle \oplus \cdots \oplus \langle \mathbf{e}_n \rangle$, then for any $J \subset \{1, \ldots, n\}$, W_J defines the $\prod_{j \in J}(s_j - 1)$ degrees of freedom for the interaction of the factors in J. The finite abelian group $G = \langle \mathbf{x}_1 \rangle \oplus \cdots \oplus \langle \mathbf{x}_m \rangle$ has a *dual* group $G^* = \langle \chi_1 \rangle \oplus \cdots \oplus \langle \chi_m \rangle$, where χ_i is a functional on G defined by

$$\chi_i\left(\sum_{j=1}^{m} d_j \mathbf{x}_j\right) = d_i \lambda / l_i \bmod \lambda,$$

with l_i being the order of $\mathbf{x}_i$ and λ the least common multiple of $l_1, \ldots, l_m$. Each element χ of G^* has a unique expression as $\sum_{i=1}^{m} c_i \chi_i$, and

$$\chi\left(\sum_{j=1}^{m} d_j \mathbf{x}_j\right) = \sum_{j=1}^{m} c_j d_j \lambda / l_j \bmod \lambda.$$

Given $\chi \in G^*$, a partition of G can be obtained by putting two elements $\mathbf{x}$ and $\mathbf{y}$ in the same class if $\chi(\mathbf{x}) = \chi(\mathbf{y})$. Let V_χ be the subspace of R^G consisting of the vectors which are constant on each class of this partition. Further, let W_χ be the orthogonal complement in V_χ of $V_{t_1\chi}, \ldots, V_{t_h\chi}$, where $t_1, \ldots, t_h$ are all the divisors, except 1, of the order of χ. Then the W_χ's give an orthogonal decomposition of R^G. When $G = S = \langle \mathbf{e}_1 \rangle \oplus \cdots \oplus \langle \mathbf{e}_n \rangle$ is the complete $s_1 \times \cdots \times s_n$ design, W_χ defines $\phi(l)$ treatment degrees of freedom, where l is the order of χ and $\phi(l)$ is the number of integers between 1 and l which are coprime to l; see Bailey, Gilchrist and Patterson (1977). In fact, in this case, for each $\chi \in S^*$, there exists $\mathbf{y} \in S$ such that

$$\chi(\mathbf{x}) = [\mathbf{x}, \mathbf{y}] \equiv \sum_{i=1}^{n} x_i y_i \mu / s_i \bmod \mu, \text{ for all } \mathbf{x} \in S,$$

where μ is the least common multiple of the s_i's. Using the notation in Bailey, Gilchrist and Patterson (1977), we shall denote these $\phi(l)$ degrees of freedom by $T_*(\mathbf{y}')$. If $y_{i_1}, y_{i_2}, \ldots, y_{i_r}$ are the nonzero components of $\mathbf{y}$, then these $\phi(l)$ degrees of freedom belong to the interaction of factors $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$. In a symmetric s^n design, where s is a prime number, this is the same as the usual decomposition of treatment degrees of freedom based on finite geometries (Bose (1947), Kempthorne (1947); also see below). The following theorem from Bailey (1985) links the two decompositions described above and is needed later in the paper.

Theorem 2.1. *Let $G = <\mathbf{x}_1> \oplus \cdots \oplus <\mathbf{x}_m>$ be a subgroup of S. For any $\chi = \sum_{i=1}^m c_i\chi_i$ in G^*, let $J(\chi) = \{i : c_i\chi_i \neq 0\}$, that is, $J(\chi) = \{i : \chi(\mathbf{x}_i) \neq 0\}$. Then W_χ is a subspace of $W_{J(\chi)}$.*

When the number of levels is a prime power, these levels can also be represented by the elements of a finite field. In this case, the classical decomposition of the treatment degrees of freedom is based on finite geometries. For example, in an s^n experiment, each treatment combination is identified with a point in $EG(n, s)$, the n-dimensional Euclidean geometry with s points per line. For any nonzero $\mathbf{b} = (b_1, b_2, \ldots, b_n)' \in \mathrm{EG}(n, s)$, and any $b_0 \in \mathrm{GF}(s)$, the finite field with s elements, the set $\{\mathbf{x} : \sum_{i=1}^n b_i x_i = b_0\}$ is called an $(n-1)$-flat. As b_0 ranges through the s elements of $GF(s)$, we obtain the s nonoverlapping $(n-1)$-flats in the pencil $P(\mathbf{b})$ determined by $\mathbf{b}$. A total of $s-1$ mutually orthogonal contrasts between these s $(n-1)$-flats can be constructed. If $b_{i_1}, b_{i_2}, \ldots, b_{i_r}$ are the nonzero elements among $b_1, b_2, \ldots, b_n$, then these contrasts represent interactions of factors $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$. The associated $s-1$ degrees of freedom are denoted by $A_{i_1}^{b_{i_1}} A_{i_2}^{b_{i_2}} \cdots A_{i_r}^{b_{i_r}}$. Unless s is a prime number, this gives a different decomposition from that obtained in Bailey, Gilchrist and Patterson (1977), and, as pointed out earlier, the generalized foldover method also leads to a different construction because field operations, instead of (2.1), are used. In Section 4, we shall discuss the construction based on finite geometries and its relation to a version of the generalized foldover method based on field operations, while in Section 5, (2.1) is used for general factorial designs.

If $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k \in \mathrm{EG}(n, s)$ are linearly independent, then we say that the k pencils $P(\mathbf{b}_1)$, $P(\mathbf{b}_2)$, $\ldots$, $P(\mathbf{b}_k)$ are linearly independent. Throughout the paper, $\mathbf{1}_m$ denotes the $m \times 1$ vector of ones.

