

# A MULTIVARIATE PROCESS CAPABILITY INDEX OVER A RECTANGULAR SOLID TOLERANCE ZONE

Hanfeng Chen

*Bowling Green State University*

*Abstract:* In this paper, we propose a multivariate process capability index (PCI) over a general tolerance zone which includes ellipsoidal and rectangular solid ones as special cases. Our multivariate PCI appears to be a natural generalization of the PCI  $C_p$  for a univariate process to a multivariate process. Computing aspects of the proposed multivariate PCI are discussed in detail, especially for a bivariate normal process. It is noted that its distributional and inferential aspects are difficult to deal with. Resampling methods and a Monte Carlo procedure are suggested to overcome this difficulty. Some examples with a set of real data are presented to illustrate and examine the proposed multivariate PCI.

*Key words and phrases:* Bivariate normal, ellipsoidal tolerance zone, multivariate process.

## 1. Introduction

Consider a manufacturing process with a  $\nu$ -dimensional characteristic  $X = (X_1, \dots, X_\nu)$ . It is common practice that there are some given engineering specifications on the characteristic. The set of the values of  $X$  within the specifications is called a tolerance zone of the process. For an ellipsoidal tolerance zone, several multivariate process capability indices (PCI's) have recently been proposed and discussed (e.g., Chan, Cheng and Spiring (1991), Pearn, Kotz and Johnson (1992), and Littig, Lam and Pollock (1992)). Not much work has been done, however, for a rectangular solid tolerance zone. This is regrettable, since in spite of the warning of Alt and Smith (1988), one actually encounters a rectangular solid zone very often in practice. The specifications for a product generally consist of a collection of individual specifications for each variable, so that the intersection of these specifications (the tolerance zone formed) would be a rectangular solid (see, e.g., Jackson (1991) for a discussion). Our purpose in writing this paper is to attack this problem.

Original motivation underlying the introduction of PCI's is to monitor the proportion of nonconforming products (Kane (1986), and Pearn et al. (1992)).

An item produced by a process is said to be nonconforming if its characteristic is not within the tolerance zone assigned in advance. The process is considered to be capable if the expected proportion of nonconforming products is small enough, say 0.27%. In the case of a univariate characteristic ( $\nu = 1$ ), a unitless PCI (Sullivan (1984)) is defined as  $C_p = d/3\sigma$ , where  $\sigma$  is the standard deviation of  $X$  and  $2d$  equals the length of the tolerance interval. A variety of PCI's for a univariate characteristic has then been proposed to deal with various manufacturing situations. See Kane (1986), Chan et al. (1988), Cheng and Spiring (1989), and Pearn et al. (1992). For a multivariate process ( $\nu \geq 2$ ), we also try to use a single quantity (multivariate PCI) to adequately describe the quality of the process; whereas Hubele, Shahriari and Cheng (1991) suggest a three-component PCI for a bivariate normal process over a rectangular zone. Motivated by the inspiring papers of Chan et al. (1991) and Pearn et al. (1992), we propose a multivariate PCI (as a single scale quantity) over a general tolerance zone that has already been established in advance by engineers.

## 2. Multivariate PCI's

We proceed to deal with a general type of tolerance zones which includes rectangular solid as a special case. Suppose that a tolerance zone is defined as

$$V = \{x \in R^\nu : h(x - T) \leq r_0\}, \quad (1)$$

where  $h(x)$  is a specific positive function with the same scale as  $x$ ,  $T \in R^\nu$  is a constant vector and  $r_0$  a positive number. Mathematically, the restriction to the same scale on  $h$  means that  $h(x)$  is a positive homogeneous function with degree one:  $h(tx) = th(x)$ , for  $t > 0$  and  $x \in R^\nu$ . A plausible rationale for this is that the function  $h$  is used to specify limits for the  $X$ -process. A more important reason for this requirement is that a multivariate PCI over  $V$  can be well-defined. The constant vector  $T$  may be a  $\nu$ -dimensional target vector of the manufacturing process on the  $\nu$ -dimensional characteristic  $X$ . As for  $r_0$ , though it can be assumed to be equal to 1 without loss of generality, we prefer to retain its general value since one may endow it with the role of *radius* of the tolerance zone.

