Proof of (A.4)

Writing

$$\begin{aligned} S_{u1} &= n^{-1}\sum_{i=1}^{n}\tilde{\pi}_i^{-2}\left(x_i-\mu_x\right) \\ S_{ux} &= n^{-1}\sum_{i=1}^{n}\tilde{\pi}_i^{-2}\left(x_i-\mu_x\right)x_i \\ S_{uy} &= n^{-1}\sum_{i=1}^{n}\tilde{\pi}_i^{-2}\left(x_i-\mu_x\right)y_i, \end{aligned}$$

the linearized term in (A.3)

$$\tilde{y}_{NPMLE} \equiv \bar{y}\left(\boldsymbol{\lambda}_0\right) - \left[\frac{\partial \bar{y}}{\partial \boldsymbol{\lambda}}\left(\boldsymbol{\lambda}_0\right)\right]' \left[\frac{\partial \mathbf{U}}{\partial \boldsymbol{\lambda}}\left(\boldsymbol{\lambda}_0\right)\right]^{-1} \mathbf{U}\left(\boldsymbol{\lambda}_0\right)$$

reduces to

$$\begin{aligned} \tilde{y}_{NPMLE} &= \bar{y}_\pi - \left(\bar{y}_\pi, S_{uy}\right) \times \left[\begin{array}{cc} 1 & S_{u1} \\ \bar{x}_\pi & S_{ux} \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \bar{x}_\pi - \mu_x \end{array}\right] \\ &= \bar{y}_\pi - \frac{S_{uy} - S_{u1}\bar{y}_\pi}{S_{ux} - S_{u1}\bar{x}_\pi}\left(\bar{x}_\pi - \mu_x\right), \end{aligned}$$

which equals to (A.4).

Proof of (4.5)

The NPMLE of the mean of y under the stratified sampling can be written

$$\bar{y}(\hat{\boldsymbol{\lambda}}) = \sum_{h=1}^{H} \frac{W_h}{n_h} \sum_{i=1}^{n_h} \frac{y_{hi}}{\hat{\lambda}_h \tilde{\pi}_{hi} + \hat{\lambda}_{H+1} m_h \left(x_{hi} - \tilde{x}_h\right)},$$

where $\tilde{\pi}_{hi} = (\hat{N}_h/n_h)\pi_{hi}$ with $\hat{N}_h = \sum_{i=1}^{n_h} \pi_{hi}^{-1}$, $m_h = (n/n_h)W_h$, $\hat{\boldsymbol{\lambda}}$ is the solution to $U\left(\boldsymbol{\lambda}\right) = \mathbf{0}$ and satisfies

$$\hat{\boldsymbol{\lambda}} = \boldsymbol{\lambda}_0 + O_p\left(n^{-1/2}\right),$$

where $\boldsymbol{\lambda}_0 = (1, 1, \cdots, 1, 0)'$. Using (A.3), we have

$$\bar{y}_{NPMLE} = \tilde{y}_{NPMLE} + O_p\left(n^{-1}\right)$$

and the linearized term is

$$\tilde{y}_{NPMLE} \equiv \bar{y}\left(\boldsymbol{\lambda}_0\right) - \left[\frac{\partial \bar{y}}{\partial \boldsymbol{\lambda}}\left(\boldsymbol{\lambda}_0\right)\right]' \left[\frac{\partial \mathbf{U}}{\partial \boldsymbol{\lambda}}\left(\boldsymbol{\lambda}_0\right)\right]^{-1} \mathbf{U}\left(\boldsymbol{\lambda}_0\right).$$

Note that

$$\bar{y}\left(\boldsymbol{\lambda}_0\right) = \sum_{h=1}^{H} \frac{W_h}{n_h} \sum_{i=1}^{n_h} \frac{y_{hi}}{\tilde{\pi}_{hi}} = \sum_{h=1}^{H} W_h \bar{y}_h = \bar{y}_\pi$$

and

$$\frac{\partial \bar{y}}{\partial \lambda_h}\left(\boldsymbol{\lambda}_0\right) = \begin{cases} -W_h n_h^{-1} \sum_{i=1}^{n_h} \tilde{\pi}_{hi}^{-1} y_{hi} = -W_h \bar{y}_h, & h = 1, \cdots, H \\ \\ -n \sum_{h=1}^{H} W_h^2 \sum_{i=1}^{n_h} d_{hi}^2 \left(x_{hi} - \bar{x}_h\right) y_{hi}, & h = H+1, \end{cases}$$

where $d_{hi} = \pi_{hi}^{-1}/\hat{N}_h$. Also, since

$$U_h\left(\boldsymbol{\lambda}\right) = \frac{1}{n_h} \sum_{i=1}^{n_h} \frac{1}{\lambda_h \tilde{\pi}_{hi} + \lambda_{H+1} m_h \left(x_{hi} - \tilde{x}_h\right)} - 1 \quad h = 1, 2, \cdots, H$$

