RELAXING LATENT IGNORABILITY IN ESTIMATING THE CACE

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Supplementary Material

S1. Proof of Theorems 4.1 and 6.1

In order to derive the asymptotic distributions for $\widehat{CACE}^{LI}$ and $\widehat{CACE},^{\overline{LI}}$ let $X_i \equiv (Y_iR_i(1-Z_i)(1-D_i), Y_iR_i(1-Z_i)D_i, Y_iR_iZ_i(1-D_i), Y_iR_iZ_iD_i, R_i(1-Z_i)(1-D_i), R_i(1-Z_i)D_i, R_iZ_i(1-D_i), R_iZ_iD_i)$ and let $p \equiv (v_{00}, v_{01}, v_{10}, v_{11}, \pi_{00}, \pi_{01}, \pi_{10}, \pi_{11})$ where X_i is a vector of bernoulli random variables with mean p. If $X_1, \ldots, X_n$ are $\overset{iid}{\sim}$ random vectors in R^8 with mean p and covariance matrix $\Sigma = E(X-p)(X-p)$ (where $E(X^TX) < \infty$), then from the multivariate central limit theorem, $\sqrt{n}(\bar{X}-p) \rightarrow_d N_d(0, \Sigma)$ where $\Sigma =$

$$\begin{pmatrix}
v_{00}(1-v_{00}) & -v_{00}v_{01} & -v_{00}v_{10} & -v_{00}v_{11} & v_{00}(1-\pi_{00}) & -v_{00}\pi_{01} & -v_{00}\pi_{10} & -v_{00}\pi_{11} \\
-v_{01}v_{00} & v_{01}(1-v_{01}) & -v_{01}v_{10} & -v_{01}v_{11} & -v_{01}\pi_{00} & -v_{01}(1-\pi_{01}) & -v_{01}\pi_{10} & -v_{01}\pi_{11} \\
-v_{10}v_{00} & -v_{10}v_{01} & v_{10}(1-v_{10}) & -v_{10}v_{11} & -v_{10}\pi_{00} & -v_{10}\pi_{01} & v_{10}(1-\pi_{10}) & -v_{10}\pi_{11} \\
-v_{11}v_{00} & -v_{11}v_{01} & -v_{11}v_{10} & v_{11}(1-v_{11}) & -v_{11}\pi_{00} & -v_{11}\pi_{01} & -v_{11}\pi_{10} & v_{11}(1-\pi_{11}) \\
(1-\pi_{00})v_{00} & -\pi_{00}v_{01} & -\pi_{00}v_{10} & -\pi_{00}v_{11} & \pi_{00}(1-\pi_{00}) & -\pi_{00}\pi_{01} & -\pi_{00}\pi_{10} & -\pi_{00}\pi_{11}) \\
-\pi_{01}v_{00} & (1-\pi_{01})v_{01} & -\pi_{01}v_{10} & -\pi_{01}v_{11} & -\pi_{01}\pi_{00} & \pi_{01}(1-\pi_{01}) & -\pi_{01}\pi_{10} & -\pi_{01}\pi_{11}) \\
-\pi_{10}v_{00} & -\pi_{10}v_{01} & (1-\pi_{10})v_{10} & -\pi_{10}v_{11} & -\pi_{10}\pi_{00} & -\pi_{10}\pi_{01} & \pi_{10}(1-\pi_{10}) & -\pi_{10}\pi_{11}) \\
-\pi_{11}v_{00} & -\pi_{11}v_{01} & -\pi_{11}v_{10} & (1-\pi_{11})v_{11} & -\pi_{11}\pi_{00} & -\pi_{11}\pi_{01} & -\pi_{11}\pi_{10} & \pi_{11}(1-\pi_{11})
\end{pmatrix}$$