Let  $\alpha$  be the allowable expected proportion of nonconforming products of a process (conventionally,  $\alpha = 0.27\%$ ). With the tolerance zone  $V$  defined in (1), a process is capable if  $P(X \in V) \geq 1 - \alpha$ , i.e.,  $P(h(X - T) \leq r_0) \geq 1 - \alpha$ . Let  $r = \min\{c : P(h(X - T) \leq c) \geq 1 - \alpha\}$ . It is clear that if the cumulative distribution function of  $h(X - T)$  is increasing in a neighborhood of  $r$ , the most common case, then  $r$  is simply the unique root of the equation  $P(h(X - T) \leq r) = 1 - \alpha$ . The process is capable if and only if  $r \leq r_0$ , i.e.,  $r_0/r \geq 1$ . This

suggests that the quantity  $r_0/r$  describes the quality of the process. We therefore define a multivariate PCI, denoted by  $MC_p$ , as

$$MC_p = r_0/r.$$

It is noted that  $MC_p$  is well-defined, since a tolerance zone like  $V$  is specified by a function  $h$  uniquely up to a scale constant and since  $MC_p$  is scale-invariant.

Our definition of  $MC_p$  is general in the sense that (i) the specifications can be as general as given by (1), (ii) its statistical interpretation does not rely on a particular form, say normality, of the distribution law of  $X$ , and (iii) the arbitrariness of  $\alpha$  provides flexibility in setting a criterion for the capability of a process. Additional comments are as follows:

1. The multivariate PCI  $MC_p$  identifies capability of the process. The hypothesis  $H_0$ : *the process is capable*, is equivalent to  $H_0 : MC_p \geq 1$ .
2. The value 1 of  $MC_p$  indicates that the expected proportion of nonconforming products of the process is exactly allowable, i.e. equal to  $\alpha$ . And the expected proportion is a decreasing function of  $MC_p$ . That is, the larger  $MC_p$ , the less the expected proportion and the more capable the process is. This ensures the original motivation of PCI's. See the discussion by Pearn et al. (1992).
3. Numerically,  $MC_p$  is the ratio of the radius of tolerance zone to that of actual zone needed to achieve the desired expected proportion of nonconforming products. Thus,  $MC_p$  is a natural generalization of the univariate PCI  $C_p$ .

When  $h(x - T) = [(x - T)'A^{-1}(x - T)]^{1/2}$  with  $A$  a  $\nu \times \nu$  positive definite matrix,  $V$  is the ellipsoidal specification given by  $\{x \in R^\nu : (x - T)'A^{-1}(x - T) \leq r_0^2\}$ . For this ellipsoidal tolerance zone and for  $\alpha = 0.27\%$  together with the normality assumption,  $MC_p$  reduces to the multivariate PCI proposed by Pearn et al. (1992). An interesting special case mentioned in their paper is  $X \sim N(T, \sigma^2 A)$  with  $\sigma$  a free parameter. In this case,  $MC_p = r_0/(\sigma \chi_{\nu, \alpha}^2)$ , where  $\chi_{\nu, \alpha}^2$  is the  $100(1 - \alpha)$ th percentile of the  $\chi^2$ -distribution with  $\nu$  degrees of freedom. Let  $X_1, \dots, X_n$  be a sample of size  $n$  from  $X$ . Then  $\sigma^2$  can be estimated by  $\hat{\sigma}^2 = (n - 1)^{-1} \sum (X_i - \bar{X})'A^{-1}(X_i - \bar{X})$  so that  $MC_p$  can be estimated by  $r_0/(\hat{\sigma} \chi_{\nu, \alpha}^2)$ .

Another example of the tolerance zone with structure (1) is the Hölder-type specification:

$$\left\{ x \in R^\nu : \left[ \sum_{i=1}^{\nu} |(x_i - T_i)/r_i|^p \right]^{1/p} \leq r_0 \right\}$$

for any  $p > 0$ ; and any norm in  $R^\nu$  can serve as an  $h$  to define a tolerance zone like (1). Therefore, the general form (1) for defining a tolerance zone is quite comprehensive and can meet various needs in practice.

