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CROSS-VALIDATORY CHOICE OF WEIGHTS FOR COMBINING INTRABLOCK AND INTERBLOCK ESTIMATES

Sharon A. Naberhuis Stehouwer and Anant M. Kshirsagar

University of Michigan

Abstract: A cross-validatory choice of weights for combining intrablock and interblock estimates of treatments in a Balanced Incomplete Block (B.I.B.) design only was proposed by Jensen and Stone. In this paper, this method is extended to any incomplete block design. The weights for a B.I.B. design are expressed in terms of the sums of squares in the analysis of variance of the B.I.B. design. This has enabled us to compare this method to other methods in the literature and establish properties such as unbiasedness and uniform betterness of the combined estimates. The weights for a regular Group Divisible design are also provided.

Key words and phrases: Cross-validation, combining inter- and intrablock information, incomplete block designs.

1. Introduction

If in an incomplete block design, the block effects are assumed to be random, the Best Linear Unbiased Estimate (BLUE) of an estimate treatment contrast is expressible as a weighted sum of two independent estimates (independent under the random block effects model), one based on Adjusted Treatment totals (called the intra-block estimate) and the other based on Block totals only (called the inter-block estimate). The weights are, of course, proportional to the reciprocals of the variances of these estimates; however, these true weights are unknown. There is a considerable literature on estimation and choice of these weights (see, e.g., Yates (1940), Graybill and Deal (1959), Seshadri (1963), Shah (1964), Stein (1966), Brown and Cohen (1974), C. R. Rao (1947), Khatri and Shah (1974)). Jensen and Stone (1976) have suggested a different approach based on cross-validation for this purpose. Stone's (1974) cross-validation method, as applied to this problem, consists in pretending every observation to be missing turn by turn, estimating it from both the intrablock and interblock models, and finding out weights to combine these estimates in such a way that the combined estimate is "close" to the true value. These weights are then used to combine intrablock and interblock estimates of any treatment contrast. The reasoning behind this choice is that, if the weights are good enough for predicting an observation, they should be good enough for estimating treatment contrasts. Jensen and Stone (1976) described and illustrated this method only for Balanced Incomplete Block (B.I.B.) designs. They have not expressed these weights in the customary fashion, explicitly in terms of the sums of squares of the Analysis of Variance (ANOVA) table of a B.I.B. design. They have observed some inequality relations when compared with other choices of weights, but only for very few particular designs. They have also not established properties such as uniform betterness or unbiasedness (see Khatri and Shah (1974)). All these shortcomings have been rectified in this paper and further the method is extended to other incomplete block designs. The particulars for another special case, a Group Divisible design, are also given. Our method enables us to prove that the weights are positive and also to compare them with the weights proposed by others in the literature.

2. Notation

Consider a general binary, proper, equireplicate, connected incomplete block design with b blocks of k plots each and v treatments each replicated r times. Let the ith treatment ($i = 1,2,\dots,v$) occur n_{ij} times (where n_{ij} is only 1 or 0) in the jth block ($j = 1,2,\dots,b$). The $v \times b$ matrix N of the elements n_{ij} is the "incidence matrix" of the design. When $n_{ij} = 1$, y_{ij} will denote the yield of the ith treatment in the jth block. B_j denotes the total of the k yields in the jth block and T_i denotes the total of the r yields of the ith treatment. The intrablock model is

$$y_{ij} = \beta_j + \tau_i + \epsilon_{ij}, \tag{2.1}$$

for all i, j with $n_{ij} = 1$. β_j is the effect of the jth block, τ_i is the effect of the ith treatment and ϵ_{ij}'s are assumed to be independent normal variables with zero means and variance σ^2 (denoted by $\mathrm{NI}(0,\sigma^2)$). The β_j, τ_i, σ^2 are all fixed unknown parameters. When interblock information is to be recovered, we further assume the β_j's to be $\mathrm{NI}(0,\sigma_b^2)$, independent of the ϵ_{ij}'s. The interblock model leads to

$$\mathrm{E}(B_j) = \sum_{u=1}^{v} n_{uj}\tau_u, \quad j = 1,2,\dots,b; \tag{2.2}$$

$$\mathrm{Var}(\boldsymbol{B}) = k(\sigma^2 + k\sigma_b^2)I_b, \tag{2.3}$$

where E is the expectation operator, Var stands for the variance-covariance matrix of the vector in the parenthesis following it, I_m is the identity matrix of order m, and $\boldsymbol{B}$ is the column vector of the B_j's.

