BOUNDARY CROSSING DISTRIBUTIONS OF RANDOM WALKS RELATED TO THE LAW OF THE ITERATED LOGARITHM

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To David Siegmund on his 70th birthday.

Abstract: A result for the first passage densities of Brownian motion as \( t \to \infty \) was given in Lerche (1986) for boundaries that grow faster than \( \sqrt{t} \) as \( t \to \infty \). From this result the Kolmogorov–Petrovski–Erdős test near infinity has been derived. Here we extend these results to first passage probabilities of random walks. The asymptotic formulas are the same as for Brownian motion and, especially, no overshoot term shows up.

Key words and phrases: Curved boundary crossing of random walks, overshoot calculations, stopping times and the law of the iterated logarithm.

1. Introduction

In several publications Robbins and Siegmund (1970, 1973) derived boundary crossing probabilities related to the law of the iterated logarithm. One of their methods was to use mixture stopping rules; these stop when a certain mixture of likelihood ratios crosses a certain level. Lai and Siegmund (1977, 1979) successfully applied this method to calculate the operating characteristics of a repeated significance test with a bounded horizon in a large deviation sense. For random walks overshoot terms show up, which the authors could handle with nonlinear renewal theory. A wide range of similar and related results can be found in the monograph of Siegmund (1985).

Motivated by these results the second author wrote his monograph Lerche (1986) using a different approach to curved boundary crossing. This approach is closely related to the Kolmogorov–Petrovski–Erdős test (see Itô and McKean (1974, p.33)) and to Strassen’s result on first exit times near zero of Brownian motion (see Strassen (1967)). Theorem 3.5 there can be roughly stated as follows: Let \( \psi \) denote a smooth boundary (with \( \psi(0) = 0 \)) that is monotone and belongs to the upper class near zero (in the sense of Itô–McKean). Let \( f \) denote the first hitting time density at \( \psi \). Then

\[
f(t) = \frac{\Lambda(t)}{t^{3/2}} \varphi \left( \frac{\psi(t)}{\sqrt{t}} \right) (1 + o(1)) \quad \text{as } t \to 0.
\] (1.1)
Here $\Lambda(t) = \psi(t) - tv'(t)$ and $\varphi(x) = (1/\sqrt{2\pi}) e^{-x^2/2}$. The expression on the right-hand side is the hitting density at time $t$ at the tangent of $\psi$ at $t$ according to the Lévy–Bachelier formula (see Ito and McKean (1974, p.25)). The tangent intercept is $\Lambda(t)$. Strassen derived from this result the difficult half of the proof of the Kolmogorov–Petrovski–Erdős test at zero.

Starting from (1.1), Lerche (1986) described a theory which leads to boundary crossing probabilities quite similar to those derived by the method of mixture stopping rules. Some results are given there (in Chapter I, Section 5 of Lerche, 1986) which imply the Kolmogorov–Petrovski–Erdős test near infinity. In contrast to Strassen’s approach, where one uses last entrance times (by time inversion), they are formulated for first passage times.

The purpose of this note is to show that similar results hold for random walks. One interesting point is that as the number of observations tends to infinity the overshoot term vanishes to the first order. The main part concentrates on the Gaussian random walk, but also some more general results are stated. Additionally, in those cases a central limit theorem effect turns up. This explains why the asymptotic expressions are the same as for the Gaussian random walk. For these more general situations one assumes that the moment generating function exists in a vicinity of zero; this leads to rather strong local and global central limit theorems. For their application see the Remark at the end of Section 4.1.

Finally we want to indicate the relations to other publications. At first we point at the book of Aldous (1989) on Poisson clumping, which discussed the tangent approximation on pg. 99–100. Further, there are more recent papers on boundary crossing that are more or less closely related, like that of Kesten and Maller (1998) on “Random Walks Crossing High Level Curved Boundaries” and that of Chan and Lai (2003) on “Saddlepoint Approximations and Nonlinear Boundary Crossing Probabilities of Markov Random Walks”. Perhaps closest in the sense of integral tests is the paper of Hambly, Kersting, and Kyprianou (2003) on “Law of the iterated logarithm for oscillating random walks conditioned to stay non-negative”. We also mention Doney and Maller (2000). Recent work on the law of iterated logarithm of random walks care more about the range in higher dimensions (see Bass and Kumagai (2002) and Hamana (2006)) or discuss more involved topics like local times of self intersections of planar random walks (see Bass, Chen, and Rosen (2005)).

2. Results

Let $X_i$, $i \geq 1$ denote a sequence of independent random variables that are identically distributed according to $N(0,1)$. Let $S_n = \sum_{i=1}^{n} X_i$, $S_0 = 0$. Let $\psi$ be a nondecreasing continuously differentiable function $\psi : [1, \infty) \to \mathbb{R}$. Let
$T_\psi = \inf\{n \geq 1 \mid S_n \geq \psi(n)\}$ denote the first passage time of the random walk $S_n$ of the boundary $\psi$, with $T_\psi = \infty$ if the infimum is taken over the empty set.

For boundaries $\psi$ that grow faster to infinity than $p_t$ if $t \to \infty$, we study the asymptotic behavior of $P(T_\psi = n)$ as $n \to \infty$. A typical example is $\psi(n) = \sqrt{2n \log \log n}$. Let $\Lambda(n) = \psi(n) - n\psi'(n)$ denote the intercept of the tangent at the curve $\psi$ in point $n$. The following result is the discrete version of Theorem 5.1 in Lerche (1986) on Brownian motion.

**Theorem 1.** Let $\psi$ satisfy the conditions:

(i) $\psi(t)/\sqrt{t} \to \infty$ as $t \to \infty$;

(ii) there exists a constant $\alpha$ with $1/2 < \alpha < 1$, such that $\psi(t)/t^\alpha$ is finally decreasing;

(iii) for all $\varepsilon > 0$ there exists a $\delta > 0$ and a $t_1 \geq 1$ such that $|s/t - 1| < \delta$ implies $|\Lambda(s)/\Lambda(t) - 1| < \varepsilon$ if $s, t \geq t_1$.

Then, as $n \to \infty$

$$P(T_\psi = n) = P(T_\psi \geq n) \frac{\Lambda(n)}{n^{3/2}} \Phi \left( \frac{\psi(n)}{\sqrt{n}} \right) (1 + o(1)).$$

(2.1)

**Remark 1.** We note that for Gaussian random walks the formula on the right-hand side is the same as that for Brownian motion, but evaluated at discrete time points instead of continuous ones. This means that an overshoot term does not show up. In Theorem 1 no restrictions are made concerning ultimate crossing: $P(T_\psi < 1) < 1$ is as possible as $P(T_\psi < 1) = 1$, which means that $P(T_\psi \geq n) \to 0$ as $n \to \infty$.

