
Coherence for Multivariate Random Fields

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Supplementary Material

This supplementary material contains proofs for the results of the manuscript.

S1 Supplemental Material: Proofs

The following Lemma is useful in proving some results of the manuscript.

Lemma 1. *Suppose* $Z_1(\mathbf{s})$ *is a stationary processes on* $\mathbb{R}^d$ *with covariance function* $C_1(\mathbf{h})$ *having spectral density* $f_1(\boldsymbol{\omega})$ *and* $Z_2(\mathbf{s}) = \int K(\mathbf{s} - \mathbf{u})Z_1(\mathbf{u})\mathrm{d}\mathbf{u}$ *where* K *is continuous, symmetric and square integrable with Fourier transform* $f_K(\boldsymbol{\omega})$*. Then* $Z_2(\mathbf{s})$ *has covariance function* $C_2(\mathbf{h}) = \int\int K(\mathbf{u}+\mathbf{v}-\mathbf{h})K(\mathbf{v})C_1(\mathbf{u})\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}$ *with associated spectral density* $f_2(\boldsymbol{\omega}) = f_1(\boldsymbol{\omega})f_K(\boldsymbol{\omega})^2$*. Additionally, the cross-covariance function between* Z_1 *and* Z_2 *is* $C_{12}(\mathbf{h}) = \int K(\mathbf{u}-\mathbf{h})C_1(\mathbf{u})\mathrm{d}\mathbf{u}$ *with spectral density* $f_{12}(\boldsymbol{\omega}) = f_1(\boldsymbol{\omega})f_K(\boldsymbol{\omega})$.

The proof of this Lemma involves straightforward calculations involving convolutions and is not included here.

We recall the spectral representation for a stationary vector-valued process $\mathbf{Z}(\mathbf{s}) \in \mathbb{R}^p, \mathbf{s} \in \mathbb{R}^d$ with matrix-valued covariance function $\mathbf{C}(\mathbf{h})$ having spectral measures $F_{ij}, i, j = 1, \ldots, p$ defined on the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}^d$. There is a set of complex random measures $\mathbf{M} = (M_1, \ldots, M_p)$ on $\mathcal{B}$ such that if $B, B_1, B_2 \in \mathcal{B}$ are disjoint, $\mathbb{E}M_i(B) = 0, \mathbb{E}(M_i(B)\overline{M_j(B)}) = F_{ij}(B)$ and $\mathbb{E}(M_i(B_1)\overline{M_j(B_2)}) = 0$ for $i, j = 1, \ldots, p$. Then $\mathbf{Z}(\mathbf{s})$ has the spectral representation

$$\mathbf{Z}(\mathbf{s}) = \int \exp(i\boldsymbol{\omega}^{\mathrm{T}}\mathbf{s})\mathrm{d}\mathbf{M}(\boldsymbol{\omega}),$$

see (Gihman and Skorohod, 1974) for details. If all F_{ij} admit associated spectral densities f_{ij}, then in shorthand we write $\mathbb{E}(\mathrm{d}M_i(\boldsymbol{\omega})\overline{\mathrm{d}M_j(\boldsymbol{\omega})}) = f_{ij}(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega}$.

Proof of Theorem 2. The spectral representation implies

$$Z_i(\mathbf{s}) = \int \exp(i\boldsymbol{\omega}^{\mathrm{T}}\mathbf{s})\mathrm{d}M_i(\boldsymbol{\omega}),$$

for complex-valued random measures M_i, $i = 1, 2$. Then if K has Fourier transform F_K,

$$\begin{aligned}\int K(\mathbf{u}-\mathbf{s})Z_2(\mathbf{u})\mathrm{d}\mathbf{u} &= \int\int K(\mathbf{u}-\mathbf{s})\exp(i\boldsymbol{\omega}^{\mathrm{T}}\mathbf{u})\mathrm{d}M_2(\boldsymbol{\omega})\mathrm{d}\mathbf{u} \\ &= \int \exp(i\boldsymbol{\omega}^{\mathrm{T}}\mathbf{s})F_K(\boldsymbol{\omega})\mathrm{d}M_2(\boldsymbol{\omega})\end{aligned}$$

by a change of variables. Then, using that $f_{ii}(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega} = \mathbb{E}|\mathrm{d}M_i(\boldsymbol{\omega})|^2$ and

$$\mathbb{E}\left(\int g(\boldsymbol{\omega})\mathrm{d}M_i(\boldsymbol{\omega})\overline{\int h(\boldsymbol{\omega})\mathrm{d}M_j(\boldsymbol{\omega})}\right) = \int g(\boldsymbol{\omega})\overline{h(\boldsymbol{\omega})}f_{ij}(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega}$$

we have

$$\begin{aligned}\mathbb{E}\left|Z_1(\mathbf{s}) - \int K(\mathbf{u}-\mathbf{s})Z_2(\mathbf{u})\mathrm{d}\mathbf{u}\right|^2 &= \int \Big(f_{11}(\boldsymbol{\omega}) - f_{12}(\boldsymbol{\omega})F_K(\boldsymbol{\omega}) - f_{21}(\boldsymbol{\omega})\overline{F_K(\boldsymbol{\omega})} + \\ &\qquad F_K(\boldsymbol{\omega})\overline{F_K(\boldsymbol{\omega})}f_{22}(\boldsymbol{\omega})\Big)\mathrm{d}\boldsymbol{\omega} \\ &= \int \mathbb{E}\left|\mathrm{d}M_1(\boldsymbol{\omega}) - F_K(\boldsymbol{\omega})\mathrm{d}M_2(\boldsymbol{\omega})\right|^2.\end{aligned}$$

The integrand is minimized for each $\boldsymbol{\omega}$ if

$$F_K(\boldsymbol{\omega}) = \frac{\mathbb{E}(\mathrm{d}M_1(\boldsymbol{\omega})\overline{\mathrm{d}M_2(\boldsymbol{\omega})})}{\mathbb{E}|\mathrm{d}M_2(\boldsymbol{\omega})|^2} = \frac{f_{12}(\boldsymbol{\omega})}{f_{22}(\boldsymbol{\omega})}.$$

That the density of $\int K(\mathbf{u}-\mathbf{s})Z_2(\mathbf{u})\mathrm{d}\mathbf{u}$ is $|f_{12}(\boldsymbol{\omega})|^2/f_{22}(\boldsymbol{\omega})$ now follows by the convolution theorem for Fourier transforms. □

Proof of Proposition 4. If $f_i(\boldsymbol{\omega})$ is the Fourier transform of $c_i, i = 1, 2$, the result immediately follows as the spectral density of $C_{ij}(\mathbf{h})$ is $f_i(\boldsymbol{\omega})f_j(\boldsymbol{\omega})$.

□

Proof of Proposition 5. This result follows directly from Lemma 1. □

Proof of Proposition 6. If W has spectral density $f_W(\boldsymbol{\omega})$ then the spectral density for Z_k is $\mathcal{F}(g_k)(\boldsymbol{\omega})^2 f_W(\boldsymbol{\omega})$ where $\mathcal{F}$ denotes the Fourier transform, by Lemma 1. The result follows by definition of coherence. □

Bibliography

Gihman, I. I. and Skorohod, A. V. (1974), *The Theory of Stochastic Processes, Vol. 1*, Springer-Verlag, Berlin.