3. Complete Factorial Design: Standard Order

Lemma 3.2 of Cheng and Jacroux (1988) showed that in a standard order of the complete 2^n design, a k-factor interaction is $(k-1)$-trend free. This key result can be extended as follows:

Theorem 3.1. *In a standard order of a complete $s_1 \times s_2 \times \cdots \times s_n$ design, a main-effect or interaction contrast of order k is $(k-1)$-trend free.*

Proof. This result can be proved in two different ways. The first method is similar to the proof of Lemma 3.2 of Cheng and Jacroux (1988). Let $N = s_1 s_2 \cdots s_n$. Then one can show that the vector $\mathbf{T}_1$ defined in (2.2) is a linear combination of $\mathbf{1}_N, \mathbf{u}_1^1, \mathbf{u}_2^1, \ldots$, and $\mathbf{u}_n^1$. The result then follows.

Another proof is based on a result of Jacroux and Saha Ray (1988), also independently reproved by Wang (1989). They showed that if $\mathbf{x}$ is a p-trend free vector and $\mathbf{y}$ is a q-trend free vector ($\mathbf{x}$ and $\mathbf{y}$ can be of different dimensions), then their Kronecker product $\mathbf{x} \otimes \mathbf{y}$ is $(p+q+1)$-trend free, $\mathbf{x} \otimes \mathbf{1}$ is p-trend free, and $\mathbf{1} \otimes \mathbf{y}$ is q-trend free. In both papers, this result was stated for the case where $\mathbf{x}$ and $\mathbf{y}$ have 1 and -1 entries, but it is clear from the proof that this restriction is not necessary. Let $\mathbf{p}_s^k$ be the $s \times 1$ vector $(P_s^k(0), P_s^k(1), \ldots, P_s^k(s-1))'$. By the definition of orthogonal polynomials, $\mathbf{p}_s^k$ is $(k-1)$-trend free. Now in the standard order, $\mathbf{u}_i^k$, the kth order main-effect contrast of factor i, is equal to $\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \cdots \otimes \mathbf{x}_n$, where $\mathbf{x}_i = \mathbf{p}_{s_i}^k$, and $\mathbf{x}_j = \mathbf{1}_{s_j}$ for all $j \neq i$. Since $\mathbf{p}_{s_i}^k$ is $(k-1)$-trend free, $\mathbf{u}_i^k$ is also $(k-1)$-trend free. An interaction contrast of order k, say $\mathbf{u}_{i_1}^{k_{i_1}} \circ \mathbf{u}_{i_2}^{k_{i_2}} \circ \cdots \circ \mathbf{u}_{i_r}^{k_{i_r}}$ with $\sum_{j=1}^r k_{i_j} = k$ can be expressed as $\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \cdots \otimes \mathbf{x}_n$, where $\mathbf{x}_i = \mathbf{p}_{s_i}^{k_i}$, $i = i_1, i_2, \ldots, i_r$, and $\mathbf{x}_i = \mathbf{1}_{s_i}$ for all $i \neq i_1, i_2, \ldots, i_r$. By repeatedly using Jacroux, Saha Ray and Wang's result, we conclude that $\mathbf{u}_{i_1}^{k_{i_1}} \circ \mathbf{u}_{i_2}^{k_{i_2}} \circ \cdots \circ \mathbf{u}_{i_r}^{k_{i_r}}$ is $(k-1)$-trend free.

Therefore, for example, in the standard order the quadratic component of a main effect is linear-trend free, and the linear $\times$ quadratic $\times$ quadratic component of a three-factor interaction is 4-trend free. Unlike the 2^n designs, when the factors have more than two levels, each main effect or interaction term carries more than one degree of freedom, and some degrees of freedom may have stronger trend-free properties than others. Theorem 3.1 determines the trend-free properties of the main-effect and interaction contrasts defined by orthogonal polynomials. For other main-effect and interaction contrasts, such detailed knowledge may not be readily available; but we have the following corollary of Theorem 3.1:

Corollary 3.2. *In a standard order of a complete $s_1 \times s_2 \times \cdots \times s_n$ design, any contrast representing a k-factor interaction is at least $(k-1)$-trend free.*

The 2^n version of Corollary 3.2 is the key result for the construction of trend-resistant two-level designs discussed in Daniel and Wilcoxon (1966) and Cheng and Jacroux (1988). To make the presentation precise and to prepare for later extension based on *design keys* (Patterson (1965), Patterson and Bailey (1978)), we write P_i, $i = 1, \ldots, n$ (or $P, Q, R, \cdots$), for dummy plot factors, and A_i, $i = 1, 2, \ldots, n$ (or $A, B, C, \cdots$), for the genuine treatment factors. For plot factors, the main-effect and interaction contrasts such as $\mathbf{u}_i^k$ and $\mathbf{u}_{i_1}^{k_1} \circ \mathbf{u}_{i_2}^{k_2} \circ \cdots \circ \mathbf{u}_{i_r}^{k_r}$ defined in Section 2 will be written as $\mathbf{v}_i^k$ and $\mathbf{v}_{i_1}^{k_1} \circ \mathbf{v}_{i_2}^{k_2} \circ \cdots \circ \mathbf{v}_{i_r}^{k_r}$, respectively.

The 2^n factorial design is constructed in standard order with respect to the plot factors $P_1, \ldots, P_n$. To make all main-effect contrasts of the treatment factors trend-resistant, one can choose n mutually orthogonal higher-order interaction contrasts $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ of the plot factors, and use each to define the levels of a treatment factor. Suppose $\mathbf{w}_i$ is identified with treatment factor A_i. Then assign high (respectively, low) level to A_i on the uth run if the uth component of $\mathbf{w}_i$ is 1 (respectively, -1). Then the main effect of treatment factor A_i is $(k-1)$-trend free if $\mathbf{w}_i$ represents an interaction of k plot factors.

The above method works for two-level designs because the interaction contrasts have 1 and -1 entries, which can be used to assign high and low levels. It may not work, however, for general $s_1 \times s_2 \times \cdots \times s_n$ designs, where the entries in an interaction contrast such as $\mathbf{v}_{i_1}^{k_1} \circ \mathbf{v}_{i_2}^{k_2} \circ \cdots \circ \mathbf{v}_{i_r}^{k_r}$ may not be the s_i entries in $\mathbf{u}_i^1$ for all $i = 1, 2, \ldots, n$. In spite of this, we give some examples where the method does work.