It is possible to generalize the statement for the case $P(T_\psi < 1) < 1$ to random walks whose increments have finite moment generating functions. This is formulated in Theorem 2. For the case $P(T_\psi < 1) = 1$ one needs an extra condition to have an inequality for the hazard functions. In the situation of Theorem 1, where one has normally distributed increments, a hazard inequality always holds:

$$h_{\psi_1}(n) \leq h_{\psi_2}(n)$$

(2.2)

with $h_i(n) = P(T_{\psi_i} = n)/P(T_{\psi_i} \geq n)$. Here $\psi_i : \{1, \ldots, n\} \to \mathbb{R}_+$ for $i = 1, 2$, with $\psi_1(j) \leq \psi_2(j)$ for $j = 1, \ldots, n - 1$ and $\psi_1(n) = \psi_2(n)$.

In more general situations one needs an extra condition for (2.2) to hold. It is the total positivity of order 2 of the Lebesgue density of the increments of the random walk. We discuss this in more detail in Section 3.

Let $X_i$, $i \leq 1$ be independent identically distributed random variables whose distribution has a density $g_1$ with respect to the Lebesgue measure. Let $S_n = \sum_{i=1}^n X_i$. A condition is crucial and is always assumed in the following:

($\ast$) $E \exp(\theta X_1) < \infty$ for all $\theta$ with $|\theta| < \theta_0$ with $\theta_0 > 0$. 
Theorem 2. Let $E(X_1) = 0$ and $\text{Var}(X_1) = 1$. Let the density $g_1$ satisfy condition ($*$) and let

there exists a $\theta_1$ with $0 < \theta_1 < \theta_0$ such that

$$\sup_{|\theta| \leq \theta_1} \sup_{x \in \mathbb{R}} \exp(\theta x) g_1(x) < \infty.$$  

Suppose the boundary $\psi$ satisfies the conditions:

(i) $\psi(t)/\sqrt{t}$ is monotone increasing for sufficiently large $t$;

(ii) there exists a constant $\alpha$ with $1/2 < \alpha < 2/3$, such that $\psi(t)/t^\alpha$ is finally decreasing;

(iii) for all $\varepsilon > 0$ there exists a $\delta > 0$ and a $t_1 \geq 1$ such that $|s/t - 1| < \delta$ implies $|\Lambda(s)/\Lambda(t) - 1| < \varepsilon$ if $s, t \geq t_1$.

If $P(T_\psi = \infty) > 0$ holds then, as $n \to \infty$,

$$P(T_\psi = n) = P(T_\psi = \infty) \frac{\Lambda(n)}{n^{3/2}} \varphi \left( \frac{\psi(n)}{\sqrt{n}} \right) (1 + o(1)). \quad (2.3)$$

Remark 2. The right-hand sides of (2.1) and (2.3) are the same. This is a kind of central limit effect in the situation of Theorem 2.

Now we formulate a result which gives up the restriction $P(T_\psi = \infty) > 0$. Families of distributions that satisfy the conditions of the next result are, for instance, the double exponential.

Theorem 3. Let the conditions ($*$), ($**$) and (I)–(III) of Theorem 1 hold, and let $g_1$ be totally positive of order 2. Then, as $n \to \infty$,

$$P(T_\psi = n) = P(T_\psi \geq n) \frac{\Lambda(n)}{n^{3/2}} \varphi \left( \frac{\psi(n)}{\sqrt{n}} \right) (1 + o(1)). \quad (2.4)$$

As a consequence of Theorems 1 and 3 one can state an asymptotic result for survival probabilities.

Corollary 1. Let $P(T_\psi < \infty) = 1$. Then

$$P(T_\psi \geq n) = \exp \left[ \left( - \sum_{m=1}^{n-1} \frac{\Lambda(m)}{m^{3/2}} \varphi \left( \frac{\psi(m)}{\sqrt{m}} \right) \right) (1 + o(1)) \right]. \quad (2.5)$$

This can be seen as follows. For

$$h_\psi(m) = \frac{P(T_\psi = m)}{P(T_\psi \geq m)},$$

one has

$$P(T_\psi \geq n) = \prod_{m=1}^{n-1} (1 - h_\psi(m)) = \exp \left( - \sum_{m=1}^{n-1} \log(1 - h_\psi(m)) \right).$$
But \( h_\psi(m) \to 0 \) by (2.3), thus \( -\log(1 - h_\psi(m)) = h_\psi(m)(1 + o(1)) \). By (2.3) statement (2.5) follows.

**Remark 3.** For \( \psi(m) = \sqrt{2m \log \log(m)} \) we obtain

\[
P(T_\psi \geq n) = \exp \left(-\frac{1}{2\sqrt{\pi}}(\log \log n)^{3/2}(1 + o(1))\right).
\]

We state a version of the Kolmogorov–Petrovski–Erdős test. For a version with last entrance times see Lerche (1986, p.87).

**Corollary 2.** Let \( P(T_\psi > n) > 0 \) for all \( n > 0 \) and let the conditions of Theorem 3 hold. Then \( P(T_\psi < \infty) < 1 \) if and only if \( \sum_{n=1}^{\infty} (\psi(n)/n^{3/2}) \varphi(\psi(n)/\sqrt{n}) < \infty \).

**Proof.** By (II) and the monotonicity of \( \psi \), \((1 - \alpha)\psi(n) \leq \Lambda(n) \leq \psi(n)\). Thus \( \sum_{n \geq 1} (\psi(n)/n^{3/2}) \varphi(\psi(n)/\sqrt{n}) < \infty \) is equivalent to \( \sum_{n \geq 1} (\Lambda(n)/n^{3/2}) \varphi(\psi(n)/\sqrt{n}) < \infty \). Let \( P(T_\psi < \infty) = 1 \). Then (2.5) implies

\[
\sum_{n \geq 1} \frac{\Lambda(n)}{n^{3/2}} \varphi(\psi(n)/\sqrt{n}) = \infty.
\]

Conversely if \( P(T_\psi < \infty) < 1 \), then \( P(T_\psi = \infty) > 0 \). Then by (2.3), as \( n' \to \infty \),

\[
P(n' < T_\psi < \infty) = \sum_{n \geq n'} P(T_\psi = n)
= P(T_\psi = \infty)(1 + o(1)) \sum_{n \geq n'} \frac{\Lambda(n)}{n^{3/2}} \varphi(\psi(n)/\sqrt{n})
\]

This implies \( \sum_{n \geq 1} (\Lambda(n)/n^{3/2}) \varphi(\psi(n)/\sqrt{n}) < \infty \).

We will give a complete proof of Theorem 1. It follows the scheme of the proof of (1.1), which is a statement for \( t \to 0 \). Strassen’s construction of the time sections also works here for \( n \to \infty \). It is combined with results of Woodroofe and with Donsker’s invariance principle.

The hazard inequality is discussed in Section 3. The proofs of Theorems 2 and 3 are rather lengthy and can be found in Kerkhoff (1990). See also the Remark at the end of Section 4.1.

