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CONSTRUCTION OF ASYMMETRICAL ORTHOGONAL ARRAYS OF THE TYPE $OA(s^k, s^m(s^{r_1})^{n_1}\cdots(s^{r_t})^{n_t})$

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Abstract: We extend the grouping scheme introduced by Wu (1989) and construct a class of saturated asymmetrical orthogonal arrays of the type $OA(s^k, s^m(s^r)^n)$, where s is a prime power and r is any positive integer. The method is generalized to construct $OA(s^k, s^m(s^{r_1})^{n_1}\cdots(s^{r_t})^{n_t})$ for any prime power s, any positive integer r_j, and some combinations of m and n_j.

Key words and phrases: Asymmetrical orthogonal arrays, fractional factorial designs, method of replacement, grouping scheme.

1. Introduction

In this paper we exploit and further expand the method of grouping (Wu (1989)) to construct some general classes of asymmetrical orthogonal arrays. An asymmetrical (or mixed-level) orthogonal array $OA(N, s_1^{k_1}s_2^{k_2}\cdots s_q^{k_q}, 2)$ of strength two is an $N\times k$ matrix, $k=\sum_1^q k_i$, $q>1$, in which k_i columns have s_i symbols such that for any two columns each possible combination of symbols appears equally often (Rao (1973)). In the language of factorial designs, these k_i columns are k_i factors each with s_i levels. When $q=1$, it is called a (symmetrical) orthogonal array. Since we only consider saturated arrays, i.e. those with $N-1=\sum_1^q k_i(s_i-1)$, the arrays have strength two. For simplicity we use the notation $OA(N, s_1^{k_1}\cdots s_q^{k_q})$ for the rest of the paper.

To explain the grouping method, let us consider an important special case. If in an $OA(N, 2^{N-1})$ we can find three columns $\{v_1, v_2, v_1+v_2\}$, where v_1+v_2 is the sum (mod 2) of v_1 and v_2 (in factorial designs v_1+v_2 represents the interaction between v_1 and v_2), we can replace these three 2-level columns by a 4-level column according to the rule

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix} \;\rightarrow\; \begin{matrix} 0 \\ 1 \\ 2 \\ 3. \end{matrix} \tag{1.1}$$

It is known (Addelman (1962)) that this 4-level column is orthogonal (in the sense defined above) to the remaining 2-level columns in the $OA(N, 2^{N-1})$. By repeating this replacement for other sets of columns of the form $\{v_1, v_2, v_1 + v_2\}$, we can construct $OA(N, 2^m 4^n)$, $m + 3n = N - 1$. When the $OA(N, 2^{N-1})$, $N = 2^k$, is the saturated 2-level fractional factorial design with $2^k - 1$ factors, its columns can be grouped into a maximum number of sets of the form $\{v_1, v_2, v_1 + v_2\}$. Then by using the replacement rule, a complete class of $OA(2^k, 2^m 4^n)$ can be constructed (Wu (1989)).

Wu's grouping scheme has, however, two restrictions. First, it only solves the problem for the $2^m 4^n$ designs. Second, even for $N = 2^k$, k odd, the grouping scheme requires the construction of two particular permutations of $\{1, \ldots, N\}$ and therefore does not render a simple construction of the arrays. The main purpose of this paper is to develop general and simple methods for constructing more general classes of asymmetrical orthogonal arrays. In Section 3, by expanding on Wu's idea, we develop a general grouping scheme and use it to construct $OA(s^k, s^m(s^r)^n)$ for any prime power s and integer r. An illustration of the construction method is given in Section 2. A detailed description of the construction steps and some examples are given in Section 4. In Section 5 we discuss the question of whether the number of the s^r-level factors in $OA(s^k, s^m(s^r)^n)$ can be further increased. The method is extended in Section 6 to construct $OA(s^k, s^m(s^{r_1})^{n_1} \cdots (s^{r_t})^{n_t})$, for any prime power s and positive integer r_j, and some combinations of m and n_j. Some discussion on the related work by Pu (1989) and Hedayat, Pu and Stufken (1990) is given near the end of Section 6.

Although some of the arrays can be constructed by using other methods such as Pu (1989), J. C. Wang and Wu (1991) and Hedayat et al. (1990), the present approach enjoys some advantages. First, the grouping method makes it easier to study the aliasing patterns of main effects and interactions in asymmetrical arrays. For an elaboration of this point, see Section 3 of Wu (1989). Second, it is quite simple. The construction steps employ some elementary algebraic tools.

The results obtained in this paper have theoretical implications for combinatorial design theory as well as some practical applications. Most of the new arrays constructed in the paper are of very large size, thus making them much less useful in the design of physical experiments. Since orthogonal arrays are used in a great variety of scientific investigations (see the review article by Hedayat and Wallis (1978)), these new arrays are potentially useful in situations in which the run size can be large. For example, they can be used to draw balanced pseudoreplicates for inference from stratified survey samples (Wu (1991)), where each stratum is treated as a "factor" and the number of units per stratum as "factor levels". The run size being the total number of pseudoreplicates can be quite large, say, up to 400, if computational cost is not an issue.

2. An Illustrative Example

We illustrate the general construction method of Section 3 by constructing the $OA(32, 2^m 4^n)$ with $m+3n = 31$ and $n \leq 9$ from the $OA(32, 2^{31})$ via a grouping scheme. Let x_1, x_2, x_3, x_4, x_5, be five independent columns of the $OA(32, 2^{31})$. In the language of experimental design, these five columns form a full factorial design with 32 runs and five factors each at two levels. Then the 31 columns of the $OA(32, 2^{31})$ can be represented as $\sum_1^5 a_i x_i$, where $a_i = 0$ or 1 (mod 2). For simplicity, each column is represented by the vector $(a_1, a_2, a_3, a_4, a_5)$. The 31 vectors for the columns are given in I_1, I_2' and K_2 in Table 1. The 24 vectors in I_1 consist of $(a_1, a_2, a_3, a_4, a_5)$ with $a_1 = 1$ or $a_2 = 1$. The three vectors in I_2' consist of $(0, 0, a_3, a_4, 0)$ with $a_3 = 1$ or $a_4 = 1$ and the four remaining vectors in K_2 consist of $(0, 0, a_3, a_4, 1)$ with $a_3 = 1$ or $a_4 = 1$ and $(0,0,0,0,1)$.