S1.1 Proof of Theorem 4.1

Using the multivariate delta method, the asymptotic distributions for $\hat{\eta}_{0c}$ and $\hat{\eta}_{0c}$ are:
$\sqrt{n}(\hat{\eta_{1c}}-\eta_{1c}) \rightarrow_d N(0, \delta^{'}\Sigma\delta)$ where $\delta^{'} = (0, \frac{-1}{\pi_{11}-\pi_{01}}, 0, \frac{1}{\pi_{11}-\pi_{01}}, 0, \frac{v_{11}-v_{01}}{(\pi_{11}-\pi_{01})^2}, 0, \frac{v_{01}-v_{11}}{(\pi_{11}-\pi_{01})^2})$
and
$\sqrt{n}(\hat{\eta_{0c}}-\eta_{0c}) \rightarrow_d N(0, \beta^{'}\Sigma\beta)$ where $\beta^{'} = (\frac{-1}{\pi_{10}-\pi_{00}}, 0, \frac{1}{\pi_{10}-\pi_{00}}, 0, \frac{v_{10}-v_{00}}{(\pi_{10}-\pi_{00})^2}, 0, \frac{v_{00}-v_{10}}{(\pi_{10}-\pi_{00})^2}, 0)$
where $V_1 = \delta^{'}\Sigma\delta$ and $V_0 = \beta^{'}\Sigma\beta$ are defined in section 4. Then noting that $\sqrt{n}(\hat{\eta}_{1c}-\eta_{1c})$ and $\sqrt{n}(\hat{\eta}_{0c}-\eta_{0c})$ are asymptotically independent since $\delta^{'}\Sigma\beta = 0$, we can use Slutsky's theorem to derive the asymptotic distribution for $\widehat{CACE}^{LI}$ as in Theorem 4.1.

S1.2 Proof of Theorem 6.1

Using the multivariate delta method, the asymptotic distributions for $\hat{\eta}_{1c}$ and $\hat{\eta}_{0c}$ are:
$\sqrt{n}(\hat{\eta_{1c}}-\eta_{1c}) \rightarrow_d N(0, \delta^{'}\Sigma\delta)$ where $\delta^{'} = (0, \delta_1, 0, \delta_2, 0, \delta_3, 0, \delta_4)$ for δ defined in Theorem 6.1 and $\sqrt{n}(\hat{\eta_{0c}}-\eta_{0c}) \rightarrow_d N(0, \beta^{'}\Sigma\beta)$ where $\beta^{'} = (0, \beta_1, 0, \beta_2, 0, \beta_3, 0, \beta_4)$ for β defined

in Theorem 6.1. Again noting that $\sqrt{n}(\hat{\eta}_{1c} - \eta_{1c})$ and $\sqrt{n}(\hat{\eta}_{0c} - \eta_{0c})$ are asymptotically independent, we can use Slutsky's theorem to derive the asymptotic distribution for $\widehat{CACE}^{\overline{LI}}$ as in Theorem 6.1.

S2. Proof of Result 6.1

For $\psi_n = P(C = n|Z = 0, D = 0)$ note that under randomization $P(C = n, Z = 0) = P(C = n, Z = 1)$ and since never-takers by definition have $D = 0$, then we expect $P(C = n, Z = 0, D = 0) = P(C = n, Z = 1, D = 0)$. Similarly for always-takers, $P(C = a, Z = 1, D = 1) = P(C = a, Z = 0, D = 1)$.
Therefore:

$$\psi_n = \frac{P(C = n, Z = 0, D = 0)}{P(Z = 0, D = 0)} = \frac{P(C = n, Z = 1, D = 0)}{P(Z = 0, D = 0)} = \frac{P(Z = 1, D = 0)}{P(Z = 0, D = 0)}$$
$$\psi_a = \frac{P(C = a, Z = 1, D = 1)}{P(Z = 1, D = 1)} = \frac{P(C = a, Z = 0, D = 1)}{P(Z = 1, D = 1)} = \frac{P(Z = 0, D = 1)}{P(Z = 1, D = 1)}$$

Thus we can obtain the following estimators for ψ_n and ψ_a:

$$\hat{\psi}_n = \tfrac{\hat{\xi}_{10}}{\hat{\xi}_{00}}; \qquad \hat{\psi}_a = \tfrac{\hat{\xi}_{01}}{\hat{\xi}_{11}}$$

