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ASYMMETRIC FRACTIONAL FACTORIAL PLANS OPTIMAL FOR MAIN EFFECTS AND SPECIFIED TWO-FACTOR INTERACTIONS

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Abstract: Fractional factorial plans for asymmetric factorial experiments are obtained. These are shown to be universally optimal within the class of all plans involving the same number of runs under a model that includes the mean, all main effects and a specified set of two-factor interactions. Finite projective geometry is used to obtain such plans for experiments wherein the number of levels of each of the factors and the number of runs is a power of m, a prime or a prime power. Methods of construction of optimal plans under the same model are also discussed for the case where the number of levels as well as the number of runs are not necessarily powers of a prime number.

Key words and phrases: Finite projective geometry, Galois field, saturated plans, universal optimality.

1. Introduction

The study of optimal fractional factorial plans has received considerable attention in the recent past, mainly because of the increased use of such plans in industrial experiments and quality control work. For a review of optimal fractional factorial plans, see Dey and Mukerjee (1999a, Chs. 2, 6, 7). Much of the work on optimal fractional factorial plans relates to situations where all factorial effects involving the same number of factors are considered equally important and, as such, the underlying model involves the general mean and all factorial effects involving up to a specified number of factors. In practice however, all factorial effects involving the same number of factors may not be equally important, and an experimenter may be interested in estimating the general mean, all main effects and only a specified set of two-factor interactions, all other interactions being assumed negligible. The issue of estimability and optimality in situations of this kind has been addressed by Hedayat and Pesotan (1992, 1997), Wu and Chen (1992) and Chiu and John (1998) in the context of two-level factorials, and by Dey and Mukerjee (1999b) and Chatterjee, Das and Dey (2002) for arbitrary factorials including the asymmetric or, mixed level factorials. Using finite projective geometry, Dey and Suen (2002) recently obtained several families of optimal plans under the stated model for symmetric factorials of the type m^n, where m is a prime or a prime power.

Continuing with this line of research, we obtain optimal fractional factorial plans for asymmetric factorials under a model that includes the mean, all main effects and a specified set of two-factor interactions, all other interactions being assumed negligible. Throughout, the optimality criterion considered is the universal optimality of Kiefer (1975); see also Sinha and Mukerjee (1982). In Section 2, concepts and results from a finite projective geometry are used to obtain optimal plans for asymmetric factorials, where the levels of the factors and the number of runs are powers of the same prime. These results generalize the ones obtained by Dey and Suen (2002) in the context of prime-powered symmetric factorials. In Section 3, we obtain some optimal plans for asymmetric experiments where the levels of the factors and the number of runs are not necessarily powers of a prime number.

The plans reported here are optimal under a model that includes the mean, all main effects and a specified set of two-factor interactions, other effects being assumed negligible. If effect(s) not included in the model are not negligible, they will bias the estimates of the factorial effects included in the model. For this reason, a more practical strategy is to look for optimal plans that allow greater flexibility in the model. Though a solution to this problem in its entire generality has yet to be found, optimal plans that exhibit a kind of model robustness under different optimality criteria have been considered e.g., by Chatterjee, Das and Dey (2002) and Ke and Tang (2003).

2. Optimal Plans Based on Finite Projective Geometry

We make use of a result of Dey and Mukerjee (1999b), giving a combinatorial characterization for a fractional factorial plan to be universally optimal. For completeness, we state this result in the form that we need.

Theorem 2.1. *Let $\mathcal{D}$ be the class of all N-run fractional factorial plans for an arbitrary factorial experiment involving n factors, $F_1, \ldots, F_n$, such that each member of $\mathcal{D}$ allows the estimability of the mean, the main effects $F_1, \ldots, F_n$ and the k two-factor interactions $F_{i_1}F_{j_1}, \ldots, F_{i_k}F_{j_k}$, where $1 \leq i_u,\ j_u \leq n$ for all $u = 1, \ldots, k$. A plan $d \in \mathcal{D}$ is universally optimal over $\mathcal{D}$ if all level combinations of the following sets of factors appear equally often in d:*

(a) $\{F_u, F_v\}$, $1 \leq u < v \leq n$;

(b) $\{F_u, F_{i_v}, F_{j_v}\}$, $1 \leq u \leq n,\ 1 \leq v \leq k$;

(c) $\{F_{i_u}, F_{j_u}, F_{i_v}, F_{j_v}\}$, $1 \leq u < v \leq k$,

where a factor is counted only once if it is repeated in (b) *or* (c).

Consider now a factorial experiment involving n factors $F_1, \ldots, F_n$, where for $i = 1, \ldots, n$, the factor F_i has m^{t_i} levels, m is a prime or a prime power and

t_i is a positive integer. We use an $(r-1)$-dimensional finite projective geometry $PG(r-1,m)$ over the finite or, Galois field, $GF(m)$ to construct m^r-run plans, r being an integer. Recall that in a $PG(r-1,m)$, a point is represented by an ordered r-tuple $(x_0,\ldots,x_{r-1})$ where, for $0 \le i \le r-1$, $x_i \in GF(m)$. Two r-tuples represent the same point in $PG(r-1,m)$ if one is a multiple of the other. A t-flat consists of points whose coordinates can be written as a linear combination of $t+1$ independent points. Thus, there are $(m^{t+1}-1)/(m-1)$ distinct points in a t-flat. A 1-flat, consisting of $m+1$ points is referred to as a line in a finite projective geometry, and a 2-flat consisting of m^2+m+1 points and m^2+m+1 lines is also called a plane. Given integers s, t, $s \le t$, there are

$$\frac{(m^{r-s-1}-1)(m^{r-s-2}-1)\cdots(m^{t-s+1}-1)}{(m^{r-t-1}-1)(m^{r-t-2}-1)\cdots(m-1)}$$

t-flats passing through an s-flat in $PG(r-1,m)$. Hence there are $(m^{r-1}-1)/(m-1)$ lines through a point and $(m^{r-2}-1)/(m-1)$ planes through a line. For more details on finite projective geometry, see Hirschfeld (1979).