### 3. The Hazard Inequality

We prove that the hazard inequality (2.2) holds when the density \( g_1 \) is totally positive of order 2 (its definition follows). This latter assumption enables one to prove monotonicity statements from which the hazard inequality follows. Let \( \psi_i : \{1, \ldots, n\} \to \mathbb{R}_+ \) for \( i = 1, 2 \), with \( \psi_1(j) \leq \psi_2(j) \). Then

\[
P(S_n > z \mid T_{\psi_1} > n) \leq P(S_n > z \mid T_{\psi_2} > n).
\]

(3.1)
We show that (3.1) implies the hazard inequality (2.1). First

\[ P(T_{\psi_1} = n \mid T_{\psi_1} \geq n) \leq P(T_{\psi_2} = n \mid T_{\psi_2} \geq n). \]

Since \( \psi_1(n) = \psi_2(n) \) we have

\[ P(T_{\psi_1} = n \mid T_{\psi_1} \geq n) = \int P(X_n + x \geq \psi_1(n)) P(S_{n-1} \in dx \mid T_{\psi_1} > n - 1) \]

\[ = \int P(X_n + x \geq \psi_2(n)) P(S_{n-1} \in dx \mid T_{\psi_1} > n - 1). \]

The integrand is nondecreasing in \( x \), thus by (3.1) we get

\[ P(T_{\psi_1} = n \mid T_{\psi_1} \geq n) = \int P(X_n + x \geq \psi_2(n)) P(S_{n-1} \in dx \mid T_{\psi_2} > n - 1) \]

\[ = P(T_{\psi_2} = n \mid T_{\psi_2} \geq n). \]

For (3.1) we appeal the notion of total positivity of order 2; for the concept see Karlin (1968).

**Definition 1.** A measurable function \( h : \mathbb{R} \to \mathbb{R}_+ \) that is different from zero for at least two points has property TP2 if

\[ \begin{vmatrix} h(x_2 - x_1) & h(y_2 - y_1) \\ h(x_2 - y_1) & h(y_2 - y_1) \end{vmatrix} \geq 0 \]

for all \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\) with \( x_1 \leq y_1 \) and \( x_2 \leq y_2 \).

We turn to (3.1).

**Lemma 1.** Let \( \psi_i : \{1, \ldots, n\} \to \mathbb{R}_+ \) with \( \psi_1 \leq \psi_2 \). Let \( X_1, \ldots, X_n \) be independent and identically distributed with density \( g_1 \) with respect to Lebesgue-measure. Let \( g_1 \) have property TP2. Then

\[ P(S_n > z \mid T_{\psi_1} > n) \leq P(S_n > z \mid T_{\psi_2} > n). \]  

**Proof.** For simplicity we write \( g \) instead of \( g_1 \). We show, that we can apply a FKG-type inequality (see e.g., Karlin and Rinott (1980)): Let \( h_i, i = 1, 2, \) denote densities on \( \mathbb{R}^n \) with respect to the \( n \)-dimensional Lebesgue-measure \( \lambda_n \). Then

\[ \int f h_1 d\lambda_n \leq \int f h_2 d\lambda_n \]

for any nondecreasing function \( f \), if

\[ h_1(x) h_2(y) \leq h_1(x \lor y) h_2(x \land y) \]

holds for all \( x, y \in \mathbb{R}^n \). To apply the inequality we put

\[ \tilde{g}(s_1, \ldots, s_n) = \prod_{i=1}^{n} g(s_i - s_{i-1}), \]
with \( s_0 = 0 \). This is the density of the joint distribution of \((S_1, \ldots, S_n)\) with respect to \( \lambda_n \). We also put, for \( i = 1, 2 \),

\[
h_i := \frac{\tilde{g} I(T_{\psi_i} > n)}{P(T_{\psi_i} > n)}
\]
and let \( f(s_1, \ldots, s_n) = I_{(x, \infty)}(s_n) \). Then \((3.2)\) follows.

It is left to show \((3.3)\) for the choice of \( h_i \). It is clear that

\[
\prod_{i=1}^n I_{(-\infty, \psi_1(i))}(x_i) I_{(-\infty, \psi_2(i))}(y_i) \leq \prod_{i=1}^n I_{(-\infty, \psi_1(i))}(x_i \wedge y_i) I_{(-\infty, \psi_2(i))}(x_i \vee y_i). \tag{3.5}
\]

By the TP2 property of \( g \) one obtains for \( i = 2, \ldots, n \),

\[
g(x_i \vee y_i - x_{i-1} \vee y_{i-1}) \geq g(x_i \vee y_i - x_{i-1} \wedge y_{i-1}) g(x_i \wedge y_i - x_{i-1} \vee y_{i-1}) = g(y_i - y_{i-1}) g(x_i - x_{i-1}). \tag{3.6}
\]

The last equality follows by distinction of cases. Using \((3.4)\) and \((3.6)\) yields

\[
\tilde{g}(x)\tilde{g}(y) \leq \tilde{g}(x \vee y)\tilde{g}(x \wedge y).
\]

This inequality combined with that of \((3.3)\) yields \((3.3)\) for our choice of \( h_i \).

**Lemma 2.** Let \( \psi_i : \mathbb{N} \to \mathbb{R}_+ \) for \( i = 1, 2 \) with \( \psi_1(j) \leq \psi_2(j) \) for \( j = 1, \ldots, n-1 \), and \( \psi_1(n) = \psi_2(n) \). If \( g_1 \) has the property TP2, then

\[
P(T_{\psi_1} = n \mid T_{\psi_1} \geq n) \leq P(T_{\psi_2} = n \mid T_{\psi_2} \geq n).
\]

We remark that normal densities have the property TP2. This follows, for instance, by a result of Schönberg [1951]. We state it without proof and point for further information to Karlin’s monograph (1968).

**Theorem 4.** A function \( h \) has property TP2 if and only if it can be written as \( h = \exp(-T) \), where \( T \) is a convex function.

4. **Proofs**

4.1. **The upper estimate**

We prove Theorem 1. At first we show the upper estimate

\[
P(T_{\psi} = n) \leq P(T_{\psi} \geq n) \frac{\Lambda(n)}{n^{3/2}} \varphi \left( \frac{\psi(n)}{\sqrt{n}} \right) (1 + o(1)) \quad \text{as } n \to \infty. \tag{4.1}
\]
Let \( \varepsilon > 0 \) with \( \alpha + \varepsilon < 1 \). Let \( s(n) := n \left( 1 - n^\varepsilon / (\psi(n))^{2\varepsilon} \right) \) and \( k(n) := \lceil s(n) \rceil \) be the largest natural number less than \( s(n) \). We apply Lemma 2 to \( \psi \) and \( \psi_k \), where \( \psi_k : \mathbb{N} \to \mathbb{R}_+ \) with \( \psi_k(i) \equiv \psi(k) \) for \( i \leq k \) and \( \psi_k(i) = \psi(i) \) if \( i > k \).

