Statistica Sinica **25** (2015), 1527-1546
doi:http://dx.doi.org/10.5705/ss.2013.155

A COLLOCATION METHOD FOR THE SEQUENTIAL TESTING OF A GAMMA PROCESS

B. Buonaguidi and P. Muliere

Bocconi University

Abstract: We study the Bayesian problem of sequential testing of two simple hypotheses about the parameter $\alpha > 0$ of a Lévy gamma process. The initial optimal stopping problem is reduced to a free-boundary problem where, at the unknown boundary points separating the stopping and continuation set, the principles of the smooth and/or continuous fit hold and the unknown value function satisfies on the continuation set a linear integro-differential equation. Due to the form of the Lévy measure of a gamma process, determining the solution of this equation and the boundaries is not an easy task. Hence, instead of solving the problem analytically, we use a collocation technique: the value function is replaced by a truncated series of polynomials with unknown coefficients that, together with the boundary points, are determined by forcing the series to satisfy the boundary conditions and, at fixed points, the integro-differential equation. The proposed numerical technique is employed in well-understood problems to assess its efficiency.

Key words and phrases: Bayes decision rule, Chebyshev polynomials, collocation method, free-boundary problem, gamma process, optimal stopping, sequential testing, smooth and continuous fit principles.

1. Introduction

Establishing the correct distributional properties of a sequentially observed stochastic process is of fundamental importance in many practical problems, as well as a challenging task from a theoretical view point. In this paper it is assumed that at time $t = 0$ we begin to follow the evolution of a Lévy gamma process $X = (X_t)_{t\geq 0}$ with parameter $\alpha > 0$: its sequential testing consists of picking a stopping time τ of X and a decision function d, expressing which of the two simple hypotheses initially formulated about α might be accepted at time τ so that a risk value function is minimized. The problem is analyzed within the Bayesian framework, where a priori distribution on the correctness of the hypotheses is given and the goal is the minimization of the sum between the expected cost of the observation process and the expected loss one suffers if a final wrong decision is made.

Problems of sequential testing for continuous time processes have been widely studied in the literature and can be distinguished in two areas depending on the sample paths of the observed process: the first area contains the works of Shiryaev (1978, Sec. 4.2), Gapeev and Peskir (2004), Gapeev and Shiryaev (2011) and Shiryaev and Zhitlukhin (2011), where solutions to the Bayesian sequential testing for the drift of a Wiener process or a more general diffusion process are provided; the second area includes the works of Peskir and Shiryaev (2000), Gapeev (2002), Dayanik and Sezer (2006), Dayanik, Poor, and Sezer (2008), Dayanik and Sezer (2012) and Ludkovski and Sezer (2012) who study problems of sequential testing for jumping processes of compound Poisson type. In the first area the analyzed processes have continuous patterns, in the second the observed processes jump a finite number of times on any finite time interval.

The novelty here is the analysis of the Bayesian sequential testing for a gamma process, a purely jump process with infinitely many positive jumps on any finite time interval. The value function and the optimal stopping boundaries of the initial optimal stopping problem for the posterior probability process are shown to be the solution of a free-boundary problem: the value function satisfies at the stopping boundaries the principles of the smooth and/or continuous fit and solves, on the continuation set, a linear integro-differential equation. Determining an explicit solution of the free-boundary problem appears to be extremely complex and requires the devising of a suitable numerical technique.

The successive approximation scheme adopted in Dayanik and Sezer (2006) for the sequential testing of a compound Poisson process cannot be applied and a collocation approach is developed. It relies on replacing the value function in the free-boundary problem with a truncated series of polynomials with unknown coefficients (in particular, Chebyshev polynomials are used) and forcing it to solve the boundary conditions and, at a fixed number of points, the integro-differential equation. The number of points is chosen so that, taking into account the boundary conditions, the number of equations coincides with the number of the coefficients of the series and the stopping boundaries. This approach is a modification of the well-known collocation method, widespread in mathematical physics and engineering for solving boundary value problems. Its efficiency is illustrated in problems where exact solutions are available.

The paper is organized as follows. In Section 2, after we recall the main properties of a gamma process and define the problem, the original optimal stopping problem for the posterior probability process is reduced to a free-boundary problem. In Section 3, we show how its numerical solution can be accurately derived by a collocation approach. In Section 4, we compare exact and collocation solutions of well-understood sequential testing problems. Section 5 contains a summary discussion. Proofs are deferred to the Appendix, as well as a basic introduction on the collocation method and Chebyshev polynomials.