We assign the factor F_i to a (t_i-1)-flat in $PG(r-1,m)$, these flats being disjoint for F_i, F_j, $i \ne j$. The two-factor interaction F_iF_j is assigned to the $(m^{t_i}-1)(m^{t_j}-1)/(m-1)$ points in the (t_i+t_j-1)-flat through the (t_i-1)-flat F_i and the (t_j-1)-flat F_j but not in F_i and F_j. Making an appeal to Theorem 2.1, one can prove the following result.

Theorem 2.2. *Let $F_1,\ldots,F_n$ be n factors of a factorial experiment, where for $u=1,\ldots,n$, the factor F_u has m^{t_u} levels, m is a prime or a prime power and t_u is a positive integer. Assign the n main effects $F_1,\ldots,F_n$ and the k two-factor interactions $F_{i_1}F_{j_1},\ldots,F_{i_k}F_{j_k}$ to points in $PG(r-1,m)$ as described in the previous paragraph. If the $\sum_{u=1}^{n}(m^{t_u}-1)/(m-1)+\sum_{u=1}^{k}(m^{t_{i_u}}-1)(m^{t_{j_u}}-1)/(m-1)$ points corresponding to $F_1,\ldots,F_n,F_{i_1}F_{j_1},\ldots,F_{i_k}F_{j_k}$ are all distinct, then we can obtain a universally optimal plan for estimating the main effects $F_1,\ldots,F_n$ and two-factor interations $F_{i_1}F_{j_1},\ldots,F_{i_k}F_{j_k}$ involving m^r runs.*

Proof. Let A_u be an $r \times t_u$ matrix with the t_u column vectors corresponding to t_u independent points in the (t_u-1)-flat F_u. Then the plan can be generated by the row space of the $r \times \sum_{u=1}^{n} t_u$ matrix $A=[A_1 \vdots \cdots \vdots A_n]$, where the t_u columns of A_u represent the levels of the factor F_u and each element of the row space of A represents a run in the plan. To prove that the plan is universally optimal, it suffices to show, as in Dey and Suen (2002), that the following matrices have full column rank:

(i) $[A_u \vdots A_v]$, $1 \le u < v \le n$;

(ii) $[A_u \vdots A_{i_v} \vdots A_{j_v}]$, $1 \le u \le n$, $1 \le v \le k$;

(iii) $[A_{i_u} \vdots A_{j_u} \vdots A_{i_v} \vdots A_{j_v}]$, $1 \leq u < v \leq k$,
where a matrix $A_u (1 \leq u \leq n)$ appears only once if it is repeated in (ii) or (iii).
Case (i) : The columns of A_u and A_v are independent since the $(t_u - 1)$-flat F_u and the $(t_v - 1)$-flat F_v are disjoint.

Case (ii) (a) : If $u = i_v$ or j_v, then the matrix reduces to $[A_{i_v} \vdots A_{j_v}]$ which has full column rank as in *Case* (i).
Case (ii) (b) : If u, i_v, j_v are distinct, then the $(t_u - 1)$-flat F_u and the $(t_{i_v} + t_{j_v} - 1)$-flat, consisting of points in F_{i_v}, F_{j_v}, and $F_{i_v} F_{j_v}$, are disjoint. Hence the columns of A_u are independent of columns of $[A_{i_v} \vdots A_{j_v}]$, and the matrix $[A_u \vdots A_{i_v} \vdots A_{j_v}]$ has full column rank.
Case (iii) (a) : If $i_u = i_v$ or j_v, then the matrix reduces to $[A_{j_u} \vdots A_{i_v} \vdots A_{j_v}]$ which has full column rank as in *Case* (ii) (b).
Case (iii) (b) : If i_u, j_u, i_v, j_v are distinct, then the $(t_{i_u} + t_{j_u} - 1)$-flat, consisting of points in F_{i_u}, F_{j_u}, and $F_{i_u} F_{j_u}$, and the $(t_{i_v} + t_{j_v} - 1)$-flat, consisting of points in F_{i_v}, F_{j_v}, and $F_{i_v} F_{j_v}$, are disjoint. Hence the columns of $[A_{i_u} \vdots A_{j_u}]$ are independent of columns of $[A_{i_v} \vdots A_{j_v}]$, and the matrix $[A_{i_u} \vdots A_{j_u} \vdots A_{i_v} \vdots A_{j_v}]$ has full column rank. This completes the proof.

Based on Theorem 2.2, we now construct specific families of optimal plans, under a model that includes the mean, all main effects and a specified set of two-factor interactions. These families of plans are constructed by a suitable choice of points in $PG(r-1, m)$ satisfying the conditions of Theorem 2.2. Most of the plans reported in this section are saturated. In the following, μ denotes the mean, F_i, the main effect of the the ith factor and $F_i F_j$, the interaction between F_i and F_j:

$$\mathcal{M}_1 \ : \ (\mu, F_1, \ldots, F_{2u}, F_1 F_2, F_3 F_4, \ldots, F_{2u-1} F_{2u});$$
$$\mathcal{M}_2 \ : \ (\mu, F_1, \ldots, F_{u+v}, F_i F_j; 1 \leq i \leq u, u+1 \leq j \leq v);$$
$$\mathcal{M}_3 \ : \ (\mu, F_1, \ldots, F_u, F_1 F_2, F_2 F_3, \ldots, F_{u-1} F_u, F_u F_1).$$

All effects not included in the model are assumed negligible.