We obtain

\[
P(T_\psi = n \mid T_\psi \geq n) \\
\leq P(T_{\psi_k} = n \mid T_{\psi_k} \geq n) \\
\leq P(T_{\psi_k} \geq n)^{-1} P(S_i < \psi_k(i) \text{ for } i = 1, \ldots, n-1, S_n \geq \psi_k(n)). \tag{4.2}
\]

We show that \( P(T_{\psi_k} \geq n) \to 1 \). Since \( \psi \) is nondecreasing, we get further

\[
P(T_{\psi_k} \geq n) \geq P(S_i < \psi(k) \text{ for } i = 1, \ldots, n-1) \\
= P\left( \frac{1}{\sqrt{n-1}} \max_{1 \leq i \leq n-1} S_i < \frac{\psi(k)\sqrt{k}}{\sqrt{k(n-1)}} \right). \tag{4.3}
\]

By Donsker’s invariance principle the right-hand side converges to the corresponding distribution of the maximum of Brownian motion, given by \( 2(\Phi(x) - 1)^+ \), where \( y^+ = \max(y, 0) \). But \( x \) here turns out to be infinity, since by the definition of \( k(n) \), we have \( k(n)/n \to 1 \) and by (I) it holds \( \psi(n)/\sqrt{n} \to \infty \). Thus the right-hand side of (4.3) tends to one. Then the inequality (4.2) can be written further as

\[
P(T_\psi = n) \leq P(T_\psi \geq n) P(S_i \leq \psi(i) \text{ for } i = k, \ldots, n-1, S_n \geq \psi(n))(1 + o(1)).
\]

We show that

\[
P(S_i < \psi(i), i = k, \ldots, n-1, S_n \geq \psi(n)) \leq \Lambda(n) \frac{\Lambda(n)}{n^{3/2}} \varphi\left( \frac{\psi(n)}{\sqrt{n}} \right)(1 + o(1)).
\]

For simplicity write

\[
g_n(x) := \frac{1}{\sqrt{n}} \varphi\left( \frac{x}{\sqrt{n}} \right).
\]

Let \( \Lambda_1 = \sup \{ \Lambda(u) \mid u \in [k, n] \} \). Let \( h \) denote the straight line determined by \( h(0) = \Lambda_1 \) and \( h(n) = \psi(n) \). Then

\[
P(S_i < \psi(i) \text{ for } i = k, \ldots, n-1, S_n \geq \psi(n)) \\
\leq P(S_i < h(i) \text{ for } i = k, \ldots, n-1, S_n \geq \psi(n)) \\
= g_n(\psi(n)) \int_0^\infty P(S_i < h(i) \mid S_n = h(n) + r) \frac{g_n(h(n) + r)}{g_n(h(n))} dr \\
= g_n(\psi(n)) \int_0^\infty I_1(r)I_2(r) dr. \tag{4.4}
\]
Now
\[ I_1 = P(S_i < h(i) \text{ for } i = k, \ldots, n - 1 \mid S_n = h(n) + r) \]
\[ = P(S_n - S_i > h(n) - h(i) + r \text{ for } i = k, \ldots, n - 1 \mid S_n = h(n) + r) \]
\[ = P(S_j > h(n) - h(n - j) + r \text{ for } j = 1, \ldots, n - k \mid S_n = h(n) + r) \]
\[ = P \left( S_i > \left( \frac{\psi(n)}{n} - \frac{\Lambda_1}{n} \right) i + r \text{ for } i = 1, \ldots, n - k \mid S_n = \psi(n) + r \right). \]

We use the fact that this conditional distribution does not depend on the drift of the underlying distribution, by sufficiency of \( S_n \). Let
\[ g_n^\theta(x) := \frac{1}{\sqrt{n}} \varphi \left( \frac{x - n\theta}{\sqrt{n}} \right). \] (4.5)

We choose as drift \( \theta_n = \psi(n)/n \), and make a change of measure from that without drift to that with drift \( \theta_n \), and indicate this by a subscript of the measure. Then
\[ I_1 = P \left( S_i > \left( \frac{\psi(n)}{n} - \frac{\Lambda_1}{n} \right) i + r \text{ for } i = 1, \ldots, n - k \mid S_n = \psi(n) + r \right) \]
\[ = P_{\theta_n} \left( S_i > \left( \frac{\psi(n)}{n} - \frac{\Lambda_1}{n} \right) i + r \text{ for } i = 1, \ldots, n - k \mid S_n = \psi(n) + r \right) \]
\[ = \int dP_{\theta_n}^{X_1, \ldots, X_{n-k}}(s_1, \ldots, s_{n-k}) I_{\{s_i > (\frac{\psi(n)}{n} - \frac{\Lambda_1}{n})i + r, i = 1, \ldots, n-k\}} \]
\[ \frac{g_n^\theta_s(\psi(n) + r - s_{n-k})}{g_n^\theta_s(\psi(n) + r)}. \]

We express \( I_2 \) also with the drift \( \theta_n \). We have from (1.3)
\[ \frac{g_n^\theta_s(x)}{g_n^\theta_s(x)} = \frac{\varphi \left( (x - n\theta_1)/\sqrt{n} \right)}{\varphi \left( (x - n\theta_2)/\sqrt{n} \right)} = \exp \left( (\theta_1 - \theta_2)x - \frac{n}{2}(\theta_1^2 - \theta_2^2) \right). \]

Thus
\[ I_2 = \frac{g_n(\psi(n) + r)}{g_n(\psi(n))} = \exp(-\theta_nr) \frac{g_n^\theta_s(\psi(n) + r)}{g_n^\theta_s(\psi(n))}. \]

Combining these expressions we get for the integral on the right-hand side of (1.4),
\[ \int_0^\infty I_1(r)I_2(r)dr \leq (1 + o(1)) \int_0^r dr \int dP_{\theta_n}^{X_1, \ldots, X_{n-k}} I_{\{s_i > (\psi(n)/n - \Lambda_1/n)i + r, i = 1, \ldots, n-k\}} \]
\[ \cdot \exp \left( - \frac{(r - s_{n-k} + [\psi(n)/n(n-k)]^2}{2k} \right) \]
\[ \leq (1 + o(1)) \int_0^r dr P_{\Lambda_1/n}(S_i > r, i = 1, \ldots, n-k) \]
\[ = (1 + o(1)) E_{\Lambda_1/n}(M^+_{n-k}). \]
Here the expectation is taken with respect to the drift $\Lambda_1/n$ and $M_n^+ := (\min_{1 \leq i \leq m} S_i)^+$ denotes the positive part of the minimum. By Lemma 3 we have, with $S_{n-k}^- := -\min(S_{n-k}, 0)$, that

$$E_{\Lambda_1/n}(M_{n-k}^+ + n - k) = \Lambda_1/n + 1/n - k E_{\Lambda_1/n}(S_{n-k}^-).$$

Here we can show that

$$\frac{1}{n - k} E_{\Lambda_1/n}(S_{n-k}^-) = o\left(\frac{\Lambda_1}{n}\right)$$

holds, so

$$\int I_1(r)I_2(r)dr \leq \frac{\Lambda_1}{n}(1 + o(1)).$$

By (I) and (III), $\Lambda_1$ can be substituted by $\Lambda(n)$ asymptotically, from which (4.1) follows.