From standard results in experimental designs, it is well known that the least squares solutions $\hat{\tau}_i$, $\hat{\beta}_j$ corresponding to the intrablock model (2.1) are

$$\hat{\tau} = C^{-}\boldsymbol{Q}, \tag{2.4}$$

$$\hat{\beta}_j = \frac{1}{k}\left(B_j - \sum_{i=1}^{v} n_{ij}\hat{\tau}_i\right), \tag{2.5}$$

where

$$\boldsymbol{\tau} = \text{the column vector of the } \tau_i\text{'s},$$

$$C = rI_v - \frac{1}{k}NN', \tag{2.6}$$

$$C^{-} = \text{any generalized inverse of } C, \tag{2.7}$$

$$\boldsymbol{Q} = \boldsymbol{T} - \frac{1}{k}N\boldsymbol{B}, \tag{2.8}$$

$$\boldsymbol{T} = \text{vector of the treatment totals } T_i. \tag{2.9}$$

Similarly, the least squares solutions $\boldsymbol{\tau}^*$ corresponding to the interblock model (2.2) are given by

$$\boldsymbol{\tau}^* = (NN')^{-}N\boldsymbol{B}, \tag{2.10}$$

where $(NN')^{-}$ is any generalized inverse of NN'. If a treatment contrast $\boldsymbol{l}'\boldsymbol{\tau}$ is estimable for both the models, its intrablock and interblock estimates will be $\boldsymbol{l}'\hat{\boldsymbol{\tau}}$ and $\boldsymbol{l}'\boldsymbol{\tau}^*$ respectively and we shall write the combined estimate as

$$(1-a)\boldsymbol{l}'\hat{\boldsymbol{\tau}} + a\boldsymbol{l}'\boldsymbol{\tau}^*. \tag{2.11}$$

This paper is concerned with the cross-validatory choice for the weight a. The method consists in pretending y_{ij} to be missing for every i and j, turn by turn, and predicting it from the models (2.1) and (2.2). If these predicted values are denoted by w_{ij} (for (2.1)) and w^*_{ij} (for (2.2)), we determine a such that y_{ij} is close to the weighted combination

$$(1-a)w_{ij} + aw^*_{ij}, \tag{2.12}$$

for all i, j and for this we minimize

$$\sum_i \sum_j n_{ij}(y_{ij} - (1-a)w_{ij} - aw^*_{ij})^2, \tag{2.13}$$

as suggested by Jensen and Stone (1976). This yields

$$a = \frac{\sum_i \sum_j n_{ij}(y_{ij} - w_{ij})(w_{ij}^* - w_{ij})}{\sum_i \sum_j n_{ij}(w_{ij}^* - w_{ij})^2}. \tag{2.14}$$

To obtain w_{ij} and w_{ij}^*, we first apply the cross-validatory method to a general linear model and use those results on (2.1) and (2.2). This is done in the next section.

3. Cross-Validatory Method for a General Linear Model

Consider the general linear model

$$\boldsymbol{y} = X\boldsymbol{\theta} + \boldsymbol{\epsilon}, \tag{3.1}$$

where $\boldsymbol{y}$ is $n \times 1$, X is $n \times p$ of rank r, $\boldsymbol{\theta}$ is the $p \times 1$ vector of unknown parameters and $\boldsymbol{\epsilon}$ is the vector of random normal independent errors. The least squares solution when all the n observations y_i in $\boldsymbol{y}$ are available is

$$\hat{\boldsymbol{\theta}} = (X'X)^- X'\boldsymbol{y}, \tag{3.2}$$

where $(X'X)^-$ is any generalized inverse of $X'X$. Let $\boldsymbol{x}_i'$ be the ith row of X, so that

$$X'X = \sum_{i=1}^{n} \boldsymbol{x}_i \boldsymbol{x}_i'. \tag{3.3}$$

Let us now pretend that y_i is missing and find the predictor of y_i from the remaining observations. We can apply the missing plot technique proposed by Yates originally and later developed by various others (see for example Chakrabarti (1962) or McKee (1980)). This consists in substituting an algebraic quantity f_i in place of y_i and determining it in such a way that the error sum of squares (s.s.) for (3.1) is minimum. This turns out to be the same as equating f_i to the expected value of y_i, namely $\boldsymbol{x}_i'\boldsymbol{\theta}$, *after* replacing $\boldsymbol{\theta}$ by its least squares solution, when the observation vector is not $\boldsymbol{y}$ but $\boldsymbol{y}$ with y_i changed to f_i. That is, f_i is a solution of

$$f_i = \boldsymbol{x}_i'(X'X)^- X'[y_1, \dots, y_{i-1}, f_i, y_{i+1}, \dots, y_n]'. \tag{3.4}$$