A plan d that is universally optimal under the above models will be denoted respectively by $d \equiv (F_1, F_2; F_3, F_4; \ldots; F_{2u-1}, F_{2u})_1$, $d \equiv (F_1, \ldots, F_u; F_{u+1}, \ldots, F_{u+v})_2$ and $d \equiv (F_1, \ldots, F_u)_3$.

Note that the optimal plans of the three types are the ones that seem to be of use in practice, and are in no way exhaustive. In principle however, it is possible to obtain optimal plans under any other model that includes specified two-factor interactions, along with the mean and all main effects, via Theorem 2.2. Throughout this section, the m^2-level factors are denoted by F_0, F_1, F_2, etc., and the m-level factors by G_0, G_1, G_2, etc. We now have the following results.

Theorem 2.3. *For any prime or prime power m, we can construct a universally optimal plan*
(a) d_1 *for an* $(m^2)^2 \times m^{m^2}$ *experiment involving* m^5 *runs where*

$$d_1 \equiv (F_0; F_1, G_1, \ldots, G_{m^2})_2;$$

(b) d_2 *for an* $(m^2) \times m^{3m^2}$ *experiment involving* m^5 *runs where*

$$d_2 \equiv \{(F_0; G_1, \ldots, G_{m^2})_2, (G_{1,1}, G_{2,1}; G_{1,2}, G_{2,2}; \ldots; G_{1,m^2}, G_{2,m^2})_1\}.$$

Both d_1 *and* d_2 *are saturated.*

Proof. (a) Let F_0 be a line disjoint from the plane K in $PG(4, m)$. Choose F_1 to be a line on the plane K and $G_1, \ldots, G_{m^2}$ to be the m^2 points on the plane K but not on the line F_1.

(b) Let H be the 3-flat containing lines F_0 and F_1 as defined in (a), and let $F_0, L_1, \ldots, L_{m^2}$ be m^2+1 lines which partition H. For $i = 1, \ldots, m^2$, choose $G_{1,i}$ and $G_{2,i}$ to be two distinct points on the line L_i.

Theorem 2.4. *For any prime or prime power m, we can construct a universally optimal saturated plan d for an $(m^2)^{m^2+1} \times m$ experiment involving m^5 runs where*

$$d \equiv (G; F_1, \ldots, F_{m^2+1})_2.$$

Proof. Let H be a 3-flat in $PG(4, m)$, and let $F_1, \ldots, F_{m^2+1}$ be $m^2 + 1$ lines which partition H. Choose G to be a point of $PG(4, m)$ not in H.

Theorem 2.5. *Let F be an m^2-level factor and G be an m-level factor of a universally optimal plan d constructed according to the method of Theorem* 2.2. *If the effects F, G and FG can be estimated via d and F has no interaction with any other factor except G, then instead of estimating F and FG via d, we can optimally estimate the following effects:*
(a) $\{G_1, \ldots, G_{m+1}, GG_j,\ 1 \le j \le m + 1\}$;
(b) $\{G_0, G_1, \ldots, G_m, G_0G, G_0G_i,\ 1 \le i \le m\}$;
(c) $\{G_1, G_2, G_1G_2, G_2G, GG_1\}$ *and the main effects of* $G_3, \ldots, G_{m^2-2m+3}$;
(d) $\{G_1, G_2, G_3, G_1G_2, G_2G_3, G_3G_1\}$ *and if $m > 2$, the main effects of* $G_4, \ldots,$ G_{m^2-2m+3}.

Proof. Let K be the plane containing the point G and the line F.
(a) Let L be a line on the plane K which does not pass through the point G. Choose $G_1, \ldots, G_{m+1}$ to be the $m + 1$ points on the line L.
(b) Let L be a line through the point G on the plane K, and let $G, G_1, \ldots, G_m$ be the $m + 1$ points on the line L. Choose G_0 to be a point on the plane K but not on the line L.

(c) Let G_1 and G_2 be points on the plane K such that G, G_1 and G_2 are not collinear. Choose $G_3, \ldots, G_{m^2-2m+3}$ to be the $(m-1)^2$ points on the plane K which are not on the three lines joining the pairs of points (G, G_1), (G, G_2), (G_1, G_2).

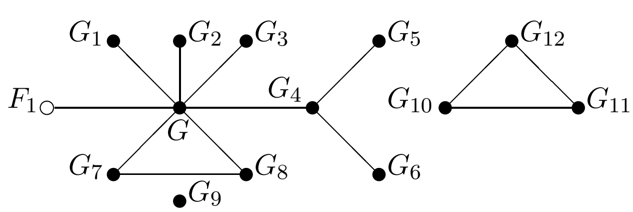
(d) Choose points G_1, G_2 and G_3 such that no three of the four points G, G_1, G_2 and G_3 are collinear. Now choose $G, G_4, \ldots, G_{m^2-2m+3}$ to be the $(m-1)^2$ points on the plane K which are not on the three lines joining the pairs of points (G_1, G_2), (G_2, G_3), (G_3, G_1).