Example 3.1. Consider a $2 \times 3 \times 4$ design where factor A_1 has 2 levels, factor A_2 has 3 levels and factor A_3 has 4 levels. Introduce plot factors P_1, P_2 and P_3 with 2, 3 and 4 levels, respectively. In the standard order with respect to P_1, P_2, P_3, we have $\mathbf{v}_1^1 = (-1,1)' \otimes \mathbf{1}_3 \otimes \mathbf{1}_4$, $\mathbf{v}_2^1 = \mathbf{1}_2 \otimes (-1,0,1)' \otimes \mathbf{1}_4$, $\mathbf{v}_3^2 = \mathbf{1}_2 \otimes \mathbf{1}_3 \otimes (1,-1,-1,1)'$, and $\mathbf{v}_3^3 = \mathbf{1}_2 \otimes \mathbf{1}_3 \otimes (-1,3,-3,1)'$. Then the $P_1\times$ quad P_3 contrast is

$$\mathbf{v}_1^1 \circ \mathbf{v}_3^2 = (-1, 1, -1, 1, -1, 1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, -1, 1, -1, 1, -1, 1)',$$

the linear $P_2\times$ quad P_3 contrast is

$$\mathbf{v}_2^1 \circ \mathbf{v}_3^2 = (-1, -1, 0, 0, 1, 1, 1, 1, 0, 0, -1, -1, 1, 1, 0, 0, -1, -1, -1, -1, 0, 0, 1, 1)',$$

and the $P_1\times$ cubic P_3 contrast is

$$\mathbf{v}_1^1 \circ \mathbf{v}_3^3 = (1, -1, 1, -1, 1, -1, -3, 3, -3, 3, -3, 3, 3, -3, 3, -3, 3, -3, -1, 1, -1, 1, -1, 1)'.$$

Then $\mathbf{v}_1^1 \circ \mathbf{v}_3^2$, $\mathbf{v}_2^1 \circ \mathbf{v}_3^2$, and $\mathbf{v}_1^1 \circ \mathbf{v}_3^3$ have the right entries for defining the levels of treatment factors A_1, A_2 and A_3, respectively: on each run, assign level 0 or 1 to factor A_1 if the corresponding entry of $\mathbf{v}_1^1 \circ \mathbf{v}_3^2$ is -1 or 1, respectively, assign level 0, 1, or 2 to factor A_2 if the corresponding entry of $\mathbf{v}_2^1 \circ \mathbf{v}_3^2$ is -1, 0, or 1, respectively, and assign level 0, 1, 2, or 3 to factor A_3 if the corresponding entry of $\mathbf{v}_1^1 \circ \mathbf{v}_3^3$ is -3, -1, 1, or 3, respectively. Then we obtain the following run order

$$a_3^2,\ a_1a_3,\ a_2a_3^2,\ a_1a_2a_3,\ a_2^2a_3^2,\ a_1a_2^2a_3,\ a_1a_2^2,\ a_2^2a_3^3,\ a_1a_2,\ a_2a_3^3,\ a_1,\ a_3^3,\ a_1a_2^2a_3^3,$$
$$a_2^2,\ a_1a_2a_3^3,\ a_2,\ a_1a_3^3,\ (1),\ a_3,\ a_1a_3^2,\ a_2a_3,\ a_1a_2a_3^2,\ a_2^2a_3,\ a_1a_2^2a_3^2. \qquad (3.1)$$

In (3.1), the main-effect contrast of A_1 is equal to $\mathbf{v}_1^1 \circ \mathbf{v}_3^2$, the linear A_2 contrast equals $\mathbf{v}_2^1 \circ \mathbf{v}_3^2$, and the linear A_3 contrast equals $\mathbf{v}_1^1 \circ \mathbf{v}_3^3$. Also, one can verify that the quadratic A_2 contrast equals $\mathbf{v}_2^2$, the quadratic A_3 contrast is $\mathbf{v}_3^2$, the cubic A_3 contrast is $-(\mathbf{v}_1^1 \circ \mathbf{v}_3^1)$, and the $A_1 \times$ linear A_2 contrast is $\mathbf{v}_1^1 \circ \mathbf{v}_2^1$. By Theorem 3.1, we conclude that in run order (3.1), the main effect of A_1 and linear A_2 are quadratic-trend free, linear A_3 is cubic-trend free, quadratic A_2, quadratic A_3, cubic A_3 and the $A_1 \times$ linear A_2 interaction are all linear-trend free.

Example 3.2. In a standard order of the complete 4^2 design, the linear $P_1 \times$ quadratic P_2 contrast is $\mathbf{v}_1^1 \circ \mathbf{v}_2^2 = (-3,-1,1,3,3,1,-1,-3,3,1,-1,-3,-3,-1,1,3)'$, and the quadratic $P_1 \times$ linear P_2 contrast is $\mathbf{v}_1^2 \circ \mathbf{v}_2^1 = (-3,3,3,-3,-1,1,1,-1,1,-1,-1,1,3,-3,-3,3)'$. Now as in Example 3.1, use $\mathbf{v}_1^1 \circ \mathbf{v}_2^2$ and $\mathbf{v}_1^2 \circ \mathbf{v}_2^1$ to define the levels of A_1 and A_2, respectively. Then we have the following run order:

$$(1),\ a_1a_2^3,\ a_1^2a_2^3,\ a_1^3,\ a_1^3a_2,\ a_1^2a_2^2,\ a_1a_2^2,\ a_2,\ a_1^3a_2^2,\ a_1^2a_2,\ a_1a_2,\ a_2^2,\ a_2^3,\ a_1,\ a_1^2,\ a_1^3a_2^3. \tag{3.2}$$

One can verify that all the main-effect contrasts and the linear $A_1 \times$ linear A_2 interaction contrast are at least linear-trend free. For example, the linear $A_1 \times$ linear A_2 contrast in (3.2) is equal to $\frac{16}{25}\mathbf{v}_1^1 \circ \mathbf{v}_2^1 + \frac{12}{25}\mathbf{v}_1^1 \circ \mathbf{v}_2^3 + \frac{12}{25}\mathbf{v}_1^3 \circ \mathbf{v}_2^1 + \frac{9}{25}\mathbf{v}_1^3 \circ \mathbf{v}_2^3$, which is a linear combination of linear-trend free contrasts and therefore is linear-trend free.