The sequential testing for a gamma process was already considered by Dvoretzky, Kiefer, and Wolfowitz (1953) but, to the best of our knowledge, a solution has never been provided. Our study is a natural continuation of the arguments contained in Buonaguidi and Muliere (2013) and is motivated by the extensive use of the gamma process in risk theory (Dufresne, Gerber, and Shiu (1991)), degradation and failure models (Lawless and Crowder (2004), Park and Padgett (2005)), maintenance and reliability (Van Noortwijk (2009)).

2. Sequential Testing of a Gamma Process

Interest in the analysis of the sequential testing for a gamma process was raised in Buonaguidi and Muliere (2013).

A gamma process $X = (X_t)_{t\geq 0}$ of parameter $\alpha > 0$ is a Lévy process with Lévy-Khintchine representation

$$E\left[e^{izX_t}\right] = \exp\left\{t\int_0^\infty (e^{izx}-1)\frac{e^{-\alpha x}}{x}\,dx\right\} = \left(\frac{\alpha}{\alpha - iz}\right)^t, \quad z \in \mathbb{R}, \tag{2.1}$$

where $v(dx) = x^{-1}e^{-\alpha x}\mathbf{1}_{(0,\infty)}(dx)$ is the so-called Lévy measure. Using standard arguments based on Sato (1999), the following properties are inferred from (2.1): X is a purely jump process; X is not a compound Poisson process and its jumping times are countable and dense in $[0,\infty)$ a.s.; the map $t \mapsto X_t$ is strictly increasing and not continuous anywhere a.s.; X has sample paths of finite variation; X_t, $t \geq 0$, has a gamma distribution with density

$$f_t(x;\alpha) = \frac{\alpha^t}{\Gamma(t)}x^{t-1}e^{-\alpha x}\mathbf{1}_{(0,\infty)}(x). \tag{2.2}$$

The second property means that, for any $t > 0$, X has infinitely many jumps on $(0,t)$ and is a direct consequence of $v(\mathbb{R}) = \infty$, while the fourth arises from $\int_{-1}^{1} |x|\, v(dx) < \infty$. For a deeper investigation on the properties of the gamma process we refer to Kyprianou (2006), James, Roynette, and Yor (2008) and Yor (2007). Figure 1 shows two simulated paths of a gamma process.

2.1. Formulation of the problem

On the filtered statistical space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \{P_1, P_0\})$ the process $X = (X_t)_{t\geq 0}$ is defined and is assumed to be a gamma process of parameter $\alpha_i > 0$ under P_i, $i = 0,1$. Let α be an $\mathscr{F}_0$-measurable random variable independent of X; under the probability measure P_π, defined by

$$P_\pi = \pi P_1 + (1-\pi)P_0, \quad \pi \in [0,1], \tag{2.3}$$

α takes value α_1 with probability π, and α_0 with probability $1-\pi$, where π is given. In order to test the hypotheses

$$H_0 : \alpha = \alpha_0 \quad Vs \quad H_1 : \alpha = \alpha_1, \quad \alpha_0 > \alpha_1, \tag{2.4}$$

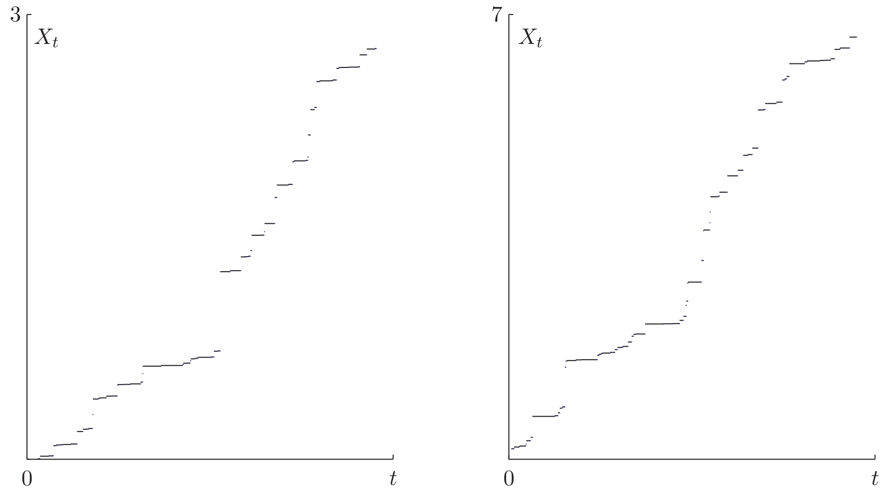

Figure 1. Simulated paths of a gamma process $X = (X_t)_{t\geq 0}$. We set $\alpha = 5$ and $\alpha = 3$ on the left and right, respectively.

we are allowed to sequentially observe X. Let $\mathscr{F}_t^X = \sigma\{X_s : 0 \leq s \leq t\}$ and denote by (τ, d) a sequential decision rule, where τ is a stopping time of X and d, a decision function, is a $\mathscr{F}_\tau^X$-measurable random variable, that, at that time τ, takes value i if H_i, $i = 0, 1$, must be accepted.