We now consider an example. To save space, only examples for $m = 2$ are given in this section. In the following, as well as in subsequent examples in this section, we use the numbers $1, \ldots, 2^r - 1$ to represent the $2^r - 1$ points in $PG(r-1, 2)$. A number α represents a point in $PG(r-1, 2)$ with coordinates $(x_0, \ldots, x_{r-1})$ such that $\sum_{i=0}^{r-1} x_i 2^i = \alpha$. For example, the number 19 represents the point $(1, 1, 0, 0, 1)$ in $PG(4, 2)$, and it represents the point $(1, 1, 0, 0, 1, 0)$ in $PG(5, 2)$. A line in $PG(r-1, 2)$ is denoted by two numbers which represent two points on this line. Linear graphs are used to demonstrate the plans, where vertices represent the main effects and an edge joining two vertices represents the interaction of the two factors representing the two vertices. A 2-level factor is denoted by a closed circle $\bullet$ in the graph, and a 4-level factor, which is represented by a line in the finite projective geometry, is denoted by an open circle $\circ$. Finally, we use the symbols $G(12), F(1, 2)$ etc. to mean that the coordinates of the point G are given by the binary represenation of 12 in an appropriate finite projective geometry and, similarly, $F(1, 2)$ denotes a line joining the points given by the binary representations of 1 and 2.

Example 2.1. With $m = 2$ in Theorem 2.4, we can construct a universally optimal plan d for a $4^5 \times 2$ experiment involving 32 runs where

$$d \equiv (G; F_1, F_2, F_3, F_4, F_5)_2$$

and $G(16), F_1(1, 2), F_2(4, 8), F_3(5, 10), F_4(6, 11), F_5(7, 9)$. Many universally optimal plans can be obtained by applying Theorem 2.5. For example, by replacing the effects (F_2, GF_2), (F_3, GF_3), (F_4, GF_4),(F_5, GF_5) by (a), (b), (c) and (d), respectively, of Theorem 2.5, we obtain a universally optimal plan for a 4×2^{13} experiment involving 32 runs, whose linear graph is shown below:



where $G_1(4)$, $G_2(8)$, $G_3(12)$, $G_4(31)$, $G_5(5)$, $G_6(10)$, $G_7(6)$, $G_8(11)$, $G_9(29)$, $G_{10}(7)$, $G_{11}(9)$, $G_{12}(30)$.

Theorem 2.6. *For any prime or prime power m, we can construct a universally optimal saturated plan*

(i) d_1 *for an* $(m^2) \times m^{m^3+m^2+m}$ *experiment involving* m^5 *runs where*

$$d_1 \equiv \{(F_1; G_1, \ldots, G_m)_2, (G_{0,1}; G_{1,1}, \ldots, G_{m,1})_2, \ldots, (G_{0,m^2}; G_{1,m^2}, \ldots, G_{m,m^2})_2\}.$$

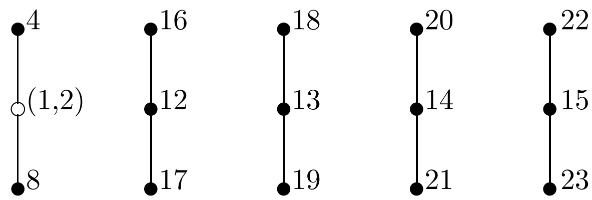
(ii) d_2 *for an* $(m^2) \times m^{m^3+2m^2-m+1}$ *experiment involving* m^5 *runs where*

$$d_2 \equiv \{(G_{0,0}; F_2, G_{1,0}, \ldots, G_{m^2-m,0})_2, (G_{0,1}; G_{1,1}, \ldots, G_{m,1})_2, \ldots, (G_{0,m^2}; G_{1,m^2}, \ldots, G_{m,m^2})_2\}.$$

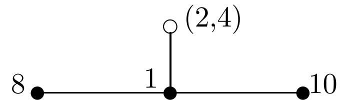
Proof. Let $G_{0,0}$ be a point on a line F_1 which is on a plane K in $PG(4,m)$. Let $L_1, \ldots, L_m, F_1$ be the $m+1$ lines through the point $G_{0,0}$ on the plane K. For $i = 1, \ldots, m$, let $G_{0,0}, G_{0,(i-1)m+1}, \ldots, G_{0,im}$ be the $m+1$ points on the line L_i. There are $m+1$ 3-flats through the plane K, say $H_0, \ldots, H_m$. For $i = 1, \ldots, m$, let $K, K_{1,i}, \ldots, K_{m,i}$ be the $m+1$ planes through the line L_i in the 3-flat H_i. For each $i = 1, \ldots, m$ and $j = 1, \ldots, m$, choose a line $L_{j,i}$ on the plane $K_{j,i}$ which does not pass through the point $G_{0,(i-1)m+j}$. Choose $G_{1,(i-1)m+j}, \ldots, G_{m,(i-1)m+j}$ to be the m points on the line $L_{j,i}$ but not on L_i. For plan (i), let L_0 be a line in the 3-flat H_0 but not on the plane K. Choose $G_1, \ldots, G_m$ to be the m points on the line L_0 but not on the plane K.

For plan (ii), let K_0 be a plane in the 3-flat H_0 which does not pass through the $G_{0,0}$. Then the line F_1 intersects K_0 at a point P_0. Choose F_2 to be a line through a point P_0 on the plane K_0, and choose $G_{1,0}, \ldots, G_{m^2-m,0}$ to be the $m^2 - m$ points on the plane K_0 which are not on the line F_2 or the plane K.