By using Fubini’s theorem one obtains (4.6) as follows:

$$\frac{1}{n - k} E_{\Lambda_1/n}(S_{n-k}^-) = \frac{1}{n - k} \int_0^\infty dx P_{\Lambda_1/n}(-S_{n-k} \geq x)$$

$$= \frac{1}{n - k} \int_0^\infty dx \int_{-\infty}^\infty du I_{\{x \leq u\sqrt{n-k-(n-k)(\Lambda_1/n)}\}} \varphi(u)$$

$$= \frac{1}{n - k} \int_{\sqrt{n-k}(\Lambda_1/n)}^\infty du \varphi(u) \left(u\sqrt{n-k-(n-k)(\Lambda_1/n)} - \left(1 - \Phi\left(\frac{\sqrt{n-k} \Lambda_1}{n}\right)\right)\right)$$

$$= \frac{\Lambda_1}{n} \left[\frac{\varphi\left(\sqrt{n-k}\Lambda_1/n\right)}{\sqrt{n-k}(\Lambda_1/n)} - \left(1 - \Phi\left(\frac{\sqrt{n-k} \Lambda_1}{n}\right)\right)\right]$$

$$= o\left(\frac{\Lambda_1}{n}\right) \text{ for } n \to \infty.$$

This follows since $\Lambda_1 \geq \Lambda(n)$ and $\Lambda(n) \geq \psi(n)(1 - \alpha)$ by (II) and, since $(\psi(n)/n)\sqrt{n-k} \to \infty$ by (I), the definition of $k(n)$.

**Remark 4.** We want to indicate how one derives the upper bound of Theorem 2 and how the central limit theorem comes in there. By (*) one can define a related exponential family of measures with drift. This allows one to apply sufficiency arguments as in the proof of Theorem 1, (4.3) and the following.

Let $s, k, r, l$ be as in the proof of Theorem 1. Then

$$P(T_\psi = n) \leq \int_{-\infty}^{\psi(l)} P(T_\psi > l, S_l \in dx) P(l, n-1, S_n \geq \psi(n)).$$
One can show similarly to the proof of Theorem 1, that

\[ P^{l,x}(S_i < \psi(i), i = k, \ldots, n - 1, S_n \geq \psi(n)) \leq g_{n-l}(\psi(n) - x) \int_0^\infty dr I_1 \cdot I_2 \]

\[ \leq g_{n-l}(\psi(n) - x) \left( \frac{\Lambda(n)}{n} + R \right), \]

with \( g_k \) the \( k \)-fold density of \( g_1 \) and \( R \) is the remainder.

At this point one applies the central limit theorem twofold, to estimate \( R \) with a global version and \( g_{n-l}(\psi(n) - x) \) with a local one.

### 4.2. The lower estimate

We want to show now, as \( n \to \infty \),

\[ P(T_\psi = n) \geq P(T_\psi \geq n) \frac{\Lambda(n)}{n^{3/2}} \varphi \left( \frac{\psi(n)}{\sqrt{n}} \right) (1 + o(1)). \] (4.7)

We partition the event \( \{T_\psi = n\} \) into three time sections. The middle section is to have low probability. The first and third sections contribute to the two terms on the right-hand side of formula (4.7).

We now define the two splitting points \( k \) and \( l \) of the sections, essentially following Strassen (1967). Let \( \varepsilon > 0 \) be chosen such that \( \alpha + \varepsilon < 1 \). Let \( s = n \left( 1 - n^\varepsilon/\psi(n)^2 \right) \) and \( k = \lfloor s \rfloor \), the largest integer smaller than \( s \). Let \( \beta \) and \( \gamma \) be chosen such that \( \alpha + \varepsilon < 2\beta - 1 < \gamma < \beta < 1 \). Let \( r \) denote the solution of the implicit equation \( \frac{\psi(k)^2}{k} \frac{l}{r^\beta} = 1 \). Let \( l = \lfloor r \rfloor \) and let \( k_l = \psi(k) \left( \frac{l}{k} \right)^\beta \).

Since \( k \sim n \) and, by (I) \( l/k \to 0 \) as \( n \to \infty \). By the definition of \( \beta \) and \( \gamma \) and (I),

\[ \frac{\psi(k)^2}{k} \left( \frac{l}{k} \right) \beta \to 0, \] (4.8)

\[ \frac{\psi(k)^2}{k} \left( \frac{l}{k} \right)^{2\beta - 1} \to \infty. \] (4.9)

Let \( P^{m,z}\{T_\psi = n\} \) denote the first passage probability of the curve \( \psi \) by the random walk \( \{S_i; i \in \mathbb{N}\} \) that starts at time \( m \) in point \( z < \psi(m) \). Then one has

\[ P(T_\psi = n) \geq \int_{-k_l}^{k_l} P(T_\psi > l, S_l \in dx) P^{l,x}(T_\psi = n) \]

\[ \geq P(T_\psi > l, |S_l| \leq k_l) \inf_{|x| \leq k_l} P^{l,x}(T_\psi = n). \] (4.10)

Now if

\[ P(T_\psi > l, |S_l| \leq k_l) \sim P(T_\psi > l), \] (4.11)
one has

\[ P(T_\psi > l, |S| \leq k_l) = (1 + o(1))P(T_\psi > l) \geq (1 + o(1))P(T_\psi > n). \]

To show (4.11), we prove

\[ \lim_{n \to \infty} P(S_l \leq k_l \mid T_\psi > l) = 1. \] (4.12)

To derive this, we apply Lemma 1 to \( \psi \) and \( \psi_l \), defined as \( \psi_l : \mathbb{N} \to \mathbb{R}^+ \) with \( \psi_l(i) = \psi(l) \) for \( i = 1, \ldots, l \), and \( \psi_l(i) = \psi(i) \) for \( i > l \). By the monotonicity of \( \psi, \psi \leq \psi_l \) on \( \{1, \ldots, l\} \) we get from Lemma 1,

\[ P(S_l \leq k_l \mid T_\psi > l) \geq P(S_l \leq k_l \mid T_{\psi_l} > l) \geq \frac{P(\max_{1 \leq i \leq l} S_i \leq k_l)}{P(\max_{1 \leq i \leq l} S_i < \psi(l))}. \] (4.13)

Now, if \( n \to \infty \), then \( l \to \infty \) and \( \psi(l)/\sqrt{l} \to \infty \) by assumption I). Further

\[ \frac{k_l}{\sqrt{l}} = \psi(k) \left( \frac{l}{k} \right)^\beta = \psi(k) \left( \frac{l}{k} \right)^{\beta - 1/2} \to \infty \]

by (4.14).

From Donsker’s invariance principle it follows, that the right-hand side of (4.14) tends to 1 as \( n \to \infty \), which implies (4.12).