Table 1. A grouping scheme for the 31 columns in the $OA(32, 2^{31})$

I_1 :	{ (1,1,0,0,1)	(1,0,1,1,0)	(0,1,1,1,1) }
	{ (1,1,0,1,1)	(1,0,1,0,1)	(0,1,1,1,0) }
	{ (1,1,1,1,1)	(1,0,0,1,0)	(0,1,1,0,1) }
	{ (1,1,1,0,1)	(1,0,0,0,1)	(0,1,1,0,0) }
	{ (1,1,1,0,0)	(1,0,1,1,1)	(0,1,0,1,1) }
	{ (1,1,1,1,0)	(1,0,1,0,0)	(0,1,0,1,0) }
	{ (1,1,0,1,0)	(1,0,0,1,1)	(0,1,0,0,1) }
	{ (1,1,0,0,0)	(1,0,0,0,0)	(0,1,0,0,0) }
I_2' :	{ (0,0,1,1,0)	(0,0,1,0,0)	(0,0,0,1,0) }
K_2 :	(0,0,1,1,1)		
	(0,0,1,0,1)		
	(0,0,0,1,1)		
	(0,0,0,0,1)		

As discussed in Section 1, any three columns in the $OA(32, 2^{31})$ whose vector representations v_1, v_2 and v_3 satisfy $v_3 = v_1 + v_2$ (mod 2) can be replaced by a 4-level column (see (1.1)) without affecting the orthogonality property. By repeating this for n sets of columns of the form $\{v_1, v_2, v_1 + v_2\}$, we get the $OA(32, 2^m 4^n)$. So the construction amounts to grouping the vectors in Table 1 into as many triplets of vectors $\{v_1, v_2, v_1 + v_2\}$ as possible. The three vectors in I_2' form one such triplet. The 24 vectors in I_1 are grouped into eight such triplets in Table 1 by using the following method. (A general version will be described in Section 3.) We can represent (a_1, a_2) in any $v \in I_1$ as $c_1(1,0) + c_2(0,1)$ with $(c_1, c_2) = (1,1)$, (1,0) or (0,1). The remaining (a_3, a_4, a_5) components of the three vectors in any triplet can be represented as $a(G + I)$, aG, a with

$$G = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and I the 3×3 identity matrix. Therefore, the three vectors in any triplet can be represented as $\{v_1 + v_2, v_1, v_2\}$ with $v_1 = (1, 0, aG)$ and $v_2 = (0, 1, a)$. Since both G and $G + I$ are nonsingular, the method partitions the 24 vectors into eight mutually exclusive triplets $\{v_1 + v_2, v_1, v_2\}$. No such grouping is possible for the vectors in K_2. In fact it is known (Wu (1989)) that nine is the maximum for n in this case and therefore the grouping scheme cannot be further improved.

3. Construction of $OA(s^k, s^m(s^r)^n)$ from $OA(s^k, s^{(s^k-1)/(s-1)})$

Let $OA(s^k, s^L)$ be a saturated orthogonal array, s a prime power, $L = (s^k - 1)/(s - 1)$, and $x_1, \ldots, x_k$ be its k independent columns. Then all its columns can be represented as $\sum_{i=1}^{k} a_i x_i$, where a_i is an element of the finite field $GF(s)$ of s elements. For simplicity, each column is represented by the k-vector $v = (a_1, \ldots, a_k)$, where the first nonzero a_i is assumed to be 1, the identity element of $GF(s)$. Consider any set of $1 + s + \cdots + s^{r-1} = (s^r - 1)/(s - 1)$ columns whose vector representations can be written as

$$H_{r,k} = \left\{ \sum_{i=0}^{r-1} c_i w_i : w_i \text{ are independent } k\text{-vectors and } c_i \text{ satisfy (3.2)} \right\}, \tag{3.1}$$

where

$$\text{“}c_i \in GF(s), \text{ at least one } c_i \text{ is nonzero and the nonzero } c_i \text{ with the largest } i \text{ is set to be 1.”} \tag{3.2}$$

Similar to (1.1), we can replace these $(s^r - 1)/(s - 1)$ columns, each of s levels, by an s^r-level column which represents the s^r level combinations of $w_0, \ldots, w_{r-1}$, and still retain orthogonality. By repeating this for other sets of columns satisfying (3.1), we can obtain $OA(s^k, s^m t^n)$, $t = s^r$ with n the number of such sets. Therefore the construction depends on finding a method to group the columns of the $OA(s^k, s^L)$ into sets of columns of the form (3.1).

For $k = rq$, we do not need any elaborate method of construction. Since $N = s^k = t^q$, $t = s^r$, we have $OA(N, t^{(N-1)/(t-1)})$. By reversing the replacement rule, we can replace any s^r-level column by $(s^r - 1)/(s - 1)$ s-level columns. By doing this repeatedly, we obtain $OA(s^k, s^m t^n)$ for any m and n with $m(s - 1) + n(s^r - 1) = s^k - 1$.