Example 2.2. With $m = 2$ in Theorem 2.6, choose the point $G_{0,0}(1)$ and the line $F_1(1,2)$. Let K be the plane through the line F_1 and the point $G_{0,1}(12)$. Let L_0 be the line consisting of points $G_1(4)$, $G_2(8)$ and $G_{0,1}$. Let L_1 be the line consisting of points $G_{0,0}$, $G_{0,1}$ and $G_{0,2}(13)$, and let L_2 be the line consisting of points $G_{0,0}$, $G_{0,3}(14)$ and $G_{0,4}(15)$. Let H_1 be the 3-flat through the plane K and the point $G_{1,1}(16)$, and let H_2 be the 3-flat through the plane K and the point $G_{1,3}(20)$. Following the procedure of Theorem 2.6 (i), we can choose the points $G_{2,1}(17)$, $G_{1,2}(18)$, $G_{2,2}(19)$, $G_{2,3}(21)$, $G_{1,4}(22)$, and $G_{2,4}(23)$ to construct the following universally optimal plan for a 4×2^{14} experiment involving 32 runs:



For plan (ii), we can choose $F_2(2,4)$, $G_{1,0}(8)$, and $G_{2,0}(10)$ to obtain the following universally optimal plan for a 4×2^{15} experiment involving 32 runs. The linear graph is the same as above except that the first component is changed to

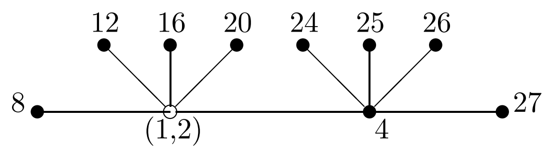


Theorem 2.7. *For any prime or prime power m and integers j, k satisfying $j + k = m + 1$, we can construct a universally optimal saturated plan d for an $(m^2) \times m^{km^2+jm+1}$ experiment involving m^5 runs where*

$$d \equiv \{(F_0; G_0, G_{1,1}, \ldots, G_{jm,1})_2, (G_0; G_{1,2}, \ldots, G_{km^2,\,2})_2\}.$$

Proof. Let K be a plane in $PG(4,m)$, and let G_0 and F_0 be a point and a line on the plane K such that G_0 is not on F_0. Let $H_1, \ldots, H_{m+1}$ be the $m + 1$ 3-flats through the plane K. For $i = 1, \ldots, j$, let L_i be a line in the 3-flat H_i which does not intersect the line F_0. Choose $G_{(i-1)m+1,1}, \ldots, G_{im,1}$ to be the m points on the line L_i which are not on the plane K. For $i = 1, \ldots, k$, let K_i be a plane in the 3-flat H_{j+i} which does not pass through the point G_0. Choose $G_{(i-1)m^2+1,1}, \ldots, G_{im^2,1}$ to be the m^2 points on the plane K_i but not on the plane K.

Example 2.3. With $m = 2, j = 2, k = 1$ in Theorem 2.7, choose the point $G_0(4)$ and the line $F_0(1,2)$. Then K is the plane through the line F_0 and the point G_0. Let H_1, H_2 and H_3 be the three 3-flats through the plane K and the points $G_{1,1}(8), G_{3,1}(16)$ and $G_{1,2}(24)$ respectively. Let L_1 be the line through the points $G_{1,1}$ and $G_{2,1}(12)$, and let L_2 be the line through points $G_{3,1}$ and $G_{4,1}(20)$. Let K_1 be the plane through the line F and the point $G_{1,2}$. Then K_1 has 4 points $G_{2,2}(25)$, $G_{3,2}(26)$, $G_{4,2}(27)$ and $G(1,2)$ which are not on the plane K. We have thus constructed the following universally optimal plan for a 4×2^9 experiment involving 32 runs:



Theorem 2.8. *For any prime or prime power m and integers j, k satisfying $j+k=m$, we can construct a universally optimal saturated plan*

(i) d_1 *for an* $(m^2)^j \times m^{m^3+km+k+1}$ *experiment involving* m^5 *runs where*

$$d_1 \equiv \{(G_{0,0}; G_{0,1}, G_1, \ldots, G_k, G_{1,0}, \ldots, G_{j(m^2-m),0}, F_1, \ldots, F_j)_2,\\ (G_{0,1}; G_{1,1}, \ldots, G_{(k+1)m^2,1})_2\}.$$

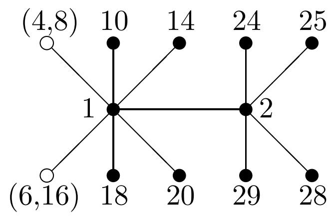
(ii) d_2 *for an* $(m^2)^j \times m^{m^3+(k+1)m+k}$ *experiment involving* m^5 *runs where*

$$d_2 \equiv \{(G_{0,0}; G_1, \ldots, G_k, G_{1,0}, \ldots, G_{j(m^2-m),0}, F_1, \ldots, F_j)_2,\\ (G_{0,1}; G'_{1,1}, \ldots, G'_{(k+1)m,1})_2, \ldots, (G_{0,m}; G'_{1,m}, \ldots, G'_{(k+1)m,m})_2\}.$$