Then note the following relationship:

$$\begin{aligned}
v_{zd} &= P(Y = 1, R = 1, Z = z, D = d) \\
&= \sum_{t \in (n,c,a)} P(R = 1, Z = z, D = d, C = t, Y = 1) \\
&= \sum_{t \in (n,c,a)} P(R = 1|Z = z, D = d, C = t, Y = 1) P(Y = 1|Z = z, D = d, C = t) \\
&\qquad P(C = t|Z = z, D = d) P(Z = z, D = d) \\
\Leftrightarrow \frac{v_{zd}}{\xi_{zd}} &= \sum_{t \in (n,c,a)} P(R = 1|Z = z, D = d, C = t, Y = 1) \\
&\qquad P(Y = 1|Z = z, D = d, C = t) P(C = t|Z = z, D = d) \\
&= \sum_{t \in (n,c,a)} \phi_{zt1} \eta_{zt} \psi_{tzd} \qquad (1)
\end{aligned}$$

Also note the following relationship:

$$\begin{aligned}
\frac{\pi_{zd}}{\xi_{zd}} &= P(R = 1|Z = z, D = d) \\
&= \sum_{t \in (n,c,a)} P(R = 1|Z = z, D = d, C = t) P(C = t|Z = z, D = d) \\
&= \sum_{t \in (n,c,a)} \Big(\sum_{y \in (0,1)} P(R = 1|Z = z, D = d, C = t, Y = y) P(Y = y|Z = z, D = d, C = t) \Big) \\
&\qquad P(C = t|Z = z, D = d)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t\in(n,c,a)} P(R=1|Z=z,D=d,C=t,Y=1)P(Y=1|Z=z,D=d,C=t) \\
&\qquad P(C=t|Z=z,D=d) + P(R=1|Z=z,D=d,C=t,Y=0) \\
&\qquad P(Y=0|Z=z,D=d,C=t)P(C=t|Z=z,D=d) \\
&= \sum_{t\in(n,c,a)} \phi_{zt1}\eta_{zt}\psi_{tzd} + \phi_{zt0}(1-\eta_{zt})\psi_{tzd} \\
&= \sum_{t\in(n,c,a)} \phi_{zt1}\eta_{zt}\psi_{tzd} + f_{zt}\phi_{zt1}(1-\eta_{zt})\psi_{tzd} \\
&= \sum_{t\in(n,c,a)} \phi_{zt1}\psi_{tzd}(\eta_{zt} + f_{zt}(1-\eta_{zt})) \qquad (2)
\end{aligned}
$$

Letting (z,d) equal $(0,1)$ and $(1,0)$ in expression (3) above and solving for ϕ_{0a1} and ϕ_{1n1} we get:

$$
\phi_{0a1} = \frac{v_{01}}{\xi_{01}\eta_a}; \qquad \phi_{1n1} = \frac{v_{10}}{\xi_{10}\eta_n} \qquad (3)
$$

Letting (z,d) equal $(0,1)$ and $(1,0)$ in expression (4) above and substituting the expressions from (5):

$$
\frac{\pi_{01}}{\xi_{01}} = \frac{v_{01}}{\xi_{01}\eta_a}(\eta_a + f_{0a}(1-\eta_a)); \qquad \frac{\pi_{10}}{\xi_{10}} = \frac{v_{10}}{\xi_{10}\eta_n}(\eta_n + f_{1n}(1-\eta_n))
$$

Then solving for η_a and η_n above, we obtain the following estimators:

$$
\hat{\eta}_a = \frac{f_{0a}\hat{v}_{01}}{\hat{\pi}_{01} + (f_{0a}-1)\hat{v}_{01}}; \qquad \hat{\eta}_n = \frac{f_{1n}\hat{v}_{10}}{\hat{\pi}_{10} + (f_{1n}-1)\hat{v}_{10}}
$$

We can then obtain estimators for ϕ_{0a1} and ϕ_{1n1}:

$$
\hat{\phi}_{0a1} = \frac{\hat{v}_{01}}{\hat{\xi}_{01}\hat{\eta}_a}; \qquad \hat{\phi}_{1n1} = \frac{\hat{v}_{10}}{\hat{\xi}_{10}\hat{\eta}_n}
$$

Using the exclusion criteria for never-takers and always-takers and (2), note that:

$$
\phi_{0n1}(\eta_n + f_{0n}(1-\eta_n)) = \phi_{1n1}(\eta_n + f_{1n}(1-\eta_n))
$$

$$
\phi_{1a1}(\eta_a + f_{1a}(1-\eta_a)) = \phi_{0a1}(\eta_a + f_{0a}(1-\eta_a))
$$