For the rest of this section we concentrate on the development of a grouping method for general k and the largest possible n. We first partition all the columns of $OA(s^k, s^L)$ into mutually exclusive sets as follows. Let H_r be as defined in (3.1) with k (in the k-vector) replaced by r, H_{p+r} be similarly defined with both

r and k replaced by $p+r$, and $\mathbf{0}_r$ be the r-vector of zeros. Define

$$\begin{aligned} I_1 &= \{(\boldsymbol{u}, a_{r+1}, \dots, a_k) : \boldsymbol{u} \in H_r,\ a_i \in GF(s)\}, \\ I_j &= \{(\mathbf{0}_{(j-1)r}, \boldsymbol{u}, a_{jr+1}, \dots, a_k) : \boldsymbol{u} \in H_r,\ a_i \in GF(s)\} \\ &\qquad \text{for } 2 \le j \le q-1, \\ I'_q &= \{(\mathbf{0}_{(q-1)r}, \boldsymbol{u}, \mathbf{0}_p) : \boldsymbol{u} \in H_r\}, \\ K_q &= \{(\mathbf{0}_{(q-1)r}, \boldsymbol{u}) : \boldsymbol{u} \in H_{p+r}\} \backslash I'_q. \end{aligned} \tag{3.3}$$

It is clear that all the L columns in $OA(s^k, s^L)$ can be partitioned into $I_1 \cup \cdots \cup I_{q-1} \cup I'_q \cup K_q$ for $p > 0$, and $I_1 \cup \cdots \cup I_{q-1} \cup I_q$ for $p = 0$, where $I_q = \{(\mathbf{0}_{(q-1)r}, \boldsymbol{u}) : \boldsymbol{u} \in H_r\}$.

Since the $(s^r-1)/(s-1)$ vectors in I'_q are of the form (3.1), they can be grouped into an s^r-level column as shown before. For each of the I_j's, $1 \le j \le q-1$, we will find a method to partition its vectors into mutually exclusive sets of $(s^r-1)/(s-1)$ vectors of the form (3.1).

Starting with I_1, suppose that we can find $r-1$ $(k-r)\times(k-r)$ matrices G_i, $i = 1, \dots, r-1$ whose elements are in $GF(s)$, such that the following matrices

$$\sum_0^{r-1} c_i G_i \text{ for all } c_i \in GF(s) \text{ and satisfying (3.2),} \tag{3.4}$$

where $G_0 = I$ the identity matrix, are nonsingular. By rewriting the $(s^r-1)/(s-1)$ vectors $\boldsymbol{u}$ in H_r as $\sum_0^{r-1} c_i \boldsymbol{u}_i$, where $\boldsymbol{u}_i$ are independent r-vectors and c_i satisfy (3.2), the $(s^r-1)/(s-1)$ vectors $(\sum_0^{r-1} c_i\boldsymbol{u}_i, \boldsymbol{a}\sum_0^{r-1} c_i G_i)$ for any given $\boldsymbol{a} = (a_{r+1}, \dots, a_k)$ satisfy (3.1) because they can be rewritten as $\sum_0^{r-1} c_i(\boldsymbol{u}_i, \boldsymbol{a}G_i)$. By repeating this for other choices of $\boldsymbol{a}$, we can partition the $(s^r-1)(s-1)^{-1}s^{k-r}$ vectors in I_1 into s^{k-r} mutually exclusive sets of vectors of the form (3.1) because of the condition (3.4).

Similarly, for I_2, if we can find $r-1$ $(k-2r)\times(k-2r)$ matrices satisfying conditions similar to (3.4), then using the same approach we can partition all the vectors in I_2 into s^{k-2r} mutually exclusive sets of vectors of the form (3.1). The same procedure can be applied to $I_3, \dots, I_{q-1}$. It is, however, inapplicable to K_q. Since K_q has $s^r(s^p-1)/(s-1)$ columns, this method will allow us to group $(s^k - s^{r+p} + s^r - 1)(s-1)^{-1}$ columns in $OA(s^k, s^L)$ into $(s^k - s^{r+p})/(s^r-1)+1$ columns, each of s^r levels.

The impossibility of getting another $(s^r-1)/(s-1)$ columns of the form (3.1) from K_q can be proved as follows. The vectors in

$$K_q \cup I'_q = \{(\mathbf{0}_{(q-1)r}, \boldsymbol{u}) : \boldsymbol{u} \in H_{p+r}\},$$

by ignoring their first $(q-1)r$ components of zeros, form an $OA(s^{p+r},$

$s^{(s^{p+r}-1)/(s-1)})$. If the statement above were true, by using replacement, we could get two orthogonal s^r-level columns within this array. That is, we could get an $OA(s^{p+r},(s^r)^2)$, which does not exist because $p<r$ implies that s^{p+r} is not a multiple of s^{2r}.

Through this analysis, we have demonstrated the construction of $OA(s^k, s^m t^n)$, $t=s^r$ for general m and n if we can find $r-1$ $(k-jr)\times(k-jr)$ matrices G_i to satisfy (3.4) for any $j\le q-1$. Since $k-jr=rq+p-jr=(q-j)r+p\ge r$ and the identity matrix I, which is G_0 in (3.4), is obviously nonsingular, we can restate (3.4) as the existence and construction of $r-1$ $\ell\times\ell$ matrices $G_1,\dots,G_{r-1}$, for *any* $\ell\ge r$, such that the $s+s^2+\cdots+s^{r-1}$ matrices

$$G_j+\sum_{i=1}^{j-1}c_{ji}G_i+c_{j0}I \tag{3.5}$$

are nonsingular for $j=1,\dots,r-1$, where c_{ji} and the elements of G_j are from $GF(s)$.

An explicit solution to (3.5) is given in the following theorem.

Theorem 1. *For any prime power s and any positive integers r and ℓ with $\ell\ge r$, we can construct an $\ell\times\ell$ matrix G_ℓ over $GF(s)$ such that all the matrices*

$$G_\ell^j+\sum_{i=1}^{j-1}c_{ji}G_\ell^i+c_{j0}I, \tag{3.6}$$

for $j=1,\dots,r-1$, $c_{ji}\in GF(s)$, are nonsingular.

Proof. The proof is based on the three lemmas given at the end of this section.