This yields, after some algebra,

$$f_i = \frac{\boldsymbol{x}_i'\hat{\boldsymbol{\theta}} - \boldsymbol{x}_i'(X'X)^- \boldsymbol{x}_i y_i}{1 - \boldsymbol{x}_i'(X'X)^- \boldsymbol{x}_i}, \quad i = 1, 2, \dots, n. \tag{3.5}$$

Actually, this formula can be derived without going through all this algebra by using the method of Ker-Chau Li (1987). Since y_i is not actually missing but only held back, we have used (3.5) which uses y_i, but different forms of f_i not using y_i are also available in the missing plot situation. Also, the denominator is non-zero, in general, and vanishes only when $n = p$ and in that case there is no unique solution for f_i and any value serves the purpose. We shall assume that $n > p$. From the literature on missing plot technique, it is known that this f_i is the BLUE of $E(y_i)$ and hence is the best predictor of y_i in a certain sense. The least squares solution $\hat{\boldsymbol{\theta}}_{(-i)}$ when y_i is deleted is the usual least squares solution when y_i is replaced by f_i, given above, in $\boldsymbol{y}$. This then gives (using (3.2))

$$\hat{\boldsymbol{\theta}}_{(-i)} = \hat{\boldsymbol{\theta}} + (f_i - y_i)(X'X)^{-}\boldsymbol{x}_i, \quad i = 1, 2, \dots, n. \tag{3.6}$$

We shall now use these results to derive w_{ij} and w_{ij}^*.

4. Intrablock and Interblock Predictors of Yields

Using (3.5) for the intrablock model (2.1), the intrablock predictor of y_{ij} (when $n_{ij} = 1$) comes out to be

$$w_{ij} = (\hat{\tau}_i + \hat{\beta}_j - p_{ij}y_{ij})/(1 - p_{ij}), \tag{4.1}$$

where $\hat{\tau}_i$, $\hat{\beta}_j$ are defined by (2.4), (2.5) and p_{ij} is nothing but $\boldsymbol{x}_i'(X'X)^{-}\boldsymbol{x}_i$ of (3.5) which reduces to

$$p_{ij} = \frac{1}{k} + \frac{1}{k^2}(N'C^{-}N)_{jj} - \frac{2}{k}(C'^{-}N)_{ij} + (C^{-})_{ii}. \tag{4.2}$$

Here $(C^{-})_{ii}$ denotes the (i,i)th element of C^{-} and similarly for other matrices. Using (2.5), (4.1) further simplifies as

$$w_{ij} = \frac{k\hat{\tau}_i - \boldsymbol{N}'_{(j)}\hat{\boldsymbol{\tau}} + B_j - kp_{ij}y_{ij}}{k(1 - p_{ij})}, \tag{4.3}$$

where $\boldsymbol{N}_{(j)}$ is the jth column of N.

To find w_{ij}^* from the interblock model (2.2), we find that the total of the jth block, with the yield y_{ij} assumed to be missing and an algebraic quantity f_{ij} substituted for it, is $B_j^o + f_{ij}$, where $B_j^o = B_j - y_{ij}$ is the total yield of all the plots in the jth block, excluding y_{ij}. Minimizing the inter-block error sum of squares with respect to f_{ij}, for the model (2.2), or from (3.5), we find that

$$f_{ij} = \frac{\boldsymbol{N}'_{(j)}\boldsymbol{\tau}^* - B_j + (1 - m_j)y_{ij}}{1 - m_j}, \tag{4.4}$$

where m_j is given by

$$m_j = N'_{(j)}(NN')^{-}N_{(j)} \tag{4.5}$$

and τ^* is given by (2.10). From (3.6), the least squares solution for τ in the interblock model, when y_{ij} is pretended to be missing, is

$$\tau^*_{(-ij)} = \tau^* + (B^o_j + f_{ij} - B_j)(NN')^{-}N_{(j)}. \tag{4.6}$$

To find the inter-block predictor w^*_{ij} of y_{ij}, from these results, we observe that, when $n_{ij} = 1$,

$$y_{ij} = \tau_i + \beta_j + \epsilon_{ij} \tag{4.7}$$

and

$$B^o_j = (k-1)\beta_j + \sum_{u=1,u\neq i}^{v} n_{uj}\tau_u + \sum_{u=1,u\neq i}^{v} \epsilon_{uj}. \tag{4.8}$$