4. Symmetric Factorial Designs with Prime-Power Number of Levels

The method described in the last section imposes severe constraints; in particular, an interaction contrast identified with an s-level factor must have exactly s distinct entries. When the factors have prime-power numbers of levels, this difficulty can be circumvented by using Corollary 3.2 in conjunction with the classical breakdown of treatment degrees of freedom via finite geometries. Here we shall consider only the case where all the factors have the same number of levels and briefly comment on the asymmetrical case, because an alternative construction for the general case will be discussed in the next section.

The fact that all pencils $P(\mathbf{b})$ in $\mathrm{EG}(n,s)$ contain the same number of $(n-1)$-flats facilitates the construction of trend-resistant run orders. To make all main-effect contrasts at least t-trend free, one can choose n linearly independent pencils $P(\mathbf{b}_1), P(\mathbf{b}_2), \ldots, P(\mathbf{b}_n)$, where each $\mathbf{b}_i = (b_{i1}, \ldots, b_{in})'$ has at least $t+1$ nonzero entries, to redefine the levels of the n factors. By Corollary 3.2, all the contrasts of the s flats in $P(\mathbf{b}_i)$ are t-trend free in the standard order. The main effect of A_i can be made t-trend free by rearranging the run order so that all runs originally occupied by treatment combinations in the same $(n-1)$-flat of $P(\mathbf{b}_i)$ are now occupied by those with the same value of x_i. This can be achieved by the following: in the standard order, if a treatment combination satisfies

$\mathbf{b}_i'\mathbf{x} = b, b \in \mathrm{GF}(s)$, then assign level b to factor A_i. In other words, the run order can be constructed, from the standard order with respect to n s-level plot factors $P_1, \ldots, P_n$, by assigning the level

$$\sum_{j=1}^{n} b_{ij} y_j \tag{4.1}$$

to treatment factor A_i on the run whose level of P_j is y_j for $j = 1, \ldots, n$. More succinctly, this can be expressed as a *design key*

$$A_i = \sum_{j=1}^{n} b_{ij} P_j. \tag{4.2}$$

This is a natural extension of the construction in Cheng and Jacroux (1988). A similar construction is reported in Wang (1991).

In a run order constructed by this method, the trend-free properties of all the interaction components can easily be determined. All runs occupied by treatment combinations in the same $(n-1)$-flat of pencil $P(\mathbf{c})$ in the new order were occupied, in the standard order, by treatment combinations in the same $(n-1)$-flat of the pencil $P(\sum_{i=1}^{n} c_i \mathbf{b}_i)$. Therefore the $s-1$ degrees of freedom of the r-factor interaction $A_{i_1}^{c_{i_1}} A_{i_2}^{c_{i_2}} \cdots A_{i_r}^{c_{i_r}}$ is t-trend free if $\sum_{j=1}^{r} c_{i_j} \mathbf{b}_{i_j}$ contains $t+1$ nonzero entries. As in Patterson (1965), the $s-1$ degrees of freedom defined by $\sum_{j=1}^{r} c_{i_j} \mathbf{b}_{i_j}$ are called the *plot alias* of $A_{i_1}^{c_{i_1}} A_{i_2}^{c_{i_2}} \cdots A_{i_r}^{c_{i_r}}$.

In a regular $(1/s^p)$-fraction of a complete s^n design, there exist $n-p$ factors, called *basic factors*, such that the design contains all the s^{n-p} combinations of these factors. The other factors are called *added factors*, whose levels can be determined from those of the basic factors by the defining relations of the fractional factorial design. The method described above can be applied to the complete s^{n-p} design of the basic factors. The plot aliases of all the factorial effects can be determined from those of the main effects of the basic factors and the defining relations. One added complication here is that after choosing $n-p$ linearly independent pencils to redefine the levels of the basic factors, defining relations must be employed to check that factorial effects involving the added factors also have desired trend-free properties. If the defining relations are not given, the tasks of selecting defining relations and constructing trend-resistant run orders can be combined by choosing appropriate interaction components of the basic factors to assign the levels of *all* factors.

Example 4.1. In the 3^2 design with treatment factors A and B and plot factors P and Q, to make the main effects linear-trend free, one can write down the design in standard order with respect to P and Q, and then use the design key

$A = PQ, B = PQ^2$ to define the levels of the two treatment factors. This results in the following run order:

$$(1),\ ab,\ a^2b^2,\ ab^2,\ a^2,\ b,\ a^2b,\ b^2,\ a.$$

For example, the fourth run in the standard order with respect to the plot factors is q, which, when considered as a point in EG(2, 3), is $(y_1, y_2)'$ with $y_1 = 0$ and $y_2 = 1$. Then $y_1+y_2 = 1$ and $y_1+2y_2 = 2$. So the fourth treatment combination in the new run order is $(1, 2)'$, that is, ab^2. In this order, the interaction component AB is not linear-trend free because its plot alias $(PQ)(PQ^2) = P^2$ is a main effect.

When s is not a prime number, one has to use the complicated operations of a finite field instead of the straightforward arithmetics modulo s for this construction. In this case, it may be more convenient to use pseudo factors.