From Lemma 2, for any positive integer ℓ, we can take an irreducible polynomial of degree ℓ over $GF(s)$, say $f(x)=x^\ell+\lambda_1x^{\ell-1}-\lambda_2x^{\ell-2}+\cdots+(-1)^{\ell-2}\lambda_{\ell-1}x+(-1)^{\ell-1}\lambda_\ell$ and the corresponding irreducible matrix G_ℓ in (3.11) such that $f(x)=\det(G_\ell+xI)$. Let $\alpha_1,\dots,\alpha_\ell$ be the characteristic roots of G_ℓ. Then by Lemma 3 we have

$$|G_\ell^u+i_1G_\ell^{u-1}+\cdots+i_{u-1}G_\ell+i_uI|=\prod_{j=1}^{\ell}g_u(\alpha_j), \tag{3.7}$$

where $g_u(x)=x^u+i_1x^{u-1}+\cdots+i_{u-1}x+i_u$, $i_1,\dots,i_u\in GF(s)$. By Lemma 1, $\alpha_1,\dots,\alpha_\ell$ are distinct and $F(\alpha_1,\dots,\alpha_\ell)$ with $F=GF(s)$ is a finite algebraic extension of $GF(s)$. Over $F(\alpha_1,\dots,\alpha_\ell)$ we have $g_u(\alpha_j)\ne0$, $j=1,\dots,\ell$, for any $u < r \leq \ell$ and hence

$$\prod_{j=1}^{\ell} g_u(\alpha_j) \neq 0. \tag{3.8}$$

Since the orders of the polynomials in (3.6) do not exceed $r - 1$, we conclude from (3.8) that the matrices in (3.6) are nonsingular, thus completing the proof.

From this proof it is clear that the crucial step in the construction is the irreducible matrix G_ℓ from (3.11). Since ℓ can be quite large, we can use the following technique to reduce the work of construction. For $\ell \geq 2r$, write $\ell = rq_1 + p = r(q_1 - 1) + (r + p)$ with $q_1 - 1 \geq 1$ and $0 \leq p < r$. Let G_r and G_{r+p} be $r \times r$ and $(r + p) \times (r + p)$ matrices respectively satisfying (3.6). Then define the $\ell \times \ell$ matrix

$$G_\ell = \begin{cases} \operatorname{diag}(G_r, \dots, G_r), & \text{if } p = 0, \\ \operatorname{diag}(G_r, \dots, G_r, G_{r+p}), & \text{if } p > 0, \end{cases} \tag{3.9}$$

where G_r repeats for q_1 times if $p = 0$, and $q_1 - 1$ times if $p > 0$. It is clear that G_ℓ in (3.9) satisfies (3.6) since G_r and G_{r+p} both satisfy (3.6). Noting that the orders of G_r and G_{r+p} do not exceed $2r$, we will construct G_r and G_{r+p} according to (3.11) and then use (3.9) to build up the larger matrix G_ℓ for any $\ell \geq 2r$. This idea of using smaller matrices to build up a bigger matrix (and hence a bigger design) is implicit in Wu (1989, proof of Theorem 2).

Using the grouping scheme given at the beginning of this section, we obtain from Theorem 1 the result that, for any prime power s and any positive integer r, starting from the saturated orthogonal array $OA(s^k, s^L)$, we can construct asymmetrical orthogonal arrays of the type $OA(s^k, s^m(s^r)^n)$. This is stated precisely in the following theorem.

Theorem 2. *For any prime power s and arbitrary positive integers r and k, where $k = rq+p$, $q \geq 1$, and $0 \leq p \leq r-1$, via the grouping scheme in this section, we can construct asymmetrical orthogonal arrays of the type $OA(s^k, s^m(s^r)^n)$, for any integers m and n satisfying*

$$\begin{aligned} & m(s-1) + n(s^r - 1) = s^k - 1 \\ \textit{and} \quad & n = 1, 2, \dots, (s^k - s^{r+p})/(s^r - 1) + 1. \end{aligned} \tag{3.10}$$

This result was also obtained by Pu (1989) and Hedayat et al. (1990) using different methods. See Section 6 for more information.

Let F be a field. A field K is said to be an extension of F if F is a subfield of K. We use $F(S)$ to denote the minimum field which includes S and F. If S

is a finite set, say, $S = \{\alpha_1, \dots, \alpha_n\}$, then we denote $F(S)$ by $F(\alpha_1, \dots, \alpha_n)$. From the theory of finite fields, an extension field K of F is a linear space over F. If the dimension of the space is finite, then we say that the extension is finite. Usually, $F[x]$ denotes the polynomial ring in x over F. A number α is said to be algebraic if there is a polynomial $f(x) \in F[x]$ such that $f(\alpha) = 0$. An extension field K of F is said to be algebraic if every element in K is algebraic over F.

Lemma 1. *Let $F = GF(s)$ and $f(x)$ be an irreducible polynomial in $F[x]$ of degree n. Let $\alpha_1, \dots, \alpha_n$ be the roots of $f(x)$. Then*
(i) *$\alpha_1, \dots, \alpha_n$ are distinct,*
(ii) *$F(\alpha_1, \dots, \alpha_n)$ is a finite algebraic extension of F,*
(iii) *for any polynomial $g(x)$ with degree less than n over F, we have $g(\alpha_i) \neq 0$, $i = 1, \dots, n$ and hence $g(\alpha_1) \cdots g(\alpha_n) \neq 0$.*

Lemma 2. *For any prime power s and positive integer ℓ, there exist an irreducible polynomial in $F[x]$ of degree ℓ, say $f(x) = x^\ell + \lambda_1 x^{\ell-1} - \lambda_2 x^{\ell-2} + \cdots + (-1)^{\ell-2}\lambda_{\ell-1}x + (-1)^{\ell-1}\lambda_\ell$, and the associated $\ell \times \ell$ matrix over $GF(s)$*

$$G_\ell = \begin{bmatrix} 0 & 0 & \dots & 0 & \lambda_\ell \\ 1 & 0 & \dots & 0 & \lambda_{\ell-1} \\ & & \dots\dots & & \\ 0 & 0 & \dots & 0 & \lambda_2 \\ 0 & 0 & \dots & 1 & \lambda_1 \end{bmatrix}, \tag{3.11}$$

whose lower left submatrix is the $(\ell-1) \times (\ell-1)$ identity matrix, such that $\det(G_\ell + xI) = f(x)$.

We call G_ℓ an *irreducible matrix* if $f(x) = \det(G_\ell + xI)$ is irreducible.