and

$$U_{H+1}\left(\boldsymbol{\lambda}\right) = \sum_{h=1}^{H} \frac{W_h}{n_h} \sum_{i=1}^{n_h} \frac{x_{hi} - \mu_x}{\lambda_h \tilde{\pi}_{hi} + \lambda_{H+1} m_h \left(x_{hi} - \tilde{x}_h\right)},$$

we have, for $h = 1, 2, \cdots, H$,

$$\frac{\partial U_h}{\partial \lambda_g}(\boldsymbol{\lambda}_0) = \begin{cases} -1 & \text{if } g = h \\ 0 & \text{if } g \neq h, g = 1, 2, \cdots, H \\ nW_h \sum_{i=1}^{n_h} d_{hi}^2 (x_{hi} - \bar{x}_h) & \text{if } g = H + 1 \end{cases}$$

and

$$\frac{\partial U_{H+1}}{\partial \lambda_h}(\boldsymbol{\lambda}_0) = \begin{cases} -W_h (\bar{x}_h - \mu_x) & \text{if } h = 1, 2, \cdots, H \\ -n \sum_{h=1}^{H} W_h^2 \sum_{i=1}^{n_h} d_{hi}^2 (x_{hi} - \bar{x}_h)(x_{hi} - \mu_x) & \text{if } h = H + 1. \end{cases}$$

Thus, the linearized term can be written

$$\begin{aligned} \tilde{y}_{NPMLE} &= \bar{y}_\pi \\ &- \left[L_y', S_{xy}\right] \times \left[\begin{array}{cc} I_H & B_x \\ L_x' & S_{xu} \end{array}\right]^{-1} \times \left[\begin{array}{c} \mathbf{0}_H \\ \sum_{h=1}^{H} W_h \bar{x}_h - \mu_x \end{array}\right] \end{aligned}$$

where I_H is the H-dimensional identity matrix, $\mathbf{0}_H$ is H-dimensional vector of zeros, and

$$\begin{aligned} (S_{xu}, S_{xy}) &= n \sum_{h=1}^{H} W_h^2 \sum_{i=1}^{n_h} d_{hi}^2 (x_{hi} - \bar{x}_h)(x_{hi} - \mu_x, y_{hi}) \\ L_x &= (W_1 (\bar{x}_1 - \mu_x), W_2 (\bar{x}_2 - \mu_x), \cdots, W_H (\bar{x}_H - \mu_x))' \\ L_y &= (W_1 \bar{y}_1, W_2 \bar{y}_2, \cdots, W_H \bar{y}_H)' \\ B_x &= n (W_1 b_{x1}, W_2 b_{x2}, \cdots, W_H b_{xH}) \end{aligned}$$

with $b_{xh} = \sum_{i=1}^{n_h} d_{hi}^2 (x_{hi} - \bar{x}_h)$. Thus,

$$\tilde{y}_{NPMLE} = \bar{y}_\pi - \left(S_{xy} - L_y' I_H^{-1} B_x\right) P^{-1} \left(\sum_{h=1}^{H} W_h \bar{x}_h - \mu_x\right)$$

where

$$\begin{aligned} P &= S_{xu} - L_x' I_H^{-1} B_x \\ &= n \sum_{h=1}^{H} W_h^2 \sum_{i=1}^{n_h} d_{hi}^2 (x_{hi} - \bar{x}_h)(x_{hi} - \mu_x) - n \sum_{h=1}^{H} W_h^2 b_{xh} (\bar{x}_h - \mu_x) \\ &= n \sum_{h=1}^{H} W_h^2 \sum_{i=1}^{n_h} d_{hi}^2 (x_{hi} - \bar{x}_h)^2 \end{aligned}$$

and

$$\begin{aligned} S_{xy} - L_y' I_H^{-1} B_x &= n \sum_{h=1}^{H} W_h^2 \sum_{i=1}^{n_h} d_{hi}^2 \left(x_{hi} - \bar{x}_h\right) y_{hi} - n \sum_{h=1}^{H} W_h^2 \sum_{i=1}^{n_h} d_{hi}^2 \left(x_{hi} - \bar{x}_h\right) \bar{y}_h \\ &= n \sum_{h=1}^{H} W_h^2 \sum_{i=1}^{n_h} d_{hi}^2 \left(x_{hi} - \bar{x}_h\right) \left(y_{hi} - \bar{y}_h\right). \end{aligned}$$

Therefore, we have

$$\tilde{y}_{NPMLE} = \bar{y}_\pi - \frac{\sum_{h=1}^{H} W_h^2 \sum_{i=1}^{n_h} d_{hi}^2 \left(x_{hi} - \bar{x}_h\right) \left(y_{hi} - \bar{y}_h\right)}{\sum_{h=1}^{H} W_h^2 \sum_{i=1}^{n_h} d_{hi}^2 \left(x_{hi} - \bar{x}_h\right)^2} \left(\sum_{h=1}^{H} W_h \bar{x}_h - \mu_x \right),$$

which proves (4.5).