### 3. $MC_p$ over a Rectangular Solid Zone

A rectangular solid tolerance zone is defined by  $V = \{x \in R^\nu : |x_i - T_i| \leq r_i, i = 1, \dots, \nu\}$ , where  $T_i$  and  $r_i$  are specific constants. Another expression for  $V$  is

$$V = \{x \in R^\nu : \max\{|x_i - T_i|/r_i, i = 1, \dots, \nu\} \leq 1\}.$$

Thus,  $V$  has the structure of (1) with  $h(x) = \max\{|x_i|/r_i, i = 1, \dots, \nu\}$ . Consequently, the multivariate PCI  $MC_p = 1/r$ , where  $r$  is such that  $P(\max\{|X_i - T_i|/r_i, i = 1, \dots, \nu\} \leq r) = 1 - \alpha$ . Let  $F$  be the cumulative distribution function of  $h(X - T)$ . Then  $r = F^{-1}(1 - \alpha)$ , i.e., the  $100(1 - \alpha)$ -th percentile of  $F$ . It follows immediately that for any  $y > 0$ ,

$$F(y) \leq \min\{P(|X_i - T_i| \leq r_i y), i = 1, \dots, \nu\}. \quad (2)$$

So, a necessary condition for a process to be capable over a rectangular solid zone is that each individual univariate process is capable with the corresponding specification limits. This seems to be expected. Indeed, this fact is the starting point of the Kocherlakota and Kocherlakota (1991) bivariate generalizations of  $C_p$ . If  $X \sim N(T, \Sigma)$ , then by Theorem 5.1.2 (or Corollary 7.2.1) of Tong (1990),  $F(y) \geq \prod_{i=1}^\nu P(|X_i - T_i| \leq r_i y)$ . Noting that the rhs is the expected proportion of conforming products of a normally distributed process with independent components, this inequality suggests that correlations between characteristics make the process more capable over a rectangular tolerance zone.

Now we turn to the issue of actually computing  $MC_p$ . Assume that  $X \sim N(\mu, \Sigma)$  with  $\mu$  and  $\Sigma$  known. If estimates  $\hat{\mu}$  and  $\hat{\Sigma}$  are computed from the observations of the process, the methods described below will be applied to the estimates. Besides  $r_i$  and  $T_i$ ,  $MC_p$  also depends on the mean vector  $\mu$  and covariance matrix  $\Sigma$  of  $X$  in a complicated way. In general, computing  $MC_p$  may be complicated and one may require a Monte Carlo method to obtain an estimate. For some special cases, however, a numerical solution to  $F(y) = 1 - \alpha$  can be easily obtained.

#### 3.1. Computing $MC_p$ : A special case

Let all  $r_i$  be identical, say equal to  $r_0$ , and let  $\mu = T$  and  $\Sigma = \sigma^2 I$ , where  $I$  is the  $\nu \times \nu$  unit matrix. It is clear that  $Y = \max\{|X_i - T_i|/r_0, i = 1, \dots, \nu\}$  has the same distribution as  $(\sigma/r_0)|Z|_{(\nu)}$ , where  $|Z|_{(\nu)}$  is the largest order statistic of the absolute values  $|Z_1|, \dots, |Z_\nu|$  of a standard normal random sample,  $Z_1, \dots, Z_\nu$  of size  $\nu$ . Let  $r$  be such that  $P\{(\sigma/r_0)|Z|_{(\nu)} \leq r\} = 1 - \alpha$ , i.e.,  $\Phi(r r_0/\sigma) = [1 + (1 - \alpha)^{1/\nu}]/2$ . Then

$$MC_p = 1/r = r_0/(\sigma z_{\nu, \alpha}), \quad (3)$$

where  $z_{\nu,\alpha}$  is the  $100\alpha_\nu$ -th percentile of  $N(0, 1)$  and  $\alpha_\nu = [1 + (1 - \alpha)^{1/\nu}]/2$ . For  $\alpha = 0.27\%$ ,  $\alpha_2 = 0.9993$  and  $z_{2,\alpha} = 3.1949$ , so  $MC_p = r_0/(3.1949\sigma)$ . For  $\alpha = 5\%$ ,  $z_{2,\alpha} = 2.2357$  and  $MC_p = r_0/(2.2357\sigma)$ . When  $\alpha$  is small,  $\alpha_\nu$  can be approximated by  $1 - \alpha/(2\nu)$ .