We show now

\[ \lim_{n \to \infty} P(S_l < -k_l \mid T_\psi > l) = 0. \] (4.14)

Let \( \psi_1 : \mathbb{N} \to \mathbb{R}^+ \) with \( \psi_1(i) = \psi(1) \) for \( i = 1, \ldots, l \) and \( \psi_1(i) = \psi(i) \) for \( i > l \). Then by the monotonicity of \( \psi, \psi \geq \psi_1 \) holds. By Lemma 1 one obtains

\[ P(S_l \leq -k_l \mid T_\psi > l) \leq P(S_l \leq -k_l \mid T_{\psi_1} > l) \leq P(S_l \leq -k_l \mid T_{\bar{\varepsilon}} > l). \]

Here \( \bar{\varepsilon} \) denotes the constant function with value \( \varepsilon \) at \( \mathbb{N} \), where \( 0 < \varepsilon \leq f(1) \).

Since the inequality holds for every such \( \varepsilon \), one obtains by symmetry for the random walk

\[
\begin{align*}
P(S_l \leq -k_l \mid T_\psi > l) &\leq P(S_l \leq -k_l \mid S_i \leq 0, \text{ for } i = 1, \ldots, l) \\
&= \frac{P(S_l \geq k_l, S_i \geq 0 \text{ for } i = 1, \ldots, l)}{P(S_i \leq 0, \text{ for } i = 1, \ldots, l)} \\
&\leq \frac{1}{k_l} \frac{E(S_l; S_i \geq 0, \text{ for } i = 1, \ldots, l)}{P(S_i \leq 0, \text{ for } i = 1, \ldots, l)} \\
&= \left( \frac{k_l}{\sqrt{l}} P(S_i \leq 0, \text{ for } i = 1, \ldots, l) \right)^{-1} \cdot E(S_l; S_i \geq 0, \text{ for } i = 1, \ldots, l).
\end{align*}
\]
But \( k_l / \sqrt{l} \to \infty \) and there exists a positive constant \( K \) with \( \sqrt{l} P(S_i \leq 0, \text{ for } i = 1, \ldots, l) \to K \). See [Feller (1971, p.415)]. By Lemma 4 the expectation term remains bounded. Thus one obtains (4.14). But (4.12) together with (4.13) yields (4.11).

To estimate \( \inf_{|x| \leq k_l} P^{l,x}(T_\psi > n) \) at (4.11), we linearize the boundary \( \psi \) as follows: Let \( \lambda_1 = \inf\{ \Lambda(u) \mid u \in [k, n] \} \) and \( \nu \in (0, 1] \), such that \( h_1 \) is a straight line with \( h_1(0) = \nu \lambda_1 \) and \( h_1(n) = f(n) \). Let \( h_2 \) denote the straight line with \( h_2(k) = \psi(k) \) and \( h_2(l) = \psi(l)(l/k)^\alpha \). Then

\[
\begin{align*}
P^{l,x}(T_\psi = n) & \geq \int_{-\infty}^{h_1(k)} P^{l,x}(T_\psi > k, S_k \in dy) P^{k,y}(T_{h_1} = n) \\
& \geq \int_{-\infty}^{h_1(k)} P^{l,x}(S_k \in dy) \\
& \quad \cdot \left[ 1 - P^{l,x}(S_i \geq h_2(i) \text{ for some } i = l+1, \ldots, k-1 \mid S_k = y) \right] P^{k,y}(T_{h_1} = n).
\end{align*}
\]

(4.15)

It now holds that

\[
P^{l,x}(S_i \geq h_2(i) \text{ for some } i = l+1, \ldots, k-1 \mid S_k = y) = o(1)
\]

(4.16)

uniformly in \( |x| \leq k_l \) and \( y \leq h_1(k) \) for \( n \to \infty \). To see this, let \( W \) denote Brownian motion. Then

\[
P^{l,x}(S_i \geq h_2(i) \text{ for some } i = l+1, \ldots, k-1 \mid S_k = y)
\]

\[
\leq P^{l,x}(W_t \geq h_2(t) \text{ for some } t \in (l, k) \mid W_k = y)
\]

\[
= \exp \left( \frac{-2(h_2(l) - x)(h_2(k) - y)}{k - l} \right)
\]

\[
\leq \exp \left( \frac{-2(h_2(l) - k_l)(h_2(k) - h_1(k))}{k - l} \right).
\]

(4.17)

Since \( l, k, k_l \) as well as \( h_1 \) and \( h_2 \) are slightly modified versions of \( r, s, \) and \( k \) in the proof of Theorem 3.5 in [Strassen (1967)], one obtains with similar estimates and an appropriate choice of \( \nu \) (tending to 1),

\[
\frac{(h_2(l) - k_l)(h_2(k) - h_1(k))}{k - l} \to \infty.
\]

This together with (4.14) implies (4.10).

It is therefore left to estimate the part regarding \( h_1 \) in (4.13). We show

\[
P^{l,x}(S_i < h_1(i) \text{ for } i = k, \ldots, n-1, S_n \geq h_1(n)) \geq \tilde{\theta}_n g_{n-i}(\psi(n) - x)(1 + o(1))
\]

(4.18)
with $\tilde{\theta}_n = \nu(\lambda_1/n) - \psi(n)/n + (\psi(n) - x)/(n - l)$ uniformly in $|x| \leq k_l$. By the definition of $k_l$ and (II),

$$\tilde{\theta}_n \geq \nu \frac{\lambda_1}{n} - \frac{k_l}{n-l} = \nu \frac{\lambda_1}{n} (1 + o(1))$$

uniformly for $|x| \leq k_l$. Since $l = o(k)$ and $(\psi(k)^2/k)(l/k)^\beta \to 0$ as $n \to \infty$, it follows that

$$g_{n-l}(\psi(n) - x) \geq g_{n-l}(\psi(n) + k_l) \geq \frac{1}{\sqrt{n}} \varphi \left( \frac{\psi(n)}{\sqrt{n}} \right) (1 + o(1))$$

uniformly for $|x| \leq k_l$. Since $\nu \to 1$ as $n \to \infty$, by (I) and (III) $\lambda_1$ can be substituted by $\Lambda(n)$. This and (11) together with (11) yield, for (11),

$$P^{l,x}(T_\psi = n) \geq \frac{\Lambda(n)}{n^{3/2}} \varphi \left( \frac{\psi(n)}{\sqrt{n}} \right) (1 + o(1))$$

uniformly for $|x| \leq k_l$. This together with (11) and (11) yields (11).