Lemma 3. *Let G be an $\ell \times \ell$ matrix over $GF(s)$, and $\alpha_1, \dots, \alpha_\ell$ be its characteristic roots. Then, for any positive integer u, the determinant*

$$|G^u + i_1 G^{u-1} + \cdots + i_{u-1}G + i_u I| = \prod_{j=1}^{\ell} g_u(\alpha_j),$$

where $g_u(x) = x^u + i_1 x^{u-1} + \cdots + i_{u-1}x + i_u$, and $i_1, \dots, i_u \in GF(s)$.

These lemmas can be readily proved by consulting any standard textbook on modern algebra, e.g., McCarthy (1976, pp. 1-27), Rotman (1984, pp. 160–170), and Lidl and Niederreiter (1983, pp. 50–51).

4. Construction Procedure and Examples

Based on the grouping scheme in Section 3, the construction of the matrices G_ℓ in the proof of Theorem 1, Lemma 2 and (3.9), we can summarize the construction of the $OA(s^k, s^m(s^r)^n)$ in Theorem 2 in the following steps.

Let $k = rq + p$, $0 \le p < r$.

(i) Use vectors $v = (a_1, \dots, a_k)$ to represent all the columns in a saturated orthogonal array $OA(s^k, s^L)$ and partition them into $I_1, \dots, I_{q-1}, I'_q$ and K_q if $p > 0$, and into $I_1, \dots, I_{q-1}$ and I_q if $p = 0$.

(ii) Find the irreducible polynomials in $F[x]$ ($F = GF(s)$):

$$f_\ell(x) = x^\ell + \sum_{i=1}^{\ell}(-1)^{i-1}\lambda_i^{(\ell)}x^{\ell-i}, \quad \lambda_i^{(\ell)} \in GF(s)$$

with $\ell = r$ if $p = 0$ and $\ell = r$ and $r + p$ if $p > 0$, from tables, for example, in Lidl and Niederreiter (1983), pp. 553-566, Tables C–F. From $f_\ell(x)$, obtain the irreducible matrices $G_\ell = [P, \lambda_\ell^T]$, where

$$P = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ & \dots & \dots & \\ 0 & 0 & \dots & 1 \end{pmatrix}, \quad \lambda_\ell = (\lambda_\ell^{(\ell)}, \lambda_{\ell-1}^{(\ell)}, \dots, \lambda_1^{(\ell)}).$$

(iii) For $k - ur = (q-u-1)r + (r+p)$, $u = 1, \dots, q-1$, define the $(k-ur)\times(k-ur)$ matrix

$$G_{k-ur} = \begin{cases} \operatorname{diag}(G_r, \dots, G_r) & \text{if } p = 0, \\ \operatorname{diag}(G_r, \dots, G_r, G_{r+p}) & \text{if } p > 0, \end{cases}$$

where G_r repeats $q - u$ times if $p = 0$ and $q - u - 1$ times if $p > 0$.

(iv) We first consider the case of $p > 0$. Group the $(s^r - 1)(s - 1)^{-1}$ vectors in I'_q into one s^r-level column. For each I_j, $1 \le j \le q - 1$, partition its $(s^r - 1)(s - 1)^{-1}s^{k-jr}$ vectors into s^{k-jr} mutually exclusive groups of $(s^r - 1)(s - 1)^{-1}$ vectors so that each group of vectors can be replaced by an s^r-level column. The s^{k-jr} groups are defined by $\sum_{i=0}^{r-1} c_i(\mathbf{0}_{(j-1)r}, \boldsymbol{u}_i, \boldsymbol{a}G^i_{k-jr})$, where c_i satisfy (3.2), $\boldsymbol{u}_i$ are r independent r-vectors that generate H_r in (3.1), G^i_{k-jr} is the ith power of G_{k-jr} given in (iii), $G^0_{k-jr} = I$ and $\boldsymbol{a} = (a_{jr+1}, \dots, a_k)$ with $a_i \in GF(s)$. For a fixed $\boldsymbol{a}$, the group has $(s^r - 1)(s - 1)^{-1}$ vectors as c_i varies. The s^{k-jr} groups are obtained as $\boldsymbol{a}$ varies over $GF(s)$. For $p = 0$, we apply the same grouping method to $I_1, \dots, I_{q-1}$ and I_q.

(v) By replacing n groups obtained in (iv) with n s^r-level columns and keeping the remaining columns of s levels, we obtain $OA(s^k, s^m(s^r)^n)$ for any $n \le (s^k - s^{r+p})(s^r - 1)^{-1} + 1$.

A crucial step in the procedure is the construction of the matrices G_ℓ. Since P within G_ℓ is fixed, G_ℓ is determined by the vector λ_ℓ. Noting that $\ell = r$ and $r + p$ in step (iii) and $0 \le p \le r-1$, we give values of λ_ℓ for selected values of ℓ and s at the end of the section.

We now illustrate the steps with the construction of $OA(64, 2^m 8^n)$. Here $k = 6$, $s = 2$ and $r = 3$.

(i) We have $q = 2$,
$I_1 = \{(\boldsymbol{u}, \boldsymbol{a}) : \boldsymbol{u}$ can be any of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_1 + \boldsymbol{u}_2$, $\boldsymbol{u}_3$, $\boldsymbol{u}_1 + \boldsymbol{u}_3$, $\boldsymbol{u}_2 + \boldsymbol{u}_3$, and $\boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3$, where $\boldsymbol{u}_1 = (1,0,0)$, $\boldsymbol{u}_2 = (0,1,0)$, $\boldsymbol{u}_3 = (0,0,1)$, and $\boldsymbol{a} = (a_4, a_5, a_6)$ with $a_i = 0$ or $1\}$, and
$I_2 = \{(0,0,0,\boldsymbol{u}) : \boldsymbol{u}$ defined in $I_1\}$.

(ii) Since $p = 0$, we only need to consider $\ell = 3$, the third degree irreducible polynomial over $GF(2)$, $f_3(x) = x^3 + x^2 + 1$ and the corresponding vector $\lambda_3 = (1,0,1)$.