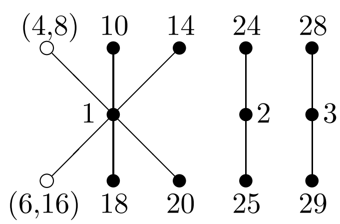
Proof. Let $G_1, \ldots, G_m$ and $G_{0,1}$ be the $m+1$ points on a line L in $PG(4,m)$, and let $G_{0,0}$ be a point not on the line L. Let K be the plane through the line L and the point $G_{0,0}$. There are $m+1$ 3-flats through the plane K in $PG(4,m)$, say $H_1, \ldots, H_{m+1}$. For $i = 1, \ldots, j$, let F_i be a line in the 3-flat H_i which passes through the point G_{k+i} but is not on the plane K. Let K_i be the plane through the lines L and F_i, and choose $G_{(i-1)(m^2-m)+1,0}, \ldots, G_{i(m^2-m),0}$ to be the m^2-m points on the plane K_i which are not on the lines L and F_i. To obtain plan (i), for $i = 1, \ldots, k+1$, let K_{j+i} be a plane in the 3-flat H_{j+i} which does not pass through the point $G_{0,1}$. Choose $G_{(i-1)m^2+1,1}, \ldots, G_{im^2,1}$ to be the m^2 points on the plane K_i but not on the plane K.

To obtain plan (ii), let L_0 be the line through the points $G_{0,0}$ and $G_{0,1}$, and let $G_{0,2}, \ldots, G_{0,m}$ be the $m-1$ other points on L_0. For $i = 1, \ldots, k+1$, let $K_{1,j+i}, \ldots, K_{m,j+i}$ and K be the $m+1$ planes through the line L_0 in the 3-flats H_{j+i}. For $u = 1, \ldots, m$, let $L_{u,j+i}$ be a line on the plane $K_{u,j+i}$ which does not pass through the point $G_{0,u}$. Now choose $G'_{(i-1)m+1,u}, \ldots, G'_{im,u}$ to be the m points on the line $L_{u,j+i}$ but not on the line L_0.

Example 2.4. With $m=2$, $j=2$, $k=0$ in Theorem 2.8, choose the point $G_{0,0}(1)$ and the line L consisting of points $G_1(4)$, $G_2(6)$, and $G_{0,1}(2)$. Then K is the plane through the line L and the point $G_{0,0}$. Choose lines $F_1(4,8)$ and $F_2(6,16)$. Let K_1 be the plane through the lines F_1 and L. Then K_1 has 2 points $G_{1,1}(10)$ and $G_{2,1}(14)$ which are not on the lines F_1 and L. Let K_2 be the plane through the lines F_2 and L. Then K_2 has 2 points $G_{3,1}(18)$ and $G_{4,1}(20)$ which are not on the lines F_2 and L. For plan (i), let K_3 be the plane through the points G_1, $G_{0,0}$, and $G_{1,2}(24)$. Then K_3 has 4 points $G_{1,2}$, $G_{2,2}(25), G_{3,2}(28)$ and $G_{4,2}(29)$ which are not on the plane K. We have thus constructed the following universally optimal plan for a $4^2 \times 2^{10}$ experiment involving 32 runs, whose linear graph is shown below:



For plan (ii), let L_0 be the line consisting of the points $G_{0,0}$, $G_{0,1}$ and $G_{0,2}(3)$. Choose $L_{1,3}$ to be the line through the points $G'_{1,1}(24)$ and $G'_{2,1}(25)$ and choose $L_{2,3}$ to be the line through the points $G'_{1,2}(28)$ and $G'_{2,2}(29)$. We have thus constructed the following universally optimal plan for a $4^2 \times 2^{11}$ experiment involving 32 runs:



Theorem 2.9. *For any prime or prime power m and an integer j, $0 \leq j \leq m+1$, we can construct a universally optimal saturated plan*

(i) d_1 *for an* $(m^2)^j \times m^{m^3+3m^2-2j+2}$ *experiment involving* m^6 *runs where*

$$d_1 \equiv \{(F_1; G_{1,1}, \ldots, G_{u_1 m^2,1})_2, \ldots, (F_j; G_{1,j}, \ldots, G_{u_j m^2,j})_2,$$
$$(G_1, G_2; \ldots; G_{2m^2-2j+1}, G_{2m^2-2j+2})_1\}, \quad \text{and} \quad \sum_{i=1}^{j} u_i = m+1.$$

(ii) d_2 *for an* $(m^2)^2 \times m^{m^3+m^2}$ *experiment involving* m^6 *runs where*

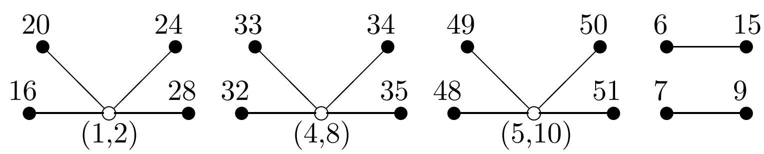
$$d_2 \equiv \{(F_1; F_2, G'_{1,1}, \ldots, G'_{jm^2,1})_2, (F_2; G'_{1,2}, \ldots, G'_{(m+1-j)m^2,2})_2\}.$$

Proof. Let $F_1, \ldots, F_{m^2+1}$ be m^2+1 lines which partition a 3-flat H in $PG(5,m)$. There are $m+1$ 4-flats through the 3-flat H in $PG(5,m)$, say $M_1, \ldots, M_{m+1}$. To obtain plan (i), for $i = 1, \ldots, j$ and $v = 1, \ldots, u_i$, let $K_{(v-1)m^2-1,i}, \ldots, K_{vm^2,i}$

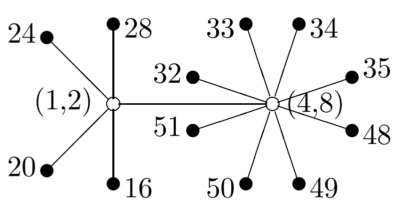
be the m^2 planes in the 4-flat M_i which pass through the line F_i but are not in the 3-flat H. For $t = 1, \dots, m^2$, choose $G_{(v-1)m^2+t,i}$ to be a point on the plane $K_{(v-1)m^2+t,i}$ but not on the line F_i. For $i = 1, \dots, m^2 - j + 1$, choose G_{2i-1} and G_{2i} to be two distinct points on the line F_{j+i}.