Example 4.2. Consider a complete 4^2 design with two 4-level factors A and B. Identify the 4 levels 0, 1, 2, 3 with the combinations 00, 10, 01, 11, respectively, of two 2-level factors. Denote the pseudo factors of A by A_1 and A_2, and the pseudo factors of B by B_1 and B_2. Construct the 2^4 design in standard order with respect to two-level plot factors P_1, P_2, P_3 and P_4. Then use the design key $A_1 = P_1P_2P_3$, $A_2 = P_2P_3P_4$, $B_1 = P_1P_2P_4$, $B_2 = P_1P_3P_4$. This gives the following run order of the 2^4 design:

$$\begin{array}{c}(1),\ a_1b_1b_2,\ a_1a_2b_1,\ a_2b_2,\ a_1a_2b_2,\ a_2b_1,\ b_1b_2,\ a_1, \\ a_2b_1b_2,\ a_1a_2,\ a_1b_2,\ b_1,\ a_1b_1,\ b_2,\ a_2,\ a_1a_2b_1b_2.\end{array} \tag{4.3}$$

The corresponding run order for the 4^2 design is

$$(1),\ ab^3,\ a^3b,\ a^2b^2,\ a^3b^2,\ a^2b,\ b^3,\ a,\ a^2b^3,\ a^3,\ ab^2,\ b,\ ab,\ b^2,\ a^2,\ a^3b^3. \tag{4.4}$$

The plot aliases of A_1A_2 and B_1B_2 are, respectively, P_1P_4 and P_2P_3. Therefore in run order (4.4), all main-effect contrasts of A and B are at least linear-trend free.

In this example, more detailed information can be obtained about the trend-free properties of various main-effect and interaction contrasts defined by orthogonal polynomials. Let $\mathbf{w}_1 = (-1, 1, -1, 1)'$, $\mathbf{w}_2 = (-1, -1, 1, 1)'$, and $\mathbf{w}_3 = \mathbf{w}_1 \circ \mathbf{w}_2 = (1, -1, -1, 1)'$. Then we have

$$\begin{aligned}\mathbf{p}_4^1 &= (-3, -1, 1, 3)' = \mathbf{w}_1 + 2\mathbf{w}_2, \\ \mathbf{p}_4^2 &= (1, -1, -1, 1)' = \mathbf{w}_1 \circ \mathbf{w}_2, \\ \mathbf{p}_4^3 &= (-1, 3, -3, 1)' = 2\mathbf{w}_1 - \mathbf{w}_2,\end{aligned}$$

where $\mathbf{p}_4^1$, $\mathbf{p}_4^2$ and $\mathbf{p}_4^3$ are as defined in the proof of Theorem 3.1. Let $\mathbf{u}_A^1$, $\mathbf{u}_A^2$, $\mathbf{u}_A^3$ be the linear A, quadratic A and cubic A contrasts and $\mathbf{u}_B^1$, $\mathbf{u}_B^2$, $\mathbf{u}_B^3$ be the linear B, quadratic B and cubic B contrasts in (4.4). Also let $\mathbf{u}_{A_1}$, $\mathbf{u}_{A_2}$, $\mathbf{u}_{B_1}$, $\mathbf{u}_{B_2}$ be the contrasts representing the main effects of A_1, A_2, B_1 and B_2 in (4.3), and $\mathbf{v}_{P_1}$, $\mathbf{v}_{P_2}$, $\mathbf{v}_{P_3}$, $\mathbf{v}_{P_4}$ be the contrasts representing the main effects of P_1, P_2, P_3 and P_4 in the standard order. Then we have

$$\mathbf{u}_A^1 = \mathbf{u}_{A_1} + 2\mathbf{u}_{A_2},\ \mathbf{u}_A^2 = \mathbf{u}_{A_1} \circ \mathbf{u}_{A_2},\ \mathbf{u}_A^3 = 2\mathbf{u}_{A_1} - \mathbf{u}_{A_2}, \tag{4.5}$$
$$\mathbf{u}_B^1 = \mathbf{u}_{B_1} + 2\mathbf{u}_{B_2},\ \mathbf{u}_B^2 = \mathbf{u}_{B_1} \circ \mathbf{u}_{B_2},\ \mathbf{u}_B^3 = 2\mathbf{u}_{B_1} - \mathbf{u}_{B_2}. \tag{4.6}$$

Furthermore,

$$\begin{aligned} \mathbf{u}_{A_1} &= \mathbf{v}_{P_1} \circ \mathbf{v}_{P_2} \circ \mathbf{v}_{P_3},\ \mathbf{u}_{A_2} = \mathbf{v}_{P_2} \circ \mathbf{v}_{P_3} \circ \mathbf{v}_{P_4}, \\ \mathbf{u}_{B_1} &= \mathbf{v}_{P_1} \circ \mathbf{v}_{P_2} \circ \mathbf{v}_{P_4},\ \mathbf{u}_{B_2} = \mathbf{v}_{P1} \circ \mathbf{v}_{P_3} \circ \mathbf{v}_{P_4}. \end{aligned}$$

From these relations and (4.5), (4.6), the linear, quadratic and cubic contrasts of the main effects of A and B, and various interaction contrasts can be expressed in terms of $\mathbf{v}_{P_1}$, $\mathbf{v}_{P_2}$, $\mathbf{v}_{P_3}$, and $\mathbf{v}_{P_4}$, and then their trend-free properties can be determined. For example, the linear $A \times$ linear B contrast

$$\begin{aligned} \mathbf{u}_A^1 \circ \mathbf{u}_B^1 &= (\mathbf{v}_{P_1} \circ \mathbf{v}_{P_2} \circ \mathbf{v}_{P_3} + 2\mathbf{v}_{P_2} \circ \mathbf{v}_{P_3} \circ \mathbf{v}_{P_4}) \\ &\quad \circ (\mathbf{v}_{P_1} \circ \mathbf{v}_{P_2} \circ \mathbf{v}_{P_4} + 2\mathbf{v}_{P_1} \circ \mathbf{v}_{P_3} \circ \mathbf{v}_{P_4}) \\ &= \mathbf{v}_{P_3} \circ \mathbf{v}_{P_4} + 2\mathbf{v}_{P_2} \circ \mathbf{v}_{P_4} + 2\mathbf{v}_{P_1} \circ \mathbf{v}_{P_3} + 4\mathbf{v}_{P_1} \circ \mathbf{v}_{P_2} \end{aligned}$$

is linear-trend free. Similarly, one can verify that linear A, cubic A, linear B and cubic B are quadratic-trend free, and quadratic A, quadratic B are linear-trend free.