(iii) We have $k - r = 3$ and

$$G_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(iv) The 56 vectors in I_1 can be partitioned into eight groups, each of 7 vectors of the form $\sum_{i=0}^{2} c_i(\boldsymbol{u}_i, \boldsymbol{a}G_3^i)$, where $c_i = 0$ or 1 and at least one 1, and $G_3^0 = I$.

(v) The eight groups in I_1 and the one group in I_2 can give at most nine 8-level columns, from which we can construct $OA(64, 2^m 8^n)$, $n \le 9$, $7n + m = 63$.

Table 2. λ_ℓ vectors for various values of s and ℓ.

s	ℓ	λ_ℓ	s	ℓ	λ_ℓ
2	2	(1,1)	7	2	(4,1)
	3	(1,0,1)		3	(1,0,1)
	4	(1,0,0,1)			
	5	(1,1,0,1,1)	8	2	(1,1)
	6	(1,0,0,0,0,1)		3	(A,0,1)
	7	(1,0,0,0,0,0,1)		4	(A,1,0,1)
	8	(1,1,0,0,0,0,1,1)		5	(A,0,1,0,1)
	9	(1,0,0,0,0,0,0,0,1)			
			9	2	(A,1)
3	2	(1,1)		3	(A,0,1)
	3	(2,0,1)			
	4	(1,0,0,1)	11	2	(4,1)
	5	(2,2,0,0,1)		3	(2,0,1)
	6	(1,1,0,0,0,1)			
	7	(2,2,0,1,0,0,1)	13	2	(8,1)
	8	(1,1,0,0,1,1,0,1)		3	(2,0,1)

	9	(2,2,0,0,1,2,0,0,1)			
			17	2	(10,1)
4	2	(A,1)		3	(7,0,1)
	3	(1,0,1)			
	4	(1,A,0,1)	19	2	(10,1)
	5	(A,0,0,0,1)		3	(1,0,1)
5	2	(4,1)			
	3	(1,0,1)			
	4	(1,0,0,1)			
	5	(4,0,0,0,1)			

Note: For $s = 4$, $A = w$, where $w^2 + w + 1$ is an irreducible polynomial over $GF(2)$, and $GF(2^2) = \{0, 1, w, w + 1\}$. For $s = 8$, $A = w$, where $w^3 + w + 1$ is an irreducible polynomial over $GF(2)$, and $GF(2^3) = \{0, 1, w, w^2, w + 1, w^2 + 1, w^2 + w, w^2 + w + 1\}$. For $s = 9$, $A = w$, where $w^2 + 1$ is an irreducible polynomial over $GF(3)$, and $GF(3^2) = \{0, 1, 2, w, w + 1, w + 2, 2w, 2w + 1, 2w + 2\}$.

5. Upper Bounds on the Number of s^r-level Columns

The arrays $OA(s^k, s^m(s^r)^n)$ given in Theorem 2 have at most $B_1 = (s^k - s^{r+p})/(s^r - 1) + 1$ columns of s^r levels, where $k = qr + p$, $0 \le p < r$. When $p = 0$, $B_1 = (s^k - 1)/(s^r - 1)$ attains the maximum possible value for n because it exhausts all the degrees of freedom. When $p \geq 1$, for this B_1 value, $m = (s^{r+p} - s^r)/(s-1)$, i.e., there are still $(s^{r+p} - s^r)/(s-1)$ columns of s levels, which according to our method cannot be grouped into additional s^r-level columns (see the discussion between (3.4) and (3.5)). A natural question to be discussed in this section is whether B_1 can be further increased.

By ignoring the s-level columns in these arrays, we can apply the Bose-Bush (1952) bound on n to the arrays $OA(s^k, (s^r)^n)$. Using their notation, we rewrite the arrays as $OA(\lambda t^2, t^n)$ where $t = s^r$, $\lambda = s^{k-2r}$. Since $\lambda - 1$ is not divisible by $t - 1$, the Bose-Bush bound on n is

$$B_2 = \left\lfloor \frac{s^k - 1}{s^r - 1} \right\rfloor - \lfloor \theta \rfloor - 1 = \frac{s^k - s^p}{s^r - 1} - \lfloor \theta \rfloor - 1, \tag{5.1}$$

where $\lfloor \theta \rfloor$ is the integer part of θ,

$$\begin{aligned} \theta &= \frac{1}{2}\{[1 + 4s^r(s^r - 1 - (s^p - 1))]^{1/2} - [2s^r - 2(s^p - 1) - 1]\} \\ &= \left[\frac{1}{4} + s^r(s^r - s^p)\right]^{1/2} - \left(s^r - s^p + \frac{1}{2}\right), \end{aligned}$$

$(s^p - 1)$ in θ is the remainder on dividing $\lambda - 1$ by $s^r - 1$, and the second equality in (5.1) holds because $s^p - 1$ is the remainder on dividing $s^k - 1$ by $s^r - 1$. Since

B_2 is an upper bound on n, we have $B_2 - B_1 \geq 0$, where

$$\begin{aligned} B_2 - B_1 &= \frac{s^{r+p} - s^p}{s^r - 1} - \lfloor\theta\rfloor - 2 \\ &= s^p - 2 + (s^r - s^p) - \lfloor\psi\rfloor = s^r - 2 - \lfloor\psi\rfloor, \end{aligned}$$

where $\psi = [\frac{1}{4} + s^r(s^r - s^p)]^{1/2} - \frac{1}{2}$. By writing $d = B_2 - B_1 = s^r - 2 - \lfloor\psi\rfloor$, we have $s^r - 2 - d = \lfloor\psi\rfloor$, or equivalently,

$$s^r - 2 - d \leq \psi < s^r - 1 - d.$$

That is, d is the smallest nonnegative integer satisfying $s^r - 2 - d \leq \psi$, which is equivalent to

$$\left(s^r - \frac{3}{2} - d\right)^2 = s^{2r} - (3 + 2d)s^r + \left(\frac{3}{2} + d\right)^2 \leq \frac{1}{4} + s^{2r} - s^{r+p},$$

which can be further reduced to

$$s^r(3 + 2d - s^p) \geq d^2 + 3d + 2 = (d+1)(d+2).$$

Therefore d is the smallest nonnegative integer satisfying

$$3 + 2d - s^p \geq s^{-r}(d+1)(d+2). \tag{5.2}$$

In particular, the B_1 value attains the Bose-Bush upper bound B_2 iff $3 - s^p \geq 2s^{-r}$, which holds iff $p = 1$ and $s = 2$. These results are summarized in the following theorem.