Then solving for ϕ_{0n1} and ϕ_{1a1} above we obtain the following estimators:

$$
\hat{\phi}_{1a1} = \frac{\hat{\phi}_{0a1}(\hat{\eta}_a + f_{0a}(1-\hat{\eta}_a))}{\hat{\eta}_a + f_{1a}(1-\hat{\eta}_a)}; \qquad \hat{\phi}_{0n1} = \frac{\hat{\phi}_{1n1}(\hat{\eta}_n + f_{1n}(1-\hat{\eta}_n))}{\hat{\eta}_n + f_{0n}(1-\hat{\eta}_n)}
$$

Letting $(z, d) = (0, 0)$ in expressions (3) and (4) and solving for η_{0c} and ϕ_{0c1} we obtain:

$$\phi_{0c1} = \frac{\phi_{0n1}\psi_n\xi_{00}(f_{0c}\eta_n + f_{0n}(1-\eta_n)) + (1-f_{0c})v_{00} - \pi_{00}}{\xi_{00}f_{0c}(\psi_n - 1)}$$

$$\eta_{0c} = \frac{f_{0c}(\phi_{0n1}\psi_n\xi_{00}\eta_n - v_{00})}{\phi_{0n1}\psi_n\xi_{00}(f_{0c}\eta_n + f_{0n}(1-\eta_n)) + (1-f_{0c})v_{00} - \pi_{00}}$$

Letting $(z, d) = (1, 1)$ in expression (3) and (4) and solving for η_{1c} and ϕ_{1c1} we obtain:

$$\phi_{1c1} = \frac{\phi_{1a1}\psi_a\xi_{11}(f_{1a}\eta_a + f_{1a}(1-\eta_a)) + (1-f_{1c})v_{11} - \pi_{11}}{\xi_{11}f_{1c}(\psi_a - 1)}$$

$$\eta_{1c} = \frac{f_{1c}(\phi_{1a1}\psi_a\xi_{11}\eta_a - v_{11})}{\phi_{1a1}\psi_a\xi_{11}(f_{1a}\eta_a + f_{1a}(1-\eta_a)) + (1-f_{1c})v_{11} - \pi_{11}}$$

Substituting in the expressions for $\hat{\phi}_{0n1}$, $\hat{\psi}_n$, $\hat{\eta}_n$, $\hat{\phi}_{1a1}$, $\hat{\psi}_a$, and $\hat{\eta}_a$ we obtain the following expressions for $\hat{\eta}_{0c}$ and $\hat{\eta}_{1c}$ (which reduce to the expressions found in Result 6.1):

$$\hat{\eta}_{0c} = \frac{f_{0c}[f_{1n}\hat{v}_{10}(\hat{v}_{00} - \hat{\pi}_{10}) + f_{0n}\hat{v}_{00}(\hat{\pi}_{10} - \hat{v}_{10})]}{(f_{0c}-1)[f_{1n}\hat{v}_{10}(\hat{v}_{00} - \hat{\pi}_{10}) + f_{0n}\hat{v}_{00}(\hat{\pi}_{10} - \hat{v}_{10})] + f_{1n}\hat{v}_{10}(\hat{\pi}_{00} - \pi_{10}) + f_{0n}(\hat{\pi}_{10}\hat{\pi}_{00} - \hat{\pi}_{10}^2 - \hat{v}_{10}\hat{\pi}_{00} + \hat{v}_{10}\hat{\pi}_{10})}$$

$$\hat{\eta}_{1c} = \frac{f_{1c}[f_{1a}\hat{v}_{01}(\hat{v}_{11} - \hat{\pi}_{01}) + f_{0a}\hat{v}_{11}(\hat{\pi}_{01} - \hat{v}_{01})]}{(f_{1c}-1)[f_{1a}\hat{v}_{01}(\hat{v}_{11} - \hat{\pi}_{01}) + f_{0a}\hat{v}_{11}(\hat{\pi}_{01} - \hat{v}_{01})] + f_{1a}\hat{v}_{01}(\hat{\pi}_{11} - \pi_{01}) + f_{0a}(\hat{\pi}_{01}\hat{\pi}_{11} - \hat{\pi}_{01}^2 - \hat{v}_{01}\hat{\pi}_{11} + \hat{v}_{01}\hat{\pi}_{01})}$$