To obtain plan (ii), for $i = 1, \dots, j$, let $K'_{(i-1)m^2+1,1}, \dots, K'_{im^2,1}$ be the m^2 planes in the 4-flat M_i which pass through the line F_1 but are not in the 3-flat H. For $t = 1, \dots, m^2$, choose $G'_{(i-1)m^2+t,1}$ to be a point on the plane $K'_{(1-1)m^2+t,1}$ but not on the line F_1. For $i = 1, \dots, m+1-j$, let $K'_{(i-1)m^2+1,2}, \dots, K'_{im^2,2}$ be the m^2 planes in the 4-flat M_{j+i} which pass through the line F_2 but are not in the 3-flat H. For $t = 1, \dots, m^2$, choose $G'_{(i-1)m^2+t,2}$ to be a point on the plane $K'_{(1-1)m^2+t,2}$ but not on the line F_2.

Example 2.5. (i) With $m = 2, j = 3$ and $u_1 = u_2 = u_3 = 1$ in Theorem 2.9 (i), we obtain the following universally optimal plan for a $4^3 \times 2^{16}$ experiment involving 64 runs:



(ii) With $m = 2, j = 1$ in Theorem 2.9 (ii), we obtain the following universally optimal plan for a $4^2 \times 2^{12}$ experiment involving 64 runs:



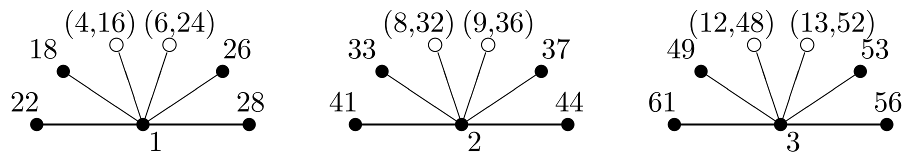
Theorem 2.10. *For any prime or prime power m, we can construct a universally optimal saturated plan d for an $(m^2)^{m^2+m} \times m^{m^4-m^2+m+1}$ experiment involving m^6 runs where*

$$\begin{aligned} d \equiv \{ & (G_{0,1}; F_{1,1}, \dots, F_{m,1}, G_{1,1}, \dots, G_{m^3-m^2,1})_2, \dots, \\ & (G_{0,m+1}; F_{1,m+1}, \dots, F_{m,m+1}, G_{1,m+1}, \dots, G_{m^3-m^2,m+1})_2 \}. \end{aligned}$$

Proof. Let L be a line in a 3-flat H in $PG(5, m)$, and let $G_{0,1}, \dots, G_{0,m+1}$ be the $m+1$ points on L. There are $m+1$ planes through the line L in the 3-flat H, say $K_1, \dots, K_{m+1}$. For $i = 1, \dots, m+1$, let L_i be a line on the plane K_i which does not pass through the point $G_{0,1}$, and let $P_{1,i}, \dots, P_{m,i}$ be the m points on

L_i but not on L. Let $M_1, \ldots, M_{m+1}$ be the $m+1$ 4-flats through the 3-flat H. For $i = 1, \ldots, m+1$, let $H_{1,i}, \ldots, H_{m,i}$, and H be the $m+1$ 3-flats through the plane K_i in the 4-flat M_i. For $j = 1, \ldots, m$, choose $F_{j,i}$ to be a line through the point $P_{j,i}$ but not on the plane K_i in the 3-flat $H_{j,i}$. Let $K_{j,i}$ be the plane through the lines $F_{j,i}$ and L_i. Choose $G_{(j-1)(m^2-m)+1,i}, \ldots, G_{j(m^2-m),i}$ to be the $m^2 - m$ points on the plane $K_{j,i}$ but not on the lines $F_{j,i}$ and L_i.

Example 2.6. With $m = 2$ in Theorem 2.10, we obtain the following universally optimal plan for a $4^6 \times 2^{15}$ experiment involving 64 runs:

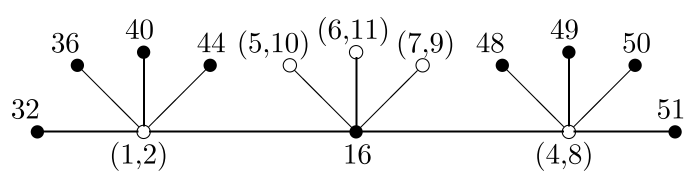


Theorem 2.11. *For any prime or prime power m and integer j, $1 \leq j \leq m$, we can construct a universally optimal saturated plan d for an $(m^2)^{m^2+1} \times m^{m^3+1}$ experiment involving m^6 runs where*

$$d \equiv \{(G_0; F_1, \ldots, F_{m^2+1})_2, (F_1; G_{1,1}, \ldots, G_{u_1 m^2,1})_2, \ldots,$$
$$(F_j; G_{1,j}, \ldots, G_{u_j m^2,j})_2\}, \quad \text{and} \quad \sum_{i=1}^{j} u_i = m.$$