Theorem 3. *Assume $k = rq + p$, $0 < p < r$. Let $d = B_2 - B_1$ be the difference between the Bose-Bush upper bound B_2 given in* (5.1) *on n for any $OA(s^k, (s^r)^n)$ and $B_1 = (s^k - s^{r+p})/(s^r - 1) + 1$, the maximum number of s^r-level columns in the $OA(s^k, s^m(s^r)^n)$ constructed by the grouping scheme. Then d is the smallest nonnegative integer satisfying* (5.2). *In particular, B_1 cannot be further increased when $p = 1$ and $s = 2$.*

Except for $p = 1$ and $s = 2$, $B_1 < B_2$. So an important unresolved question is whether the Bose-Bush bound can be made sharper for this class of problems or the construction method can be improved to increase the value of B_1. Since the Bose-Bush bound was developed for symmetrical arrays, we think it can be improved for asymmetrical arrays such as those considered in the paper.

To conclude this section, we give, in Table 3, values of s, p and r for which $d = B_2 - B_1 \leq 5$. We illustrate the calculation on $d = 1$. From (5.2), we have $5 - s^p \geq 6s^{-r}$, which is satisfied by (i) $p = 1$, $s = 3$, $r \geq 2$, (ii) $p = 1$, $s = 4$,

$r \geq 2$, and (iii) $p = 2$, $s = 2$, $r \geq 3$. The case $p = 1$ and $s = 2$ is ruled out because it has already satisfied (5.2) for $d = 0$. Another interesting case is $p = 3$ and $s = 2$. For $r \geq 5$, $d = 3$ because $9 - 2^3 = 1 \geq 20(2^{-r})$. For $r = 4$, d takes the larger value 4 because $11 - 2^3 \geq 30(2^{-4})$.

Table 3. Values of s, p, r for selected values of $d = B_2 - B_1$.
In the column for r, ≥ 2 means $r \geq 2$.

d	s	p	r
0	2	1	≥ 2
1	3	1	≥ 2
1	4	1	≥ 2
1	2	2	≥ 3
2	5	1	≥ 2
2	6	1	≥ 2
3	7	1	≥ 2
3	8	1	≥ 2
3	2	3	≥ 5
4	9	1	≥ 2
4	10	1	≥ 2
4	3	2	≥ 3
4	2	3	4
5	11	1	≥ 2
5	12	1	≥ 2

6. Extension to the Construction of $OA(s^k, s^m(s^{r_1})^{n_1} \cdots (s^{r_t})^{n_t})$

By generalizing the construction method of Section 3 to allow r to vary with the set I_j, we can construct more general orthogonal arrays of the type $OA(s^k, s^m(s^{r_1})^{n_1} \cdots (s^{r_t})^{n_t})$.

Extend the definition of I_j in (3.3) by allowing the dimension of its $\boldsymbol{u}$ vector to be r_j, $j = 1, \ldots, t$, where $r_j (\geq 2)$ are *not necessarily distinct.* Define $R_j = \sum_{i=1}^{j} r_i$. Following the method of Section 3, we know that if $k - R_j \geq r_j$, the $(s^{r_j} - 1)(s-1)^{-1} s^{k-R_j}$ vectors in I_j can be partitioned into s^{k-R_j} mutually exclusive sets of vectors of the form (3.1) (with r replaced by r_j), from which we obtain s^{k-R_j} columns, each of s^{r_j} levels. If $k - R_j < r_j$, by drawing analogy to I'_q in (3.3), we define

$$I'_j = \{(\mathbf{0}_{R_{j-1}}, \boldsymbol{u}, \mathbf{0}_{k-R_j}) : \boldsymbol{u} \in H_{r_j}\}.$$

The $(s^{r_j} - 1)(s-1)^{-1}$ vectors in I'_j satisfy (3.1) and can be replaced by an s^{r_j}-level column. In constructing the grouping scheme for I_j, we use the matrices constructed in Theorem 1. We can repeat this procedure for $j = 1, \ldots, t$ with $\sum_{j=1}^{t} r_j \leq k$. Since $r_j \geq 2$, we may terminate the procedure when $\sum_{j=1}^{t} r_j =$

$k-1$ or k. The constructed arrays have properties summarized in the following theorem.

Theorem 4. *For any prime power s, any positive integer k and any $r_j \geq 2$ (not necessarily distinct) satisfying $\sum_{j=1}^{t} r_j \leq k$, we can construct $OA(s^k, s^m(s^{r_1})^{n_1}\cdots(s^{r_t})^{n_t})$, where*

$$\text{(i)}\quad m(s-1)+\sum_{j=1}^{t} n_j(s^{r_j}-1)=s^k-1,$$

$$\begin{aligned}\text{(ii)}\quad & n_j \leq s^{k-\Sigma_{i=1}^{j} r_i} && \text{if } k-\sum_{i=1}^{j} r_i \geq r_j,\\ & n_j \leq 1 && \text{if } k-\sum_{i=1}^{j} r_i < r_j.\end{aligned} \tag{6.1}$$

Remark. The condition (i) comes from matching the degrees of freedom. The condition (ii) is apparent from the construction method.