Therefore,

$$y_{ij} = \tau_i + \frac{1}{k-1}\left(B^o_j - \sum_{u=1,u\neq i}^{v} n_{uj}\tau_u\right) + \epsilon_{ij} - \frac{1}{k-1}\sum_{u=1,u\neq i}^{v} \epsilon_{uj} \tag{4.9}$$

$$= \frac{1}{k-1}\left\{(B_j - y_{ij}) + k\tau_i - \sum_{u=1}^{v} n_{uj}\tau_u\right\} + \text{error terms}. \tag{4.10}$$

Hence, if $k\tau_i - N'_{(j)}\tau$ is estimable in the model (2.2) when y_{ij} is deleted, the interblock predictor of w^*_{ij} of y_{ij}, as used by Jensen and Stone, from all the observations excluding y_{ij}, is

$$w^*_{ij} = \frac{1}{k-1}\left\{(B_j - y_{ij}) + k\tau^*_{i(-j)} - \sum_{u=1}^{v} n_{uj}\tau^*_{u(-j)}\right\}. \tag{4.11}$$

On using (4.4), (4.6) and the notation

$$B^*_j = \sum_{u=1}^{v} n_{uj}\tau^*_{u(-j)}, \tag{4.12}$$

we find, after some algebra, that

$$(k-1)w^*_{ij} = k\tau^*_i + (1-m_j)^{-1}(B^*_j - N'_{(j)}\tau^*)\left[1 - k((NN')^{-}N)_{ij}\right] - y_{ij}. \tag{4.13}$$

Jensen and Stone did not explicitly state the estimability condition about $k\tau_i - \sum_{u=1}^{v} n_{uj}\tau_u$ (where $n_{ij} = 1$), when they used this method for some symmetric Balanced Incomplete Designs. (See the correspondence by Kshirsagar (1980) and Jensen and Stone (1980) in this connection.) To satisfy the estimability condition, the vector

$$\boldsymbol{h} = \boldsymbol{N}_{(j)} - k[0 \cdots 0\ 1\ 0 \cdots 0]' \tag{4.14}$$

(where 1 is in the ith position) must be linearly dependent on all the column vectors of N, excluding $\boldsymbol{N}_{(j)}$. Note that $\boldsymbol{h}$ is orthogonal to $\boldsymbol{N}_{(j)}$, as $n_{ij} = 1$, and

$$\begin{aligned}
\boldsymbol{h}'\boldsymbol{N}_{(j)} &= \boldsymbol{N}'_{(j)}\boldsymbol{N}_{(j)} - k \\
&= \sum_u n_{uj}^2 - k \\
&= \sum_u n_{uj} - k \\
&= k - k \\
&= 0.
\end{aligned}$$

When $b = v$ and the rank of N is b, it is obvious that $\boldsymbol{h}$ will be a linear combination of all the columns of N, excluding $\boldsymbol{N}_{(j)}$. In other cases, it may not be so and an interblock predictor may not exist. In that case, we include only those contrasts that are estimable in the interblock model.

For a general design, if we now substitute the values w_{ij} and w_{ij}^* given by (4.3) and (4.13), in (2.14), we shall get the value of the weight a. It is a complicated expression and it is worthwhile doing so only for particular designs where NN' and $(NN')^-$ or C^- are known. Such particular cases are considered in the next sections.

5. B.I.B. Designs

For a B.I.B. design with parameters b, k, v, r, λ,

$$NN' = (r - \lambda)I_v + \lambda E_{vv}, \tag{5.1}$$

where E_{pq} is a $p \times q$ matrix of unit elements. For this design it is well known (see for example John (1971)) that the intrablock and interblock least squares solutions are

$$\hat{\tau}_i = \frac{k}{v\lambda} Q_i, \quad i = 1, 2, \ldots, v, \tag{5.2}$$

and

$$\tau_i^* = \frac{1}{r - \lambda}\left(\sum_{j=1}^{b} n_{ij}B_j - \lambda \sum_{j=1}^{b} B_j/r\right), \quad i = 1, 2, \ldots, v. \tag{5.3}$$