**3.2. Computing  $MC_p$ : Bivariate normal process**

Let  $\nu = 2$  and  $\Sigma = (\sigma_{ij})$ , where  $\sigma_{11} = \sigma_1^2, \sigma_{22} = \sigma_2^2$  and  $\sigma_{12} = \rho\sigma_1\sigma_2$ . Then we have (see Tong (1990))

$$F(y) = P\left(\left|\frac{X_1 - T_1}{r_1}\right| \leq y, \left|\frac{X_2 - T_2}{r_2}\right| \leq y\right) = \int_{-\infty}^{\infty} g(u; y)e^{-u^2} du, \tag{4}$$

with  $g(u; y) = \pi^{-1/2} \prod_{i=1}^2 \{\Phi(a_{i1}) - \Phi(a_{i2})\}$ , where

$$a_{i1} = \frac{\text{sgn}^{i-1}(\rho)\sqrt{2|\rho|}u + (r_i y - \delta_i)/\sigma_i}{\sqrt{1 - |\rho|}}, \quad a_{i2} = \frac{\text{sgn}^{i-1}(\rho)\sqrt{2|\rho|}u - (r_i y + \delta_i)/\sigma_i}{\sqrt{1 - |\rho|}},$$

and  $\delta_i = \mu_i - T_i, i = 1, 2$ . Nowadays, programming subroutines and computer softwares are widely available to handle the numerical integrals (e.g., the IMSL Library and the software, *Mathematica*). Therefore, the function  $F(y)$  can be numerically realized by utilizing an appropriate computing facility. Newton-Raphson's method can be used to find a numerical solution to the equation  $F(y) = 1 - \alpha$ . From (3), we suggest using  $x_0 = (\sigma_1 \wedge \sigma_2)z_{2,\alpha}/(r_1 \vee r_2)$  as a starting point in the iteration, where  $s \wedge t = \min(s, t)$  and  $s \vee t = \max(s, t)$ . The derivative of  $F(y)$  is  $F'(y) = \int_{-\infty}^{\infty} \{dg(u; y)/dy\}e^{-u^2} du$ , where

$$\frac{dg}{dy} = g(u; y) \sum_{i=1}^2 \frac{\phi(a_{i1}) - \phi(a_{i2})}{\Phi(a_{i1}) - \Phi(a_{i2})} \frac{r_i}{\sigma_i \sqrt{1 - |\rho|}}.$$

**3.3. Computing  $MC_p$ : Arbitrary  $\nu$  with special  $\Sigma$**

For general  $\nu (\geq 2)$ , assume that the correlation coefficient  $\rho_{ij}$  between  $X_i$  and  $X_j$  has the special structure:  $\rho_{ij} = \lambda_i \lambda_j$  for  $\lambda_i \in [-1, 1]$ , for all  $i \neq j$ . (For some interesting applications of the model, see Curnow and Dunnett (1962).)

Similarly to the case of bivariate normal, we have  $F(y) = \int_{-\infty}^{\infty} g_\nu(u; y)e^{-u^2} du$ , with  $g_\nu(u; y) = \pi^{-1/2} \prod_{i=1}^\nu \{\Phi(b_{i1}) - \Phi(b_{i2})\}$ , where  $b_{i1} = [\lambda_i u + (r_i y - \delta_i)/\sigma_i](1 - \lambda_i^2)^{-1/2}$ ,  $b_{i2} = [\lambda_i u - (r_i y + \delta_i)/\sigma_i]/(1 - \lambda_i^2)^{-1/2}$ , and  $\delta_i = \mu_i - T_i, i = 1, \dots, \nu$ . Since  $F(y)$  involves the integral of only one variable, it is rather convenient to evaluate its values numerically on a computer. The derivative can be obtained similarly to the case of bivariate normal.