Proof. Let $F_1, \ldots, F_{m^2+1}$ be m^2+1 lines which partition a 3-flat H in $PG(5, m)$. There are $m+1$ 4-flats through the 3-flat H in $PG(5, m)$, say $M_0, \ldots, M_m$. Choose G_0 to be a point in the 4-flat M_0 but not in the 3-flat H. For $i = 1, \ldots, j$ and $v = 1, \ldots, u_i$, let $K_{(v-1)m^2+1,i}, \ldots, K_{vm^2,i}$ be the m^2 planes in the 4-flat $M_{u_1+\cdots+u_{i-1}+v}$ through the line F_i but not in the 3-flat H. For $t = 1, \ldots, m^2$, choose $G_{(v-1)m^2+t,i}$ to be a point on the plane $K_{(v-1)m^2+t,i}$ but not on the line F_i.

Example 2.7. With $m = j = 2$, $u_1 = u_2 = 1$ in Theorem 2.11, we obtain the following universally optimal plan for a $4^5 \times 2^9$ experiment involving 64 runs:



Remark. The plans constructed in this section have some factors at m^2 levels and the others at m levels, where m is a prime or a prime power. In principle, the methods described so far can be extended to obtain optimal plans for experiments of the type $(m^{n_1}) \times \cdots \times (m^{n_u})$ in m^r runs where the $\{n_i\}$ and r are integers. However, such plans generally have too many levels and runs to be attractive to the experimenters and we do not report them.

3. Some More Optimal Plans for Asymmetric Experiments

The plans obtained in the previous section are such that the number of levels for each of the factors and the number of runs is a power of m, which itself is a prime or a prime power. Such plans are restrictive in nature in the sense that (i) except for $m = 2$, the number of levels and the number of runs generally become too large to be attractive to experimenters, and (ii) the methods cannot be used for obtaining optimal plans for experiments in which the number of levels of the factors and the number of runs are not powers of the same prime; for example, the methods described in the previous section cannot produce optimal plans for the practically important experiments of the type $3^{n_1} \times 2^{n_2}$. In this section, we propose two methods of construction of optimal plans for asymmetric experiments where the number of levels of different factors and the number of runs are not necessarily powers of the same prime. We make use of orthogonal arrays.

Recall that an orthogonal array $OA(N, n, m_1 \times \cdots \times m_n, g)$, having N rows, n columns, $m_1, \ldots, m_n (\geq 2)$ symbols and strength $g (< n)$, is an $N \times n$ matrix with elements in the ith column from a set of m_i distinct symbols $(1 \leq i \leq n)$, in which all possible combinations of symbols appear equally often as rows in every $N \times g$ submatrix. If $m_1 = \cdots = m_n = m$, then we have a symmetric orthogonal array, which will be denoted by $OA(N, n, m, g)$.

Consider an orthogonal array $OA(N, n, m_1 \times \cdots \times m_n, 2)$ of strength two, say A, and suppose for $1 \leq j \leq n$, $m_j = t_{j1} t_{j2} \ldots t_{jk_j}$, where $t_{ji} \geq 2,\ 1 \leq i \leq k_j$ are integers. Replace the m_j-symbol column in A by k_j columns, say $F_{j1}, \ldots, F_{jk_j}$, having $t_{j1}, \ldots, t_{jk_j}$ symbols respectively, and call the derived array B. It is not hard to see that B is an $OA(N, \sum_{j=1}^n k_j, \prod_{j=1}^n \prod_{u=1}^{k_j} t_{ju}, 2)$. We then have the following result whose proof is straight forward.

Theorem 3.1. *The fractional factorial plan d represented by the orthogonal array B is universally optimal in the class of all N-run plans under a model that includes the mean, all main effects and the two-factor interactions $F_{ji} F_{ji'}$, $1 \leq i < i' \leq k_j, 1 \leq j \leq n$.*

We next discuss another class of plans. Suppose there exists a plan d^* for an $m_1 \times \cdots \times m_n$ factorial in N/t runs, where $N, t \geq 2$ are integers such that d^* satisfies the conditions of Theorem 2.1. Thus d^* is universally optimal in a relevant class of competing designs for the estimation of the mean, all main effects and the two-factor interactions $G_{i_1}G_{j_1}, \ldots, G_{i_k}G_{j_k}$, where $1 \leq i_u, j_u \leq n$ for all $u = 1, \ldots, k$. Here, for $1 \leq u \leq n$, the factor G_u is at m_u levels. Let the treatment combinations of d^* be represented by an $(N/t) \times n$ matrix A. Let B be an orthogonal array $OA(t, p, s_1 \times \cdots \times s_p, 2)$ of strength two. Form N treatment combinations of an $s_1 \times \cdots \times s_p \times m_1 \times \cdots \times m_n$ factorial as $[B \otimes \mathbf{1}_{N/t} \vdots \mathbf{1}_t \otimes A]$, where for a pair of matrices E, F, $E \otimes F$ denotes their Kronecker (tensor) product. Let d be the plan represented by these N treatment combinations. Furthermore, for $1 \leq i \leq p$, let F_i denote the factor at s_i levels. Then, one can prove the following result.

Theorem 3.2. *The N treatment combinations forming the fractional factorial plan d is universally optimal for estimating the mean, all main effects and the interactions F_iG_j; $1 \leq i \leq p, 1 \leq j \leq n$ and $G_{i_u}G_{j_u}$, $1 \leq u \leq k$.*

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