The Bayesian problem of sequentially testing (2.4) requires computing

$$V(\pi) = \inf_{(\tau,d)} E_\pi \left[\tau + a\mathbf{1}_{(d=0,\alpha=\alpha_1)} + b\mathbf{1}_{(d=1,\alpha=\alpha_0)}\right], \quad a, b > 0, \tag{2.5}$$

and determining the π-Bayes decision rule $(\tau_\pi^\star, d_\pi^\star)$ at which the infimum is attained. By means of standard arguments based on Shiryaev (1978), one can show that (2.5) is equivalent to the optimal stopping problem

$$V(\pi) = \inf_\tau E_\pi \left[\tau + g_{a,b}(\pi_\tau)\right], \tag{2.6}$$

where $(\pi_t)_{t\geq 0}$, with $\pi_t = P_\pi(\alpha = \alpha_1 | \mathscr{F}_t^X)$, is the posterior probability process, $g_{a,b}(\pi) = a\pi \wedge b(1-\pi)$, and the π-Bayes decision rule is $d_\pi^\star = 1$ if $\pi_{\tau_\pi^\star} \geq c$, and $d_\pi^\star = 0$ if $\pi_{\tau_\pi^\star} < c$, $c = b/(a+b)$.

Denoted by $D = \{\pi \in [0,1] : V(\pi) = g_{a,b}(\pi)\}$, the structure of the value function (2.6) and the general theory of optimal stopping (see, e.g., Peskir and Shiryaev (2006) or Shiryaev (1978)) imply that $\tau_\pi^\star = \inf\{t \geq 0 : \pi_t \in D, \pi_0 = \pi\}$, and that there exist two points A and B, $0 < A \leq c \leq B < 1$, such that $D = [0, A] \cup [B, 1]$. D is called the stopping set, its complement (A, B) is the continuation set.

Let $(\varphi_t)_{t\geq 0}$ be the likelihood ratio process, defined by $\varphi_t = d\left(P_1|\mathscr{F}_t^X\right)/d(P_0|\mathscr{F}_t^X)$; according to Sato (1999),

$$\begin{aligned}\varphi_t &= \exp\left\{(\alpha_0-\alpha_1)X_t - t\int_0^\infty \left(e^{(\alpha_0-\alpha_1)x}-1\right)\frac{e^{-\alpha_0 x}}{x}\,dx\right\}\\ &= \exp\left\{(\alpha_0-\alpha_1)X_t - \log\left(\frac{\alpha_0}{\alpha_1}\right)t\right\},\end{aligned} \tag{2.7}$$

where we used, under the appropriate assumptions, the well-known Frullani's formula

$$\int_0^\infty \frac{f(px)-f(qx)}{x}\,dx = [f(0)-f(\infty)]\log\left(\frac{q}{p}\right). \tag{2.8}$$

For further reference set

$$Y_t = (\alpha_0-\alpha_1)X_t - \log\left(\frac{\alpha_0}{\alpha_1}\right)t. \tag{2.9}$$

A simple application of Bayes theorem shows that

$$\pi_t = \frac{\pi e^{Y_t}}{1+\pi(e^{Y_t}-1)}. \tag{2.10}$$

Let $\mu^X\left((0,t]\times H\right) = \sum_{s\leq t}\mathbf{1}(\Delta X_s\in H)$, $H\in\mathscr{B}(\mathbb{R}^+\backslash\{0\})$, be the measure of jumps of the process X; then, (2.7) and (2.10), together with a straightforward application of Itô's formula for purely jump Lévy processes, lead to the stochastic differential equations:

$$d\varphi_t = -\log\left(\frac{\alpha_0}{\alpha_1}\right)\varphi_{t^-}dt + \varphi_{t^-}\int_0^\infty\left(e^{(\alpha_0-\alpha_1)x}-1\right)\mu^X(dx,dt),\ \varphi_0=1, \tag{2.11}$$

$$\begin{aligned}d\pi_t &= -\log\left(\frac{\alpha_0}{\alpha_1}\right)\pi_{t^-}(1-\pi_{t^-})dt + \int_0^\infty \frac{\pi_{t^-}(1-\pi_{t^-})\left(e^{(\alpha_0-\alpha_1)x}-1\right)}{1+\pi_{t^-}\left(e^{(\alpha_0-\alpha_1)x}-1\right)}\,\mu^X(dx,dt),\\ \pi_0 &= \pi.\end{aligned} \tag{2.12}$$

2.2. Reduction of the optimal stopping problem to a free-boundary problem

We reduce the optimal stopping problem (2.6) to a free-boundary problem for the value function $V(\pi)$ and the boundaries A and B defining the stopping region D. To accomplish this we need to determine the infinitesimal operator of $(\pi_t)_{t\geq 0}$ and show some properties of the function $V(\pi)$.