#### 4. Inference for Capability Based on $MC_p$

Let  $\widehat{MC}_p$  be the method of moments estimate for  $MC_p$  based on a random sample of size  $n$  from the process  $X$ . Similar to a shortcoming of all existing multivariate PCI's, a drawback of  $MC_p$  is the fundamental difficulty in studying the distributional and inferential properties of  $\widehat{MC}_p$ . Nevertheless, for  $n$  large, resampling methods can be employed to obtain an asymptotic confidence interval for  $MC_p$ . Let  $\sqrt{n}(\widehat{MC}_p - MC_p)$  have a normal limit distribution. When  $n$  is large, one can have an asymptotic confidence interval for  $MC_p$ :  $\widehat{MC}_p \pm z_\gamma \hat{\tau}_n$ , where  $2\gamma$  is the nominal confidence level,  $z_\gamma$  the  $100(1 - \gamma)$ -th percentile of the standard normal, and  $\hat{\tau}_n$  a certain estimate for the standard deviation  $\tau_n$  of  $\widehat{MC}_p$ . Two popular resampling estimates for  $\tau_n$  are bootstrap and jackknife. For a general description and discussion about the two resampling methods, we refer readers to Efron (1982).

When  $\Sigma$  is known, say  $\Sigma_0$ , the  $p$ -value can be determined for testing capability. To test the null-hypothesis  $H_0 : MC_p \geq 1$ , one would reject  $H_0$  if  $\widehat{MC}_p$  is small. It can be seen that the expected proportion of nonconforming products attains its minimum when  $T = \mu$ . Therefore, if the value of  $MC_p$  with  $T = \mu$  is greater or equal to 1, in which case the null hypothesis is true, then the  $p$ -value of the test is  $p = P(\widehat{MC}_p \leq \widehat{mc}_p)$ , where  $\widehat{mc}_p$  is the observed value on  $\widehat{MC}_p$ , and the probability is taken under  $X \sim N(T, \Sigma_0)$ . The  $p$ -value provides meaningful information about location departure of the process from the target.

Table 1. The data of brinell hardness ( $H$ ) and tensile strength ( $S$ ) of output of a bivariate process (Sultan (1986))

$H$	143	200	160	181	148	178	162	215	161	141	175	187	187
$S$	34.2	57.0	47.5	53.4	47.8	51.5	45.9	59.1	48.4	47.3	57.3	58.5	58.2
$H$	186	172	182	177	204	178	196	160	183	179	194	181	
$S$	57.0	49.4	57.2	50.6	55.1	50.9	57.9	45.5	53.9	51.2	57.5	55.6	

#### 5. Examples

All the computations in this section were carried out on double precision by a VAX 11/8650 computer at the Bowling Green State University. The numerical evaluations of the integral (4) are obtained by calling the subroutine DGQRUL from the IMSL Library with the number of quadratures equal to 50.

Chan et al. (1991) use the Sultan (1986) bivariate process data to examine their definition of a multivariate PCI over an ellipsoidal zone. In this process, the brinell hardness ( $H = X_1$ ) and the tensile strength ( $S = X_2$ ) of the output of a process are of interest. We shall also use the same data, but consider rectangular regions. The data are presented in Table 1. (Sultan (1986) discusses it in the

context of control chart.) A rectangular tolerance zone is a consequence of specifying requirements for each individual variable separately. Such a rectangular zone seems to be a convenient and appropriate choice, especially when there is no prior information available about associations between the variables.