Since the standard order can be obtained by applying the generalized foldover method to the particular generator sequence $a_1, a_2, \ldots, a_n$, use of n linearly independent pencils of $(n-1)$-flats to redefine factor levels amounts to replacement of the generators. Therefore run orders constructed by the method described above can also be obtained by the generalized foldover method. Notice that here we refer to the generalized foldover method based on field operations as in Coster and Cheng (1988), not the one described in Section 2. In (4.2), let $\mathbf{B}$ be the $n \times n$ matrix $[b_{ij}]$, whose rows correspond to the plot aliases of the main effects of the treatment factors. Then the generators are precisely the column vectors of $\mathbf{B}$. The generalized foldover method can be used, and there is no need to go through the tedious process of determining the values of $\mathbf{b}'\mathbf{x}$ for all $\mathbf{x}$ and all the pencils $P(\mathbf{b})$ used to redefine factor levels.

Example 4.3. In Example 4.1, a run order of a 3^2 experiment is obtained by using $A = PQ$ and $B = PQ^2$ to define the levels of A and B. Here $\mathbf{b}_1 = (1, 1)'$

and $\mathbf{b}_2 = (1,\ 2)'$. The column vectors of $\mathbf{B} = (\mathbf{b}_1,\ \mathbf{b}_2)'$ are $(1,\ 1)'$ and $(1,\ 2)'$, respectively. It follows that the two generators are ab and ab^2.

One can also reverse the above process. For any generator sequence $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_n$, let $\mathbf{B}$ the matrix with $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ as its column vectors, then the row vectors of $\mathbf{B}$ correspond to the plot aliases of the main effects of the treatment factors. A useful and interesting byproduct of this is a simple rule for determining the trend-resistant properties of all the factorial effects from the generator sequence. This allows one to bypass the selection of linearly independent pencils, and search directly for generator sequences which will produce run orders with desired trend-resistant properties.

Theorem 4.1. *Consider a complete s^n design, where s is a prime power. In a run order constructed by applying the generalized foldover method to the generator sequence $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_n$, where $\mathbf{x}_i = (x_{i1}, \ldots, x_{in})'$, we have the following:*
(a) *all the main-effect contrasts of a factor are t-trend free if that factor appears at nonzero levels in $t+1$ generators;*
(b) *all the contrasts associated with the r-factor interaction component $A_{i_1}^{c_{i_1}} A_{i_2}^{c_{i_2}} \cdots A_{i_r}^{c_{i_r}}$ are t-trend free if there are t+1 generators such that $\mathbf{c}'\mathbf{x}_i \neq 0$, where $\mathbf{c}$ has the i_1th, $\ldots$, i_rth entries equal to $c_{i_1}, \ldots, c_{i_r}$, respectively, and all the other entries are 0.*

Proof. Part (a) can be found in Coster and Cheng (1988). A simpler and perhaps more insightful proof can be based on the remark in the paragraph preceding the statement of this theorem. It follows that the plot alias of the main effect component A_i is $P_1^{x_{1i}} \cdots P_n^{x_{ni}}$. If A_i appears at nonzero levels in $t+1$ generators, i.e., $t+1$ of $x_{1i}, \ldots, x_{ni}$ are nonzero, then $P_1^{x_{1i}} \cdots P_n^{x_{ni}}$ is a $(t+1)$-factor interaction. The result then follows.

Part (b) can be similarly proved. The plot alias of $A_{i_1}^{c_{i_1}} A_{i_2}^{c_{i_2}} \cdots A_{i_r}^{c_{i_r}}$ is $P_1^{z_1} \cdots P_n^{z_n}$, where $z_i = \mathbf{c}'\mathbf{x}_i$. So if there are $(t+1)$ generators such that $\mathbf{c}'\mathbf{x}_i \neq 0$, then the plot alias of $A_{i_1}^{c_{i_1}} A_{i_2}^{c_{i_2}} \cdots A_{i_r}^{c_{i_r}}$ is a $(t+1)$-factor interaction and is t-trend free.

Example 4.4. In a complete 3^4 experiment with four 3-level factors A, B, C and D, a run order is constructed by applying the generalized foldover method to the generator sequence $abcd$, a^2b^2cd, a^2bc^2d, a^2bcd^2. Here $\mathbf{x}_1 = (1,1,1,1)'$, $\mathbf{x}_2 = (2,2,1,1)'$, $\mathbf{x}_3 = (2,1,2,1)'$, and $\mathbf{x}_4 = (2,1,1,2)'$. The row vectors of $\mathbf{B} = [\mathbf{x}_1\,\mathbf{x}_2\,\mathbf{x}_3\,\mathbf{x}_4]$ are $(1,2,2,2,)'$, $(1,2,1,1)'$, $(1,1,2,1)'$ and $(1,1,1,2)'$. It follows that the plot aliases of the main effects A, B, C and D are $PQ^2R^2S^2$, PQ^2RS, PQR^2S and $PQRS^2$, respectively. So, all the main-effect contrasts are at least cubic-trend free. The trend-tree properties of all the interaction contrasts can be determined by identifying their plot aliases. One can also obtain the same conclusion by using Theorem 4.1. For example, all the main-effect contrasts are cubic-trend free, since each of a, b, c and d appears in all the four generators. Denote the levels of A and B in a generator by $q(A)$ and $q(B)$, respectively. To determine the trend-free property of the interaction component AB, we need to count the number of generators such that $q(A) + q(B) \neq 0 \pmod 3$. There are two such generators: $abcd$ and a^2b^2cd. So AB is linear-trend free. Since there are also two generators with $q(A) + 2q(B) \neq 0 \pmod 3$, AB^2 is also linear-trend free. Therefore all the contrasts representing the interaction of A and B are linear-trend free.