It remains to show (11). The argument for that is similar to that for the upper estimate of (11). We have

$$P^{l,x}(S_i < h_1(i) \text{ for } i = k, \ldots, n-1, S_n > h_1(n)) = g_{n-l}(\psi(n) - x) \int_0^\infty I_1 I_2 dr$$

with

$$I_1 = P(S_i > h_1(n) - h_1(n - i) + r \text{ for } i = 1, \ldots, n - k \mid x + S_{n-l} = h_1(n) + r) = P \left( S_i > \left( \frac{\psi(n)}{n} - \nu \frac{\lambda_1}{n} \right) i + r, i = 1, \ldots, n - k \mid S_{n-l} = \psi(n) + r - x \right),$$

$$I_2 = \frac{g_{n-l}(\psi(n) + r - x)}{g_{n-l}(\psi(n) - x)} = \exp(-\theta_n r) \frac{g_{n-l}(\psi(n) + r - x)}{g_{n-l}(\psi(n) - x)}.$$

Now choose the drift $\theta_n = (\psi(n) - x)/(n - l)$. Evaluating $I_1$, noting $n - l \sim k - l$ and rewriting $I_2$ with drift $\theta_n$, yields

$$I_1 I_2 = (1 + o(1)) \exp(-\theta_n r) \int dP^{X_1,\ldots,X_{n-k}}_{\theta_n} I_{\{s_i > (\psi(n)/n - \nu(\lambda_1/n)i + r, i = 1, \ldots, n - k\}}$$

$$\cdot \sqrt{2\pi} \varphi \left( \frac{\psi(n) + r - x - s_{n-k} - (k - l)\theta_n}{(k - l)^{1/2}} \right).$$
With $\gamma_n = \frac{\sqrt{n-k}}{\sqrt{k-l}}$,

$$\tilde{S}_{n-k} = \frac{S_{n-k} - (n-k)\tilde{\theta}_n}{\sqrt{n-k}},$$

$$r_n = \sqrt{n-k},$$

$$\tilde{\theta}_n = \nu \frac{\lambda_1}{n} - \frac{\psi(n)}{n} + \frac{\psi(n) - x}{n-l},$$

$$\int_0^\infty dr I_1I_2 \geq (1 + o(1)) \int_0^{r_n} dr \exp \{-\theta_n r\}$$

$$\cdot \int dP_{\tilde{\theta}_n}^{X_1,\ldots,X_{n-k}} I_{\{s_i > r, i = 1, \ldots, n-k\}} \left(2\pi\right)^{1/2} \varphi \left(\gamma_n \left(-\tilde{S}_{n-k} + \frac{r}{(n-k)^{1/2}}\right)\right)$$

$$\geq (1 + o(1)) \int_0^{r_n} dr \exp \{-\theta_n r\}$$

$$\cdot \int dP_{\tilde{\theta}_n}^{X_1,\ldots,X_{n-k}} I_{\{s_i > r, i = 1, \ldots, n-k\}} \left(2\pi\right)^{1/2} \varphi (\gamma_n (|\tilde{S}_{n-k}| + 1)).$$

By the definition of $l$ and $k$ and assumption (I), $\gamma_n \to 0$ as $n \to \infty$. Let $(\varepsilon_n; n \geq 1)$ be a sequence with $\varepsilon_n > 0$, $\varepsilon_n \to 0$, and $\varepsilon_n/\gamma_n \to \infty$ as $n \to \infty$. Then

$$\int_0^\infty dr I_1I_2 \geq (1 + o(1)) \left[ \int_0^{r_n} dr \exp \{-\theta_n r\} P_{\tilde{\theta}_n} (S_i > r \text{ for } i = 1, \ldots, n-k) + W_n \right]$$

with

$$|W_n| \leq \int_0^{r_n} dr \int dP_{\tilde{\theta}_n}^{X_1,\ldots,X_{n-k}} (1 - (2\pi)^{1/2} \varphi (\gamma_n (|\tilde{S}_{n-k}| + 1))) I_{\{s_i > r, i = 1, \ldots, n-k\}}.$$

Now we study $W_n$. One has

$$|W_n| \leq K_1 \sup_{|u| \leq \varepsilon_n} |(2\pi)^{1/2} \varphi (u) - 1| \int_0^\infty dr P_{\tilde{\theta}_n} I_{\{s_i > r, i = 1, \ldots, n-k\}}$$

$$+ K_2 \int_0^\infty dr \int dP_{\tilde{\theta}_n}^{X_1,\ldots,X_{n-k}} I_{\{\gamma_n (|\tilde{S}_{n-k}| + 1) > \varepsilon_n\}} I_{\{s_i > r, i = 1, \ldots, n-k\}} \leq K (R_1 + R_2).$$

Since $\varepsilon_n \to 0$, by Lemma 3 and (1.10) with $\Lambda_1/n$ replaced by $\tilde{\theta}_n$, we obtain $R_1 = o(\tilde{\theta}_n)$. Note that $\tilde{\theta}_n \sqrt{n-k} \to \infty$ since $\tilde{\theta}_n \geq \nu (\lambda_1/n) - x/(n-l) \geq \nu (\lambda_1/n) - k_l/(n-l)$. One further has $\nu \to 1$, $(\Lambda_1/n) \sqrt{n-k} \to \infty$, and $k_l/\sqrt{n} = \ldots$
\[ \psi(k) / \sqrt{n}(l/k)^{3} \rightarrow 0 \] by (4.8). The estimate of \( R_2 \) is done as follows:

\[
R_2 = E_{\tilde{\theta}_n} (M_{n-k}^{+} I_{\{ \gamma_n (|\tilde{S}_{n-k}| + 1) > \varepsilon_n \}})
\]

\[
= E_{\tilde{\theta}_n} (M_{n-k}^{+} I_{\{ S_{n-k} > (\varepsilon_n/\gamma_n - 1) \sqrt{n-k} + (n-k)\tilde{\theta}_n \}})
\]

\[
+ E_{\tilde{\theta}_n} (M_{n-k}^{+} I_{\{ 0 < S_{n-k} < (\varepsilon_n/\gamma_n - 1) \sqrt{n-k} + (n-k)\tilde{\theta}_n \}})
\]

\[ = R_{2,1} + R_{2,2}. \]

Applying Lemma 5 and Fubini’s theorem to \( R_{2,1} \) yields

\[
R_{2,1} = \frac{1}{n-k} \left( \tilde{\theta}_n (n-k) + \left( \frac{\varepsilon_n}{\gamma_n} - 1 \right) \sqrt{n-k} \right) P_{\tilde{\theta}_n} \left\{ \tilde{S}_{n-k} > \frac{\varepsilon_n}{\gamma_n} - 1 \right\}
\]

\[ + (n-k)^{-1/2} \int_{\varepsilon_n/\gamma_n - 1}^{\infty} du P_{\tilde{\theta}_n} \{ \tilde{S}_{n-k} > u \} = o(\tilde{\theta}_n), \]

since \( \varepsilon_n/\gamma_n \rightarrow \infty \) and \( \tilde{\theta}_n \sqrt{n-k} \rightarrow \infty \) as \( n \rightarrow \infty \). By Lemma 5 and Fubini’s theorem, since \( \varepsilon_n/\gamma_n \rightarrow \infty \),

\[ R_{2,2} = (n-k)^{-1/2} \int_{(-\varepsilon_n/\gamma_n,-(\varepsilon_n/\gamma_n-1))} du P_{\tilde{\theta}_n} \left( - \left( \frac{\varepsilon_n}{\gamma_n} - 1 \right) > \tilde{S}_{n-k} > u \right) = o(\tilde{\theta}_n). \]

Thus we have shown that \( |W_n| = o(\tilde{\theta}_n) \), and, from (4.21),

\[
\int_{0}^{r_n} dr I_1 I_2 \geq (1 + o(1)) \int_{0}^{r_n} \exp(-\theta_n r) P_{\tilde{\theta}_n} (S_i > r \text{ for all } i \in \mathbb{N}) dr + o(\tilde{\theta}_n). \]

Now we want to increase the right-hand side by \( O_n \) to have an integral from 0 to infinity. By Theorem 2.7 of Woodroofe (1982) we see that the resulting error is \( o(\tilde{\theta}_n) \).