From the data, we have  $n = 25$ ,  $\hat{\sigma}_1 = 18.38$ ,  $\hat{\sigma}_2 = 5.80$ ,  $\hat{\rho} = 0.8341$ ,  $\hat{\mu}_1 = 177.2$  and  $\hat{\mu}_2 = 52.32$ . As the first example, we study a rectangular tolerance zone with the target values  $T_1 = 177$  for  $H$ ,  $T_2 = 53$  for  $S$ , and  $r_1 = 3.5\hat{\sigma}_1 = 64.33$ ,  $r_2 = 3.5\hat{\sigma}_2 = 20.30$ , i.e.,

$$V = \{(H, S) : 112.7 \leq H \leq 241.3, 32.70 \leq S \leq 73.30\}. \quad (5)$$

By taking the range of three and a half standard deviations, we anticipate that the process will be capable over the zone [see also the comment following the equation (2)]. Figure 1 presents a portion of  $F(x)$  graph. For  $\alpha = 0.27\%$ ,  $F(0.9063) = 1 - \alpha$ , giving  $\widehat{MC}_p = 1.103 > 1$ . The jackknife estimate for the standard deviation of  $\widehat{MC}_p$  is 0.1454. Figure 2 is a plot of the data with the rectangular specification boundaries. It is visually evident that the process is capable since all the observations are clustered around the target, except for the observation 1 (143, 34.2).

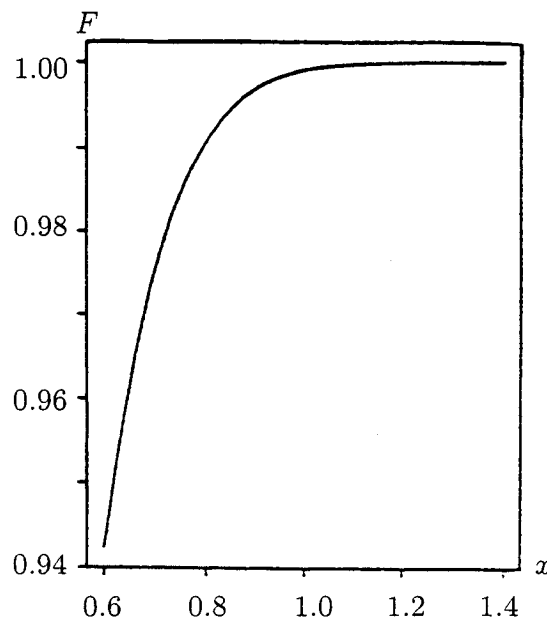


Figure 1. A portion of  $F(x)$  for the rectangular region (5). The solution of  $F(x) = 1 - 0.0027$  is found to be 0.9063, giving  $\widehat{MC}_p = 1.103$ .

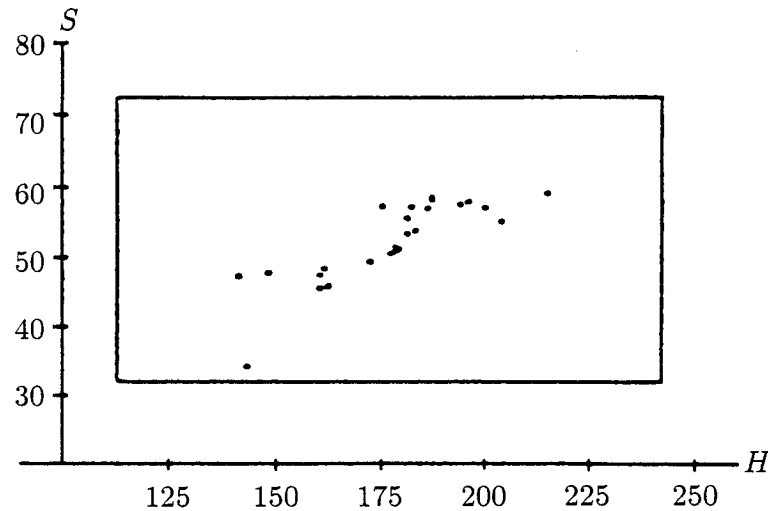


Figure 2. Observations and the rectangular boundaries with  $T_1 = 177$ ,  $T_2 = 53$ , and  $r_1 = 3.5\hat{\sigma}_1 = 64.33$ ,  $r_2 = 3.5\hat{\sigma}_2 = 20.29$ . ( $\widehat{MC}_p = 1.103$ .)