Proof. We only need to prove that the constructed arrays are orthogonal with strength two. Suppose that L_i is an s^{r_i}-level column and L_j is an s^{r_j}-level column in any constructed array. Here we allow $r_i = r_j$, $r_i = 1$ or $r_j = 1$. Let S_i and S_j be the sets of columns in $OA(s^k, s^L)$ that are grouped to get L_i and respectively L_j. Since the vectors in S_i (and resp. in S_j) and the zero vector form a linear subspace over $GF(s)$ and S_i and S_j are disjoint, it is easy to show that any r_i linearly independent vectors in S_i and r_j linearly independent vectors in S_j are jointly linearly independent in the linear space generated by S_i and S_j. Therefore any level combination of the $r_i + r_j$ vectors appears equally often, which implies that L_i and L_j are orthogonal to each other.

Note that in the construction method of Section 3, the set I'_q (which contributes only one s^r-level column) is defined at the end because r_j are constant and the condition $k - jr \geq r$ is only violated for the largest possible j. Such is not true for general r_j. For example, if a very large r_j occurs for an intermediate value of j, the condition $k - \sum_1^j r_i \geq r_j$ may be violated and as a result only one s^{r_j}-level column is obtained. Therefore we may assign the r_j value at a large j or to satisfy $k - \sum_1^j r_i < r_j$ if only few s^{r_j}-level columns are desired. The *flexibility* in the choice of r_j is indeed a major advantage of the proposed construction method, which will be elaborated in the following.

By choosing r_j in any order, we obtain a very rich collection of asymmetrical orthogonal arrays $OA(s^k, s^m(s^{r_1})^{n_1}\cdots(s^{r_t})^{n_t})$. For example, if it is desired to have more s^r-level columns for a given r, we can set $r_1 = r_2 = \cdots = r_c = r$; then the number of s^r-level columns can be as large as $s^{k-r} + \cdots + s^{k-cr} =$

$(s^k - s^{k-cr})(s^r - 1)^{-1}$. In general the r_j's do not have to be monotone. One can choose r_j for some prescribed values of n_j by following the formulas in (6.1) for n_j. If they are chosen to satisfy $r_1 \le r_2 \le \cdots \le r_t$, then more columns with smaller numbers of levels are obtained.

The method can be further enchanced by the use of the reverse replacement method. As illustrated in the following example, the construction in Theorem 4 imposes some restrictions on the combinations of r_j and n_j. This is due to the nature of grouping which replaces s-level columns by s^r-level columns, $r > 1$, but not vice versa. By *reversing* the procedure with s^r-level columns replaced by columns with fewer levels, e.g., 16 replaced by 4^5 or $4^4 \cdot 2^3$, we can obtain richer collections of asymmetrical arrays. See, for example, the arrays in (vii) of the example at the end of this section.

We now compare our work with those of Pu (1989) and Hedayat et al. (1990). Pu's approach uses tools from finite projective geometry while Hedayat et al. use combinatorial techniques such as difference matrix and resolvability. The latter can handle the construction of arrays with $2s^k$ runs, which are not covered by our approach. On the other hand, our grouping approach allows the aliasings between main effects and interactions to be easily studied, which is important for statistical analysis. Furthermore, the arrays constructed in Theorem 4 do not have the restrictive condition such as r_{j+1} being a multiple of r_j which is assumed in both of their papers. Of course this does not preclude the possibility that an improved version of Hedayat et al. can dispense with this condition. Although Pu's approach is different from ours, some of the mathematical steps are equivalent. For example, our (3.6), (3.9) and (3.11) have analogous results in Pu (1989, Sections 2.4 and 3.3). Since Pu's procedure can be viewed as a grouping scheme, with some additional work it can allow the aliasing structure to be studied as our approach does.

We conclude this section by illustrating the construction steps and use of the reverse replacement method with the construction of some 64-run arrays.

(i) $OA(2^6, 2^m \cdot 4^{n_1})$, $n_1 \le 21$, $m + 3n_1 \le 63$.
(ii) $OA(2^6, 2^m \cdot 8^{n_1})$, $n_1 \le 9$, $m + 7n_1 \le 63$.
(iii) $OA(2^6, 2^m \cdot 32)$, $m \le 32$.
(iv) $OA(2^6, 2^m \cdot 4^{n_1} \cdot 16)$, $n_1 \le 16$, $m + 3n_1 \le 48$.
(v) $OA(2^6, 2^m \cdot 4^{n_1} \cdot 8)$, $n_1 \le 16$, $m + 3n_1 \le 56$.
(vi) $OA(2^6, 2^m \cdot 4 \cdot 8^{n_2})$, $n_2 \le 8$, $m + 7n_2 \le 60$.

Construction of the arrays in (ii) is explained in detail in Section 4. Those for (i) and (iii) are obtained by taking $r_1 = r_2 = r_3 = 2$ and respectively $r_1 = 5$. For the arrays in (iv), we take $r_1 = 2$ and $r_2 = 4$. There are $3 \times 2^{6-2} = 3 \times 16 = 48$ vectors in I_1 which can be partitioned into 16 groups, giving $n_1 \le 16$. The 15 vectors in I_2' can be grouped into a 16-level column. For the arrays in (v), we take $r_1 = 2$ and $r_2 = 3$. As in (iv), the 48 vectors in I_1 can be partitioned into 16 groups. The seven vectors in I_2' can be grouped into an 8-level column. For the arrays in (vi), we reverse the order by taking $r_1 = 3$ and $r_2 = 2$. In I_1, there are $7 \times 2^{6-3} = 56$ vectors which can be partitioned into 8 groups, giving $n_2 \leq 8$. The three vectors in I_2' can be grouped into a 4-level column.

Note that none of these arrays can accommodate $4^{n_1} \cdot 8^{n_2}$ with both n_1 and $n_2 \geq 2$. This problem can be alleviated by the use of the reverse replacement rule, that is, to replace an 8-level column in the arrays in (vi) by $4 \cdot 2^4$. By repeating this for x 8-level columns, we obtain from (vi) the following arrays,
(vii) $OA(2^6, 2^{m+4x} \cdot 4^{x+1} \cdot 8^{n_2-x})$, $0 \leq x \leq n_2 \leq 8$, $m + 7n_2 \leq 60$.

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