Let \( \tau_+ = \inf\{ n \geq 1 \mid S_n > 0 \} \). By the definition of \( r_n \) and since \( \theta_n \sqrt{n-k} \rightarrow \infty \),

\[
O_n = \int_{r_n}^{\infty} dr \exp(-\theta_n r) P_{\tilde{\theta}_n} \{ S_i > r \text{ for all } i \in \mathbb{N} \}
\]

\[
= \tilde{\theta}_n \int_{r_n}^{\infty} dr \exp(-\theta_n r) (E_{\tilde{\theta}_n} S_{\tau_+})^{-1} P_{\tilde{\theta}_n} \{ S_{\tau_+} > r \}
\]

\[ \leq \tilde{\theta}_n (r_n \theta_n)^{-1} = o(\tilde{\theta}_n). \]

Thus we have

\[
\int_{0}^{\infty} dr I_1 I_2 \geq (1 + o(1)) \int_{0}^{\infty} dr \exp(-\theta_n r) P_{\tilde{\theta}_n} (S_i > r \text{ for all } i \in \mathbb{N}) + o(\tilde{\theta}_n). \quad (4.22)
\]
We evaluate the right-hand side of (4.22) using Woodroffe’s formula and the estimate $\exp(x) \geq 1 + x$.

\[
\int_0^\infty dr \exp(-\theta_n r) P_{\theta_n}(S_i > r \text{ for all } i \in \mathbb{N})
\]
\[
\geq \int_0^\infty dr P_{\theta_n}(S_i > r \text{ for all } i \in \mathbb{N}) - \theta_n \int_0^\infty dr r P_{\theta_n}(S_i > r \text{ for all } i \in \mathbb{N})
\]
\[
= \tilde{\theta}_n \int_0^\infty dr P_{\theta_n}(\tau_+ > r)(E_{\tilde{\theta}_n} S_{\tau_+})^{-1} - \tilde{\theta}_n \theta_n \int_0^\infty dr r P_{\theta_n}(\tau_+ > r)(E_{\tilde{\theta}_n} (S_{\tau_+}))^{-1}
\]
\[
= \tilde{\theta}_n \left( 1 - \theta_n \frac{1}{2} E_{\tilde{\theta}_n} (S_{\tau_+})^2 \right).
\]

(4.23)

Now it holds

\[E_{\theta}(S_{\tau_+}^i) < \infty \text{ with } i = 1, 2, \text{ for all } 0 < \theta < \theta_0 \text{ and } \lim_{\theta \to 0} E_{\theta}(S_{\tau_+}^i) = E_0(S_{\tau_+}^i)\]

by dominated convergence. Since $E_0(S_{\tau_+}) > 0$ the term $E_{\tilde{\theta}_n} (S_{\tau_+}^2)/E_{\tilde{\theta}_n} (S_{\tau_+})$ remains bounded as $n \to \infty$. Since $\tilde{\theta}_n$ and $\theta_n$ tend to zero as $n \to \infty$, it follows from the right-hand side of (4.22) and (4.23) that $\int I_1 I_2 dr \geq \tilde{\theta}_n (1 + o(1))$. By (4.18) we get (4.19) and the proof is completed.

5. Several Lemmata

**Lemma 3.** Let $X_i, 1 \leq i \leq m$ be independent identically distributed random variables with $E|X_1| < \infty$. Then

\[E(M_m^+) = E(X_1) + \frac{1}{m} E(S_m^-).\]

**Proof.** One has $E(M_m) = E(M_m^+) - E(M_m^-)$ and

\[E(M_m) = E(X_1) + E\left( \min_{1 \leq j \leq m} (S_j - X_1) \right) = E(X_1) - E(M_{m-1}^-).\]

From

\[-M_m = \min_{0 \leq i \leq m} S_i \text{ and } E\left( \min_{0 \leq i \leq m} S_i \right) = -\sum_{k=1}^m \frac{1}{k} E(S_k^-),\]

the statement of the lemma follows. For the last equation see Siegmund (1983, p.187).

**Lemma 4.** Let $X_i, i \geq 1$ be independent identically distributed random variables with $E(X_1) = 0$ and $\text{Var}(X_1) = 1$. Let $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$. Let $N = \inf\{n \geq 1 \mid S_n < 0\}$. Then

\[\lim_{n \to \infty} E(S_n I_{\{S_i \geq 0 \text{ for } i=1,...,n\}}) = -E(S_N).\]
Proof. One has
\[-S_n I_{\{S_{i} \geq 0, i=0, \ldots, n\}} = - \sum_{j=0}^{n-1} \left( S_j I_{\{S_{i} \geq 0, i=0, \ldots, j\}} - S_{j+1} I_{\{S_{i} \geq 0, i=0, \ldots, j+1\}} \right).\]

Now take expectations on both sides. Since \( \{S_i, 1 \leq i \leq j\} \) and \( X_{j+1} \) are independent and since \( \mathbb{E}X_{j+1} = 0 \) we can write \( S_{j+1} \) instead of \( S_j \) in the first term on the right-hand side. We get
\[-E(S_n I_{\{S_{i} \geq 0, i=1, \ldots, n\}}) = \sum_{j=0}^{n-1} E(S_j I_{\{S_{i} \geq 0, i=1, \ldots, j-1, S_{j+1}<0\}}) = E(S_N I_{\{N \leq n\}}).\]

For \( n \to \infty \) the statement of the lemma follows.

Lemma 5. Let \( X_i, i = 1, \ldots, m \) be identically distributed with \( \mathbb{E}(X_1) < \infty \). Let \( M_m = \min_{1 \leq i \leq m} S_i \). Then, for any real \( z > 0 \),
\[E(M_m^+I_{\{S_m > z\}}) = E(X_1I_{\{S_m > z\}}) = E\left(\frac{S_m}{m} I_{\{S_m > z\}}\right).\]
The equality holds if \( \{0 < S_m < z\} \) replaces \( \{S_n > z\}\).

Proof. On the one hand one has
\[E(M_m^+I_{\{S_m > z\}}) = E(M_mI_{\{S_m > z\}}) + E(M_{m-1}^-I_{\{S_m > z\}}),\]
on the other hand,
\[E(M_mI_{\{S_m > z\}}) = E(X_1I_{\{S_m > z\}}) + E\left(\min_{1 \leq j \leq m} (S_j - X_1)I_{\{S_m > z\}}\right)\]
\[= E(X_1I_{\{S_m > z\}}) - E(M_{m-1}^-I_{\{S_m > z\}}).\]
This implies the first equation, the second follows by conditioning.

Acknowledgement

We thank Qiwei Yao for his encouragement and support.

References


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(Received February 2012; accepted December 2012)