Proposition 1. *If $f\in C^1[0,1]$, then*

$$f(\pi_t) = f(\pi) + \int_0^t (\mathbb{L}f)(\pi_{s^-}) + \mathscr{M}_t, \tag{2.13}$$

where $\mathbb{L}$ *is the infinitesimal operator of* $(\pi_t)_{t\geq 0}$,

$$(\mathbb{L}f)(\pi) = -\log\left(\frac{\alpha_0}{\alpha_1}\right) f'(\pi)\pi(1-\pi) + \int_0^\infty \left[f\left(\frac{\pi e^{-\alpha_1 x}}{(1-\pi)e^{-\alpha_0 x}+\pi e^{-\alpha_1 x}}\right)-f(\pi)\right] \frac{(1-\pi)e^{-\alpha_0 x}+\pi e^{-\alpha_1 x}}{x} dx, \tag{2.14}$$

and $\mathscr{M} = (\mathscr{M}_t)_{t\geq 0}$, *given by*

$$\mathscr{M}_t = \int_0^t \int_0^\infty \left[f\left(\frac{\pi_{s-} e^{-\alpha_1 x}}{(1-\pi_{s-})e^{-\alpha_0 x}+\pi_{s-} e^{-\alpha_1 x}}\right)-f(\pi_{s-})\right] \times \left(\mu^X(dx,ds) - \frac{(1-\pi_{s-})e^{-\alpha_0 x}+\pi_{s-} e^{-\alpha_1 x}}{x}\, dx\, ds\right), \tag{2.15}$$

is a local martingale with respect to $(\mathscr{F}_t^X)_{t\geq 0}$ *and* P_π, $\forall \pi \in [0,1]$.

Proposition 2. *The map* $\pi \mapsto V(\pi)$ *in* (2.6) *is concave and thus continuous on* $[0,1]$.

Proposition 3. *If the optimal stopping boundary* A *is strictly less than* $c = b/(a+b)$, *then* $V(\pi)$ *at* (2.6) *is differentiable from the right at* A *and*

$$V'(A+) = a. \tag{2.16}$$

Propositions 2 and 3 formally justify the so-called principles of the smooth and continuous fit, stating that the value function $V(\pi)$ must be smooth at A and just continuous at B. The discovery of the continuous fit condition as a variational principle alike the smooth fit is due to Peskir and Shiryaev (2000). It can be explained by noticing that the process $(\pi_t)_{t\geq 0}$, defined through (2.10) and (2.12), creeps downward and jumps upward, so that the boundary A is continuously crossed, while B, at which the smooth fit breaks down, is passed by jumps only (see Figure 2).

These facts, the strong Markov property of $(\pi_t)_{t\geq 0}$, evident from (2.12), and the general theory of optimal stopping (see, e.g., Peskir and Shiryaev (2006) and Shiryaev (1978)) lead to the formulation of a free-boundary problem for the unknown function V and the unknown boundaries A and B:

$$\mathbb{L}V = -1 \qquad \text{for} \quad \pi \in (A,B), \tag{2.17}$$

$$V = g_{a,b} \qquad \text{for} \quad \pi \notin (A,B), \tag{2.18}$$

$$V < g_{a,b} \qquad \text{for} \quad \pi \in (A,B), \tag{2.19}$$

$$V(A+) = aA \qquad (\textit{continuous fit}), \tag{2.20}$$

$$V'(A) = a \qquad (\textit{smooth fit}), \tag{2.21}$$

$$V(B-) = b(1-B) \qquad (\textit{continuous fit}). \tag{2.22}$$

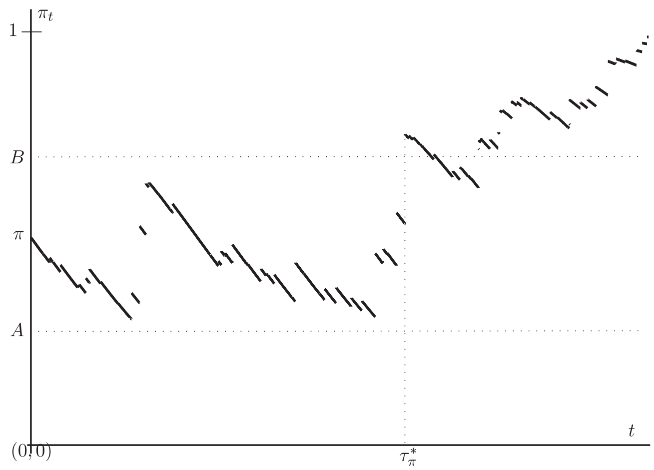

Figure 2. A simulated path of the posterior probability process $(\pi_t)_{t\geq 0}$ as defined by (2.10) and (2.12), with $\alpha_0 = 5$ and $\alpha_1 = 3$. It is assumed that the true hypothesis is $\alpha = \alpha_1$.