After considerable algebra, it can be shown that for B.I.B. designs, the Jensen and Stone weight a obtained in the last section can be expressed as

$$a = \frac{\text{MSE}}{l_1\text{MSE} + l_2\text{AB} + l_3\text{IE}}, \tag{5.4}$$

where MSE is the error mean square (intrablock) based on

$$f = bk - b - v + 1 \tag{5.5}$$

degrees of freedom (d.f.), AB stands for the Blocks (adjusted for treatments) S.S., and IE is the interblock error S.S. given by

$$\text{IE} = \text{Unadjusted Block S.S.} - \frac{k}{r-\lambda}\left(\sum_{i=1}^{v} Q_{li}^2 - \frac{g^2}{v}\right), \tag{5.6}$$

where $g =$ the grand total of all the yields, and

$$Q_{li} = \frac{1}{k}\sum_{j=1}^{b} n_{ij}B_j.$$

The constants l_1, l_2, l_3 are as follows:

$$l_1 = \frac{v-1}{b(k-1)}; \quad l_2 = -l_3 = \frac{fk}{b(k-1)(v-k)}. \tag{5.7}$$

The advantage of the form (5.4) is that one need not calculate the bk intrablock and bk interblock predictors of the y_{ij}'s for getting a. Also, all the known weights in the literature are exactly of the same form as (5.4). For a ready comparison of this Jensen and Stone weight a with others, the values of l_1, l_2 and l_3 are given in Table 1 for them. (5.4) shows that the weight a is always positive, a fact Jensen and Stone did not establish.

Since $\sigma^2 + k\sigma_b^2 \geq \sigma^2$, it is recommended in the literature that if the estimate of σ^2 is found to be negative, it should be truncated at zero. This amounts to using

$$a_T = \text{Min}\left\{a, \frac{r-\lambda}{rk}\right\}. \tag{5.8}$$

Shah (1971) has shown that this truncation reduces the variance of the combined estimate for the weights proposed by Graybill and Deal, Seshadri, Shah and Stein. Applying this truncation condition, we find that $a_T = a$, if

$$\text{MSE} \leq \frac{kf}{(v-1)[kf + v(k-1)]}(\text{AB} - \text{IE}). \tag{5.9}$$

Table 1. Values of l_1, l_2, l_3 for interblock weights for a B.I.B. design

	l_1	l_2	l_3
Yates or Rao	$\frac{r-k}{r-1}$	$\frac{k(k-1)}{(v-k)(r-1)}$	0
Graybill and Deal	1	0	$\frac{v(k-1)}{(v-k)(b-v)}$
Seshadri	0	$\frac{(v-1)k}{(v-k)(v-3)}$	$\frac{-(v-1)k}{(v-k)(v-3)}$
Shah	0	$\frac{k}{v-k}$	$\frac{-k}{v-k}$
Stein	0	$\frac{k(v-1)(f+2)}{(v-k)(v-3)f}$	$\frac{-k(v-1)(f+2)}{(v-k)(v-3)f}$
Khatri and Shah	1	$\frac{k(v-1)(f+2)}{(v-k)(b-3)f}$	$\frac{-(f+2)}{(b-3)f}$
Brown and Cohen	$\frac{(f+2)(b+1)}{f(b-3)}$	$\frac{k(v-1)(f+2)}{(v-k)(b-3)f}$	$\frac{-(f+2)}{(b-3)f}$
Jensen and Stone	$\frac{v-1}{b(k-1)}$	$\frac{kf}{b(k-1)(v-k)}$	$\frac{-kf}{b(k-1)(v-k)}$

When this condition holds, it can be verified by direct substitution that

$$\text{Jensen-Stone's } a > \text{ Shah's } a > \text{ Seshadri's } a > \text{ Stein's } a \tag{5.10}$$

for all B.I.B. designs. If a B.I.B. design is symmetric,

$$\begin{aligned} &\text{Jensen-Stone's } a > \text{Yates'-Rao's } a = \text{Shah's } a > \text{Seshadri's } a \\ &\quad > \text{Stein's } a > \text{Khatri-Shah's } a > \text{Brown-Cohen's } a \end{aligned} \tag{5.11}$$

(Graybill-Deal's a is not defined in this case). Thus the cross-validatory choice assigns a larger weight to the interblock estimate. Jensen and Stone have observed this for the four particular designs they considered but did not prove this in general.