For comparison with the Chan et al. (1991) elliptical specifications, we assume, as they did, that  $\Sigma$  is known to be  $\Sigma_0 = (\sigma_{ij})$  with  $\sigma_{11} = 324$ ,  $\sigma_{12} = 65$  and  $\sigma_{22} = 25$ . The value of  $MC_p$  for  $X \sim N(T, \Sigma_0)$  appears to be 1.173. By running 500 Monte Carlo trials, we found that the value  $\widehat{MC}_p = 1.103$  is between the 183th and the 184th smallest Monte Carlo estimates for  $MC_p$  when the process is  $N(T, \Sigma_0)$ . Thus, with 2 times the maximum standard error of the 500 Monte Carlo trials equal to  $1/\sqrt{500} = 4.5\%$ , the Monte Carlo  $p$ -value is:  $\hat{p} = 183/500 = 37\%$ . This strongly suggests that the process is capable.

By using elliptical specifications, Chan et al. (1991) obtained a more compact tolerance zone over which the process is still capable with a Type I error of 0.05 in the sense of their multivariate PCI. Their elliptical tolerance zone is  $\{(x_1, x_2) : (x - T)' \Sigma_0^{-1} (x - T) \leq 11.829\}$ , where  $T_1 = 177$ ,  $T_2 = 53$ . It is interesting to note that one observation is even excluded from this tolerance zone. On the other hand, one may find it not surprising that our rectangular zone (5) contains all observations. In fact, for a sample size as small as 25; the chance that at least one observation is excluded from a tolerance zone of a capable process (with the expected proportion of nonconforming products 0.27%) is just  $1 - (0.9973)^{25} = 6.5\%$ .

As the second example, we again consider the same process, but with a shift of the target away from  $\hat{\mu}$ . Suppose that the target values are 15% less than those in the first example, i.e.,  $T_1 = 150.45$  and  $T_2 = 45.05$ . Figure 3 presents the plot of the rectangular specifications. As evident from this plot, most of the observations are located in the upper-right quarter of the tolerance region. The plot suggests the process is not capable. Indeed, we find  $\widehat{MC}_p = 0.8101$  with the jackknife estimate of 0.0657 for the standard deviation of  $\widehat{MC}_p$ . This is a clear



indication that the process is not capable. With an asymptotic confidence level of 95%, a confidence interval for  $MC_p$  is  $0.8101 \pm 0.1288$ , the whole interval being far below 1.

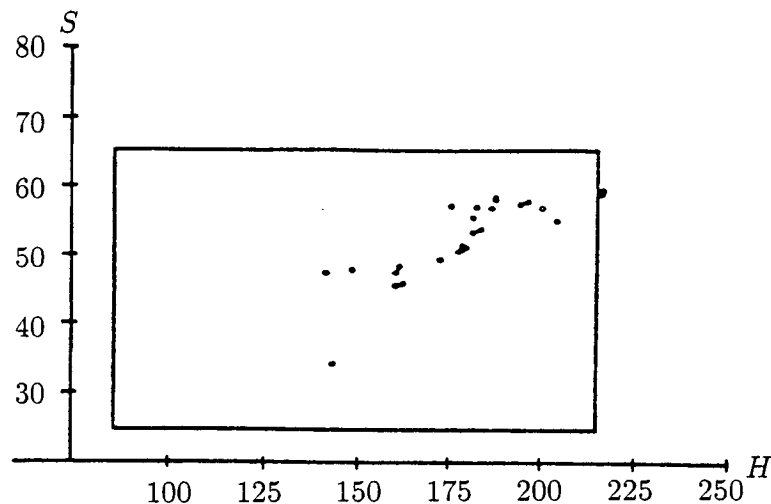


Figure 3. Observations and the rectangular boundaries with  $T_1 = 150.45$ ,  $T_2 = 45.05$ , and  $r_1 = 3.5\hat{\sigma}_1 = 64.33$ ,  $r_2 = 3.5\hat{\sigma}_2 = 20.29$ . ( $\widehat{MC}_p = 0.8101$ .)

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Department of Mathematics Statistics, Bowling Green State University, Bowling Green, OH 43403, U.S.A.

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