2.3. Existence, uniqueness and optimality of the solution

We find that if a solution to the free-boundary problem (2.17)−(2.22) exists, then it is unique and coincides with the one of the optimal stopping problem (2.6).

For a fixed $B > c$, consider on the interval $I_B = (0, B]$ the integro-differential equation defined by (2.14) and (2.17). Denote by $V(\pi; B)$, $\pi \in I_B$, its solution. The function

$$S(\pi, x) = \frac{\pi e^{-\alpha_1 x}}{(1-\pi)e^{-\alpha_0 x} + \pi e^{-\alpha_1 x}}, \quad \pi \in I_B, \quad x \geq 0, \tag{2.23}$$

appearing in (2.14), is increasing in x, $\lim_{x\to\infty} S(\pi, x) = 1$ and, according to (2.18) and (2.22), leads us to set $V(S(\pi, x); B) = b(1 - S(\pi, x))$ whenever $\pi \in I_B$ and $x \geq \log\big([(1-\pi)/\pi][B/(1-B)]\big)\big/(\alpha_0 - \alpha_1) =: x^\star(\pi; B)$. Hence, $V(\pi; B)$ satisfies

$$(\mathbb{L}_B V)(\pi; B) = 0, \quad \pi \in I_B, \tag{2.24}$$

$$V(B; B) = b(1 - B), \tag{2.25}$$

where $\mathbb{L}_B$ is the operator defined by

$$
\begin{aligned}
(\mathbb{L}_B f)(\pi) = &-\log\left(\frac{\alpha_0}{\alpha_1}\right) f'(\pi)\pi(1-\pi) \\
&+ b(1-\pi)\int_{x^\star(\pi;B)}^{\infty} \frac{e^{-\alpha_0 x}}{x} dx - f(\pi)\int_{x^\star(\pi;B)}^{\infty} \frac{(1-\pi)e^{-\alpha_0 x}+\pi e^{-\alpha_1 x}}{x} dx \\
&+ \int_0^{x^\star(\pi;B)} \big[f(S(\pi,x)) - f(\pi)\big] \frac{(1-\pi)e^{-\alpha_0 x}+\pi e^{-\alpha_1 x}}{x} dx + 1, \ \pi \in I_B.
\end{aligned} \tag{2.26}
$$

Proposition 4. *For any fixed $B > c$,* (2.24)–(2.25) *has a unique continuously differentiable solution $V(\pi;B)$, $\pi \in I_B$.*

The map $\pi \mapsto V(\pi;B)$, $\pi \in I_B$, hits the map $\pi \mapsto b(1-\pi)$ at B, because of the continuous fit principle (2.22) and (2.25). The condition ensuring the existence of a special (unique) pair of points $A^\star$ and $B^\star$, at which the map $\pi \mapsto V(\pi;B^\star)$ smoothly hits $\pi \mapsto a\pi$ and hits $\pi \mapsto b(1-\pi)$, respectively, is given in the next proposition.

Proposition 5. *There exist a unique function V and a unique pair of points $A^\star$ and $B^\star$, which solve the free-boundary problem* (2.17)–(2.22)*, defined through* (2.14)*, if and only if*

$$\lim_{B\downarrow c} V'(B-;B) < a. \tag{2.27}$$

In this case we have

$$V(\pi) = \begin{cases} V(\pi;B^\star) & \text{for} \quad \pi \in (A^\star, B^\star), \\ g_{a,b}(\pi) & \text{for} \quad \pi \in [0,A^\star] \cup [B^\star,1], \end{cases} \tag{2.28}$$

where the map $\pi \mapsto V(\pi;B)$, $\pi \in I_B$, is the unique continuously differentiable solution of (2.24)–(2.25) *and $A^\star$ and $B^\star$ uniquely solve*

$$V(A^\star;B^\star) = aA^\star, \quad V'(A^\star;B^\star) = a. \tag{2.29}$$

The next result connects the free-boundary problem (2.17)–(2.22) with the optimal stopping problem (2.6).