For two-level designs, part (b) of Theorem 4.1 simplifies to the following: an r-factor interaction $A_{i1} \cdots A_{i_r}$ is t-trend free if there are $t+1$ generators each of which contains an odd number of letters out of $a_{i_1}, \ldots, a_{i_r}$. This is Theorem 4.1 of Cheng (1990).

It is easy to see that Theorem 4.1 also holds for all the *estimable* main effects and interactions in a run order of a regular fractional factorial design obtained by the generalized foldover method.

5. Generalized Foldover Method for General Factorial Designs

In an asymmetrical design, the method of Section 4 can be applied separately to each subdesign consisting of factors with the same number of levels. The generators of the resulting run order are such that only factors with the same number of levels can appear at nonzero levels in the same generator. Using (2.1), however, one can apply the generalized foldover method to an arbitrary generator sequence in any design. An important issue is how to determine the trend-free properties of the factorial effects. Corollary 3.2 and Theorem 2.1 together provide a solution.

Throughout this section, we shall consider subgroups of the abelian group $S = < \mathbf{e}_1 > \oplus \cdots \oplus < \mathbf{e}_n >$ under operation (2.1). Unlike in Section 4, even for complete symmetric factorial designs, the number of generators is not necessarily equal to the number of factors.

Suppose $G = < \mathbf{x}_1 > \oplus \cdots \oplus < \mathbf{x}_m >$ is a subgroup of S, where $\mathbf{x}_i$ has order l_i. Let $P_1, \ldots, P_m$ be m plot factors, with P_i having l_i levels. Then it is clear that the run order obtained by applying the generalized foldover method to the generator sequence $\mathbf{x}_1, \ldots, \mathbf{x}_m$ can be constructed from the standard order with respect to $P_1, \ldots, P_m$ through the design key

$$A_i = \sum_{j=1}^{m} x_{ji} P_j.$$

Here, unlike (4.1) in which field operations are used, treatment factor A_i takes level

$$\sum_{j=1}^{m} x_{ji} y_j \ \bmod\ s_i$$

on the run whose level of P_j is y_j, $j = 1, \ldots, m$.

Theorem 5.1. *Let $G = <\mathbf{x}_1> \oplus \cdots \oplus <\mathbf{x}_m>$ be a subgroup of S. Then for any $\chi \in G^*$, if there are t+1 generators such that $\chi(\mathbf{x}_i) \neq 0$, then in the run order of G obtained by applying the generalized foldover method to the sequence of generators $\mathbf{x}_1, \ldots, \mathbf{x}_m$, all the vectors in W_χ are t-trend free.*

Proof. It follows from Corollary 3.2 that for any subset J of $\{1, \ldots, m\}$ of size $t + 1$, all the vectors in W_J are t-trend free. For any $\chi \in G^*$, let $J(\chi) = \{i : 1 \leq i \leq m,\ \chi(\mathbf{x}_i) \neq 0\}$. From Theorem 2.1, W_χ is a subspace of $W_{J(\chi)}$. Therefore if $J(\chi)$ contains $t + 1$ elements, then all the vectors in W_χ are t-trend free.

For complete factorial designs, we have the following

Corollary 5.2. *Suppose $G = S = <\mathbf{e}_1> \oplus \cdots \oplus <\mathbf{e}_n>$ is a complete $s_1 \times \cdots \times s_n$ design. Let $\mathbf{y}$ be any elememt of G with order l. In the run order obtained by applying the generalized foldover method to another set of generators $\mathbf{x}_1, \ldots, \mathbf{x}_m$, the $\phi(l)$ degrees of freedom in $T_*(\mathbf{y}')$ are t-trend free if there are $t + 1$ $\mathbf{x}_i$'s such that $[\mathbf{x}_i, \mathbf{y}] \neq 0$, where $[\mathbf{x}, \mathbf{y}] = \sum_{i=1}^{n} x_i y_i \lambda / s_i \bmod \lambda$ and λ is the least common multiple of $s_1, \ldots, s_n$.*

Example 5.1. Consider the run order of a complete $2 \times 2 \times 3 \times 6$ experiment obtained by applying the generalized foldover method to the sequence of generators $(1, 0, 0, 3)'$, $(1, 1, 0, 3)'$, $(0, 1, 0, 3)'$, $(0, 0, 1, 2)'$, $(0, 0, 2, 2)'$. These five generators have orders 2, 2, 2, 3 and 3, respectively. Corollary 5.2 can be used to verify that all the main-effect contrasts are at least linear-trend free. For example, for $\mathbf{y} = (0, 0, 0, 3)'$, there are three generators $\mathbf{x}$ such that $[\mathbf{x}, \mathbf{y}] \neq 0$; therefore the one degree of freedom in $T_*(0, 0, 0, 3)$ is 2-trend free. Similarly, the two degrees of freedom in $T_*(0, 0, 0, 1)$ and the two degrees of freedom in $T_*(0, 0, 0, 2)$ are 4- and 1-trend free, respectively. It follows that all the five degrees of freedom of the main effect of the 6-level factor are at least linear-trend free. The trend-resistant properties of all the interaction components can be similarly determined. For example, the two degrees of freedom in $T_*(0, 0, 1, 1)$ are 3-trend free since there are four generators $\mathbf{x}$ with $[\mathbf{x}, \mathbf{y}] \neq 0$, where $\mathbf{y} = (0, 0, 1, 1)'$. However, the two degrees of freedom in $T_*(0, 0, 1, 2)$ are not linear-trend free. In fact, no 2-factor interaction involving the 6-level factor has all its contrasts linear-trend free, although all the other 2-factor interactions are at least linear-trend free.

When s_i is a prime number, all the $s_i - 1$ degrees of freedom in the main effect of factor A_i is in $T_*(\mathbf{e}_i')$. From Corollary 5.2, the following is evident:

Corollary 5.3. *In the run order of a complete $s_1 \times \cdots \times s_n$ design obtained by applying the generalized foldover method to a set of generators, if s_i is a prime number, then all the main-effect contrasts of factor A_i are t-trend free if A_i appears at nonzero levels in at least $t+1$ generators.*

A result similar to Corollary 5.3 was also obtained by Coster (1988) for his generalized foldover method. Besides the fact that our construction is more general, results such as Theorem 4.1 or Corollary 5.2 for determining trend-resistant properties of interaction effects from the generators are not available in Coster (1988). His (partial) results on interaction effects are difficult to use. Our proof based on Theorem 3.1 is also more transparent.