Roy and Shah (1962) have shown that, if $(1-a)/a$ has a particular form, the combined estimate is unbiased in the random block effects model. The Jensen and Stone's a satisfies this requirement.

6. Uniformly Better Estimates

We now prove the following theorem.

Theorem. *For a Balanced Incomplete Block Design with parameters (b, v, r, k, λ), the combined intra- and interblock estimates of treatment contrasts, using the Jensen and Stone weights have a uniformly smaller variance (when blocks are random) than that of the intrablock estimate alone, if and only if*

$$\frac{b(k-1)}{v-1} \leq \frac{2f^2(v-3)}{(f+2)(v-1)^2}, \tag{6.1}$$

where $f = bk - b - v + 1$, the degrees of freedom of the intrablock error sum of squares.

To prove this theorem, consider any $v-1$ treatment contrasts $\boldsymbol{\xi}'_s\boldsymbol{\tau}$ $(s = 1, 2, \ldots, v-1)$ with

$$\boldsymbol{\xi}'_s\boldsymbol{\xi}_s = 1, \quad \boldsymbol{\xi}'_s\boldsymbol{\xi}_l = 0 \ (s \neq l), \quad \boldsymbol{\xi}'_s\mathrm{E}_{vl} = 0. \tag{6.2}$$

Further, let

$$W_s = (\boldsymbol{\xi}'_s\hat{\boldsymbol{\tau}} - \boldsymbol{\xi}'_s\boldsymbol{\tau}^*)^2, \tag{6.3}$$

where $\hat{\boldsymbol{\tau}}$, $\boldsymbol{\tau}^*$ are given by (5.2) and (5.3). Then

$$\frac{W_s}{(k\sigma^2/\lambda v) + k(\sigma^2 + k\sigma_b^2)/(r-\lambda)}, \quad s = 1, 2, \ldots, v-1, \tag{6.4}$$

are independent χ^2 variables with 1 d.f. each. Also, the quantity AB$-$IE occurring in a of (5.4) is

$$\text{AB}-\text{IE} = (rk^2)^{-1}(r-\lambda)\lambda v \sum_{s=1}^{v-1} W_s. \tag{6.5}$$

The combined estimator of $\boldsymbol{\xi}'_s\boldsymbol{\tau}$ is

$$\hat{\mu} = (1-a)\boldsymbol{\xi}'_s\hat{\boldsymbol{\tau}} + a\boldsymbol{\xi}'_s\boldsymbol{\tau}^*. \tag{6.6}$$

Bhattacharya (1980) has shown that $\hat{\mu}$ is uniformly better, if and only if

$$\frac{b(k-1)}{v-1} \leq 2\inf_{\Upsilon}\left(\frac{\mathrm{E}(\gamma)}{\mathrm{E}(\gamma^2)}\right), \tag{6.7}$$

where

$$\Upsilon = \frac{\lambda v}{r-\lambda}\left(\frac{\sigma^2 + k\sigma_b^2}{\sigma^2}\right), \tag{6.8}$$

$$\gamma = \frac{(1+\Upsilon)a^*(v-1)}{b(k-1)}, \tag{6.9}$$

and a^* is the same as a except that the W_s in a is replaced by another W_s^* having the same distribution as W_s of (6.4) but with d.f. 3 and not 1. (See also Khatri and Shah (1974) for this.) Then

$$\gamma = \frac{(1+\Upsilon)Z_1}{Z_1 + mZ_2(1+\Upsilon)}, \tag{6.10}$$

where

$$Z_1 = (\text{Error S.S.})/\sigma^2, \tag{6.11}$$

$$Z_2 = \left(\sum_{i=1, i\neq s}^{v-1} W_i + W_s^*\right) \Big/ \left(\frac{k\sigma^2}{\lambda v} + \frac{k(\sigma^2 + k\sigma_b^2)}{r-\lambda}\right), \tag{6.12}$$

$$m = f^2/(v-1)^2. \tag{6.13}$$

Differentiating $\mathrm{E}(\gamma)/\mathrm{E}(\gamma^2)$ with respect to Υ, we get

$$\frac{\partial}{\partial\Upsilon}\left(\frac{\mathrm{E}(\gamma)}{\mathrm{E}(\gamma^2)}\right) = \frac{\mathrm{E}^2(\gamma^2) - 2\mathrm{E}(\gamma^3)\mathrm{E}(\gamma)}{(1+\Upsilon)^2\mathrm{E}^2(\gamma^2)}, \tag{6.14}$$