Theorem 1. *The π-Bayes decision rule $(\tau^\star_\pi, d^\star_\pi)$ for the sequential testing of the two simple hypotheses* (2.4) *concerning the parameter α of a gamma process:*

(I) *if* (2.27) *and $\partial(\mathbb{L}V)(\pi)/\partial\pi \le 0$, $\pi \in [0, A^\star)$, hold, is given by $\tau^\star_\pi = \inf\{t \ge 0 : \pi_t \notin (A^\star, B^\star)\}$, $d^\star_\pi = 0$ (accept $H_0 : \alpha = \alpha_0$), if $\pi_{\tau^\star_\pi} \le A^\star$, and $d^\star_\pi = 1$ (accept $H_1 : \alpha = \alpha_1$), if $\pi_{\tau^\star_\pi} \ge B^\star$. The stopping boundaries $0 < A^\star < c < B^\star < 1$ and the value function V in* (2.6) *are given by means of* (2.28) *and* (2.29)*;*

(II) *if* (2.27) *does not hold, becomes trivial:* $\tau^\star_\pi = 0$, $d^\star_\pi = 0$, *if* $\pi < c$, *and* $d^\star_\pi = 1$, *if* $\pi \geq c$. *The value function* $V(\pi)$ *is then equal to* $g_{a,b}(\pi)$, *for* $\pi \in [0,1]$.

Proofs can be found in Appendix A.

3. A Collocation Method for the Free-Boundary Problem

Explicitly finding $V(\pi; B)$ is not an easy task. This is due to the presence of the integration variable x in the denominator of the fraction in the last integral of (2.26), which makes the integro-differential equation (2.24) extremely difficult to solve. The source of this complication is the Lévy measure of a gamma process.

In this section we approach the free-boundary problem (2.17)−(2.22) numerically. In particular, we propose a modified version of the collocation method based on Chebyshev polynomials: this technique allows us to get very accurate solutions. We refer to Appendix B for an introduction to the collocation method and to Chebyshev polynomials.

3.1. Identifying the continuation set

Let $\{T^\star_i\}_{i\geq 0}$ be the family of shifted Chebyshev polynomials on the interval $I = [0,1]$, $T^\star_i = T^I_i$, being T^I_i, $i \geq 0$, defined at (B.15) in Appendix B.2. For a fixed $B > c$ and a sufficiently large $n \geq 0$, consider the approximation $V_n(\pi; B)$ of $V(\pi; B)$ given by

$$V(\pi; B) \approx V_n(\pi; B) = \sum_{i=0}^{n} w_i(B) T^\star_i(\pi). \tag{3.1}$$

As discussed in Appendix B.1 and according to (2.24)−(2.25), the $n+1$ coefficients $w_i(B)$ can be determined as solution of the linear system of $n+1$ equations

$$(\mathbb{L}_B V_n)(\pi_i; B) = 0, \quad i = 1, \ldots, n, \tag{3.2}$$

$$V_n(B; B) = b(1 - B), \tag{3.3}$$

where $\mathbb{L}_B$ is defined in (2.26) and $\{\pi_1, \ldots, \pi_n\}$ are n collocation nodes in $I_B = (0, B]$. As n increases, the uniform convergence of $V_n(\pi; B)$ to $V(\pi; B)$ on any compact interval is ensured by the Waierstrass approximation theorem and the continuity of $V(\pi; B)$, as stated in Proposition 4; the latter also guarantees that the coefficients $w_i(B)$, solution to (3.2)−(3.3), are well identified, due to the uniqueness of $V(\pi; B)$.

Solving (3.2)−(3.3) for several values of B allows us to check if (2.27) is satisfied and, in this case, to have a plausible idea on the continuation set $(A^\star, B^\star)$. Let us explain this claim by means of two examples.

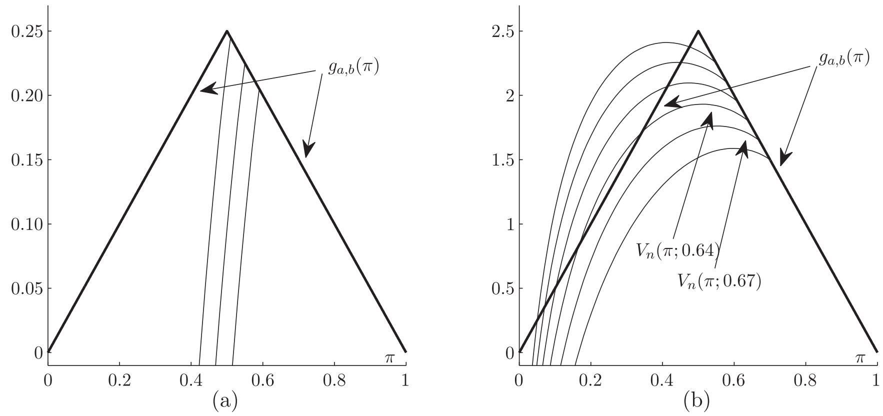

Figure 3. Two computer drawings of the maps $\pi \mapsto V_n(\pi; B)$ solving (3.2)−(3.3), with $n = 8$. On the left, the maps $\pi \mapsto V_n(\pi; B)$ never cross $\pi \mapsto a\pi$, even when $B \downarrow c$: the free-boundary problem (2.17)−(2.22) does not have a solution, so that the optimal stopping problem (2.6) becomes trivial; on the right, the condition (2.27) holds: it is evident that there exist $A^\star \in (0.2, 0.3)$ and $B^\star \in (0.64, 0.67)$ such that $\pi \mapsto V_n(\pi; B^\star)$ hits smoothly $\pi \mapsto a\pi$ at $A^\star$.