For fractional factorial designs, the result in Corollary 5.2 still holds so long as the alias relations are taken into account.

6. Quantitative Factors

The results in Section 4 are useful for constructing run orders in which the interaction components defined via finite geometries have desired trend-resistant properties. When treatment factors are quantitative, these interaction components bear no simple relation to the interaction contrasts defined by orthogonal polynomials, the contrasts typically of interest. Some examples where trend-resistant properties can be established for such contrasts are given in Section 3 (Example 3.1) and Section 4 (Example 4.2).

Suppose all the factors have s levels, where s is a prime number. When all factor levels are equally spaced, a contrast such as linear $A_1\times$linear A_2 is partially confounded with all the interaction components $A_1A_2, A_1A_2^2, \ldots, A_1A_2^{s-1}$. In order to use the methods of Section 4 to construct a run order such that the lin $A_1\times$lin A_2 contrast is t-trend free, we need to make all of $A_1A_2, A_1A_2^2, \ldots, A_1A_2^{s-1}$ t-trend free. For quantitative factors, Bailey (1982) suggested taking the actual level corresponding to nominal level $q(A_i)$ of factor A_i $(0 \le q(A_i) \le s-1)$ to be

$$k_1 + k_2 \sin q(A_i)\theta, \text{ for } \theta = 2\pi/s,$$

where k_1 and k_2 are constants, possibly different from factor to factor. The advantage of choosing such levels is that the corresponding orthogonal polynomials for a factor A are

$$\begin{aligned}
\text{linear } A &= \mathrm{s}_1(A), \\
\text{quadratic } A &= \mathrm{c}_2(A), \\
&\cdots\cdots \\
(2h)\text{th degree orthogonal polynomial in } A &= \mathrm{c}_{2h}(A), \\
(2h+1)\text{st degree orthogonal polynomial in } A &= \mathrm{s}_{2h+1}(A),
\end{aligned}$$

where $\mathbf{c}_h(A)$ and $\mathbf{s}_h(A)$ are $s\times 1$ vectors, $\mathbf{c}_h(A)$ has entries $\cos(h\theta q(A))$, and $\mathbf{s}_h(A)$ has entries $\sin(h\theta q(A))$, $0 \le h \le s-1$, $0 \le q(A) \le s-1$. Contrasts $\mathbf{c}_h$ and $\mathbf{s}_h$ are similarly defined for interaction terms: $\mathbf{c}_h(A_iA_j^k)$ has entries $\cos(h\theta(q(A_i)+kq(A_j)))$, and $\mathbf{s}_h(A_iA_j^k)$ has entries $\sin(h\theta(q(A_i)+kq(A_j)))$. Then it can be shown that

$$\begin{aligned}
\mathbf{c}_h(A_i)\otimes\mathbf{c}_l(A_j) &= \frac{1}{2}\mathbf{c}_h(A_iA_j^g)+\frac{1}{2}\mathbf{c}_h(A_iA_j^{-g})\\
\mathbf{c}_h(A_i)\otimes\mathbf{s}_l(A_j) &= \frac{1}{2}\mathbf{s}_h(A_iA_j^g)-\frac{1}{2}\mathbf{s}_h(A_iA_j^{-g})\\
\mathbf{s}_h(A_i)\otimes\mathbf{c}_l(A_j) &= \frac{1}{2}\mathbf{s}_h(A_iA_j^g)+\frac{1}{2}\mathbf{s}_h(A_iA_j^{-g})\\
\mathbf{s}_h(A_i)\otimes\mathbf{s}_l(A_j) &= -\frac{1}{2}\mathbf{c}_h(A_iA_j^g)+\frac{1}{2}\mathbf{c}_h(A_iA_j^{-g}),
\end{aligned}$$

where $gh = l \bmod s$; see Bailey (1982). Factors A_i and A_j above can be replaced by any generalized interactions. Therefore any Kronecker product of c- and s-contrasts from r factors has components in 2^{r-1} of the $(s-1)^{r-1}$ sets into which the corresponding r-factor interaction is divided by the method of finite geometries. For example, the linear $A_i\times$linear A_j contrast is $\mathbf{s}_1(A_i)\otimes\mathbf{s}_1(A_j) = -\frac{1}{2}\mathbf{c}_1(A_iA_j)+\frac{1}{2}\mathbf{c}_1(A_iA_j^{s-1})$. Thus to make linear $A_i\times$linear A_j t-trend free, it is enough to require A_iA_j and $A_iA_j^{s-1}$ to be t-trend free.

7. Concluding Remarks

In this paper, a method of constructing trend-resistant run orders of factorial designs, which goes back to Daniel and Wilcoxon (1966), is discussed. The run orders constructed by this method can also be obtained by the generalized foldover method, which is much easier to execute. The generalized foldover method is useful in its own right. The trend-resistant properties of all the main-effect and interaction components can easily be determined from the generator sequence. This can be used to search for generator sequences which will produce run orders with desired trend-resistant properties. Extensions to general asymmetrical factorial designs is also presented.

It is interesting to note that the order in which the generators appear is not important in the conditions of Theorem 4.1 and Corollary 5.2. In other words, it does not affect the trend-resistant properties of any main-effect or interaction component. This provides much needed flexibility for achieving other secondary design goals. It can be used, for example, to perform some limited randomization. Another possible application is to the case where the cost of level changes is an important consideration. When the experimenter wants to minimize the cost of level changes, the order in which the generators appear becomes important.

Cheng (1985) considered the problem of minimizing the total cost of level changes. Sometimes it is possible to construct run orders which provide required protection of important factorial effects against time trend effects as well as achieve minimum cost of level changes.

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