where it is assumed that the expected values on the right side of (6.14) all exist and hence the interchange of expectation and derivation is justified. By Liapounoff's inequality, therefore, the derivative (6.14) is negative and so $\mathrm{E}(\gamma)/\mathrm{E}(\gamma^2)$ decreases as Υ increases. Therefore

$$\begin{aligned} \mathrm{Inf}_\Upsilon \frac{\mathrm{E}(\gamma)}{\mathrm{E}(\gamma^2)} &= \lim_{\Upsilon\to\infty} \frac{\mathrm{E}(\gamma)}{\mathrm{E}(\gamma^2)} = \frac{\mathrm{E}(\lim\gamma)}{\mathrm{E}(\lim\gamma^2)} \\ &= \frac{\mathrm{E}(Z_1/mZ_2)}{\mathrm{E}(Z_1/mZ_2)^2} = \frac{m(v-3)}{f+2}. \end{aligned} \tag{6.15}$$

Substituting this in (6.7), the theorem follows. We examined this condition for all the B.I.B. designs listed by Cochran and Cox (1957), Fisher and Yates (1963), and Raghavarao (1971) and found out that this condition never holds if $v \leq 5$

Table 2. Designs satisfying (6.15) with $6 \le v \le 9$

v	b	r	k	λ
6	15	4	10	6
7	7	6	6	5
8	14	4	7	3
8	8	7	7	6
9	36	2	8	1
9	18	4	8	3
9	12	6	8	5
9	9	8	8	7
9	18	5	10	5

and always holds if $v \ge 10$. For $6 \le v \le 9$, it is satisfied for the above designs only.

The exact variance of a class of estimates of the form (5.4) has been studied by Khatri and Shah (1974). Jensen and Stone's estimate belongs to this class, but their formula is too complicated for any direct comparison of these variances analytically.

7. A Regular Group Divisible Design

We now derive the Jensen and Stone weight for a regular Group Divisible (G.D.) design with m groups of n treatments each. If the v treatments are written in the order of groups so that the first n treatments belong to the first group, the next n belong to the second group and so on, it is well known (see for example John (1971)) that

$$NN' = (r - \lambda_1)I_m \otimes I_n + (\lambda_1 - \lambda_2)I_m \otimes E_{nn} + \lambda_2 E_{mm} \otimes E_{nn},$$

where $\otimes$ denotes Kronecker Product. From this and from the intrablock and interblock least squares solutions $\hat{\tau}$ and τ^* as given by John (1971), it can be shown after considerable algebra that, when $\lambda_1 = 0$,

$$a = \frac{\text{MSE}}{\text{MSE} + \dfrac{kf\{\frac{1}{k}\sum_s(rk - \phi_s)(\xi_s'\hat{\tau} - \xi_s'\tau^*)^2 - (v-1)\text{MSE}\}}{rv(k-1)(v-1)}}, \tag{7.1}$$

where ϕ_s $(s = 1, 2, \ldots, v-1)$ are the nonzero eigenvalues of NN' and ξ_s are the corresponding unit mutually orthogonal eigenvectors. When $\lambda_1 \ne 0$, the expressions are complicated and are not reproduced here but the interested reader is referred to Stehouwer (1984). It should be noted that the Jensen and Stone method gives only one weight for every eigenvector contrast $\boldsymbol{\xi}'_s\boldsymbol{\tau}$, while Rao's or Khatri and Shah's estimators use different a's for different contrasts. Thus, Khatri and Shah's a_s in this case is

$$\frac{\mathrm{MSE}}{\mathrm{MSE} + \frac{1}{\phi_s}(rk - \phi_s)\frac{f+2}{f(b-3)}\left\{\mathrm{IE} + \frac{1}{k}\sum_s \phi_s(\boldsymbol{\xi}'_s\hat{\boldsymbol{\tau}} - \boldsymbol{\xi}'_s\boldsymbol{\tau}^*)^2\right\}} \tag{7.2}$$

and Rao's a_s is

$$\frac{\mathrm{MSE}}{\mathrm{MSE} + \frac{1}{\phi_s}(rk - \phi_s)\frac{k\mathrm{AB} - (v-k)\mathrm{MSE}}{v(r-1)}}. \tag{7.3}$$

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Biostatistics Department, University of Michigan, Ann Arbor, Michigan 48109, U.S.A.

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