In the first, we set $a = b = 0.5$ (hence $c = 0.5$), $\alpha_0 = 5$, $\alpha_1 = 1$, and we fix $n = 8$ in (3.1); the Figure 3-a shows that even for values of B very close to c (we used $B = 0.51, 0.55, 0.59$), the maps $\pi \mapsto V_n(\pi; B)$, $\pi \in I_B = (0, B]$, obtained as solutions of (3.2)−(3.3) (we used as collocation nodes a set of n equally spaced nodes in $[0.1, B]$), never intersect the map $\pi \mapsto a\pi$. It means that (2.27) fails to hold: the free-boundary problem does not admit a solution and the solution of the optimal stopping problem (2.6) becomes trivial (see point (II) of Theorem 1).

In the second example, we set $a = b = 5$ (hence $c = 0.5$), $\alpha_0 = 5$, $\alpha_1 = 1$, and $n = 8$ in (3.1); then, the system (3.2)−(3.3) has been solved for $B = 0.55, 0.58, 0.61, 0.64, 0.67, 0.70$ (again, a set of n equally spaced collocation nodes in $[0.1, B]$ has been used). The associated maps $\pi \mapsto V_n(\pi; B)$ are shown in Figure 3-b: one can observe that (2.27) is satisfied, since there exist values of $B > c$ for which $\pi \mapsto V_n(\pi; B)$ intersects $\pi \mapsto a\pi$; thus, moving B on $(c, 1)$ from the left to the right, one can notice the existence of a unique pair of points $A^\star$ and $B^\star$ at which the continuous and smooth fit conditions (2.20)−(2.22) hold. We observe that $A^\star \in (0.2, 0.3)$ and $B^\star \in (0.64, 0.67)$ (of course, we can make these intervals more precise by solving (3.2)−(3.3) for values of $B \in (0.64, 0.67)$).

3.2. Extension of the collocation method

Once we have checked that, for the free-boundary problem (2.17)−(2.22), (2.27) is satisfied, we have to compute the optimal boundary points $A^\star$, $B^\star$ and the map $\pi \mapsto V(\pi; B^\star)$, $\pi \in (A^\star, B^\star)$. This requires an extension of the collocation method presented in Appendix B and adopted in the previous subsection, because the interval $(A^\star, B^\star)$ on which $V(\pi; B^\star)$ is defined is unknown, as well as $V(\pi; B^\star)$ itself.

For a sufficiently large $n \geq 0$, let $V_n(\pi; B^\star)$ be an approximation of $V(\pi; B^\star)$, expressed as linear combination of the first $n+1$ shifted Chebyshev polynomials on $[0, 1]$:

$$V(\pi; B^\star) \approx V_n(\pi; B^\star) = \sum_{i=0}^{n} w_i(B^\star) T_i^\star(\pi), \quad \pi \in I_{B^\star} = (0, B^\star]. \tag{3.4}$$

Solving the free-boundary problem (2.17)$-$(2.22) reduces to determining the $n+1$ coefficients $w_i(B^\star)$ and the two points $A^\star$ and $B^\star$. Since the map $\pi \mapsto V(\pi; B^\star)$ solves the integro-differential equation (2.24)$-$(2.25) on $I_{B^\star}$ and satisfies (2.29), our problem boils down to solving the system of $n+3$ non-linear equations

$$(\mathbb{L}_{B^\star} V_n)(\pi_i; B^\star) = 0, \quad i = 1, \ldots, n, \tag{3.5}$$
$$V_n(A^\star; B^\star) = aA^\star, \tag{3.6}$$
$$V_n'(A^\star; B^\star) = a, \tag{3.7}$$
$$V_n(B^\star; B^\star) = b(1 - B^\star), \tag{3.8}$$

where $\mathbb{L}_{B^\star}$ is defined by (2.26) and the n collocation nodes $\{\pi_1, \ldots, \pi_n\}$ are chosen so that they are less than $B^\star$. Even though $B^\star$ is not known, the procedure developed in Subsection 3.1 for identifying the continuation set allows us to reasonably establish an open neighbourhood of $B^\star$, say (k_1, k_2). Then, we can fix $\pi_i \leq k_1$, $i = 1, \ldots, n$. The system (3.5)$-$(3.8) can be handled by means of standard numerical techniques: the $n+1$ coefficients $w_i(B^\star)$ and $A_n^\star$ and $B_n^\star$, approximating the true values $A^\star$ and $B^\star$, are well identified and rapidly computed, as consequence of the uniqueness argument of Proposition 5.

Once the solution to (3.5)$-$(3.8) has been determined , according to Theorem 1 (I), an approximated π-Bayes decision rule can be used to test the two simple hypotheses (2.4) for a gamma process of parameter α:

$$\tau_{n,\pi}^\star = \inf\{t \geq 0 : \pi_t \notin (A_n^\star, B_n^\star)\}, \tag{3.9}$$

$$d_{n,\pi}^\star = \begin{cases} 0 \quad (\text{accept } H_0) \quad \text{if} \quad \pi_{\tau_{n,\pi}^\star} \leq A_n^\star, \\ 1 \quad (\text{accept } H_1) \quad \text{if} \quad \pi_{\tau_{n,\pi}^\star} \geq B_n^\star. \end{cases} \tag{3.10}$$

The value function $V(\pi)$ from (2.6) and (2.28) can be approximated by

$$V_n(\pi) = \begin{cases} V_n(\pi; B_n^\star) & \text{for} \quad \pi \in (A_n^\star, B_n^\star), \\ g_{a,b}(\pi) & \text{for} \quad \pi \in [0, A_n^\star] \cup [B_n^\star, 1]. \end{cases} \tag{3.11}$$

Table 1.

n	I	$A_n^\star$-$B_n^\star$	M_n	ρ_n
4	$[0.1, 0.64]$	0.2577-0.6457	0.1251	-
6	$[0.1, 0.64]$	0.2525-0.6503	0.0207	0.0157
8	$[0.1, 0.64]$	0.2541-0.6510	0.0143	0.0013
40	$[0.01, 0.64]$	0.2541-0.6511	0.0060	1.8×10^{-4}

As in Appendix B.3, we can assess the quality of the approximation in two ways: the first relies on the fact that $V_n(\pi; B_n^\star)$ must satisfy $(\mathbb{L}_{B_n^\star} V_n)(\pi; B_n^\star) \approx 0$, for any $\pi \in [A_n^\star, B_n^\star)$. Then, we can increase n until

$$M_n = \sup_{\pi \in [A_n^\star, B_n^\star)} |(\mathbb{L}_{B_n^\star} V_n)(\pi; B_n^\star)| < \epsilon, \quad \epsilon > 0. \tag{3.12}$$

The second is based on the convergence of $\{V_n\}$: if

$$\rho_n = \sup_{\pi \in (0,1)} \left| \frac{V_n(\pi) - V_{n-1}(\pi)}{V_{n-1}(\pi)} \right|, \quad n \geq 1, \tag{3.13}$$

is the maximum relative distance between V_n and V_{n-1}, we can increase n until $\rho_n < \delta$, $\delta > 0$.

To illustrate the procedure, we continue the analysis of the second example in the previous subsection where we checked that the free-boundary problem $(2.17)-(2.22)$ admits a unique solution when $a = b = 5$, $\alpha_0 = 5$ and $\alpha_1 = 1$; we found that $A^\star \in (0.2, 0.3)$ and $B^\star \in (0.64, 0.67)$. For different values of n in (3.4) and n equally spaced collocation nodes in the interval I, Table 1 shows the values of $A_n^\star$, $B_n^\star$, obtained as solution of $(3.5)-(3.8)$, M_n and ρ_n.

From Table 1, the value function V_n and the boundaries $A_n^\star$ and $B_n^\star$ are almost the same when $n = 8$ and $n = 40$; this is due to the rapid convergence of the series of Chebyshev polynomials. Figure 4-a shows the maps $\pi \mapsto V_n(\pi; B_n^\star)$ and $\pi \mapsto V_n(\pi)$ when $n = 8$; Figure 4-b shows that $(\mathbb{L}V)(\pi) \approx (\mathbb{L}V_n)(\pi)$ is decreasing on $(0, A_n^\star)$, then Theorem 1 (I) applies.

4. Use of the Collocation Method in Well Known Problems

In this section, we apply the collocation approach illustrated in Section 3 to four problems of sequential testing for which explicit solutions are available. In particular, we consider the sequential testing of two simple hypotheses for a Wiener process with drift (Shiryaev (1978)), a Poisson process (Peskir and Shiryaev (2000)), a compound Poisson process with exponential jumps (Gapeev (2002)) and a negative binomial process (Buonaguidi and Muliere (2013)).

As in Subsection 2.1, let P_0 and P_1 be the probability measures under which the hypotheses H_0 and H_1 we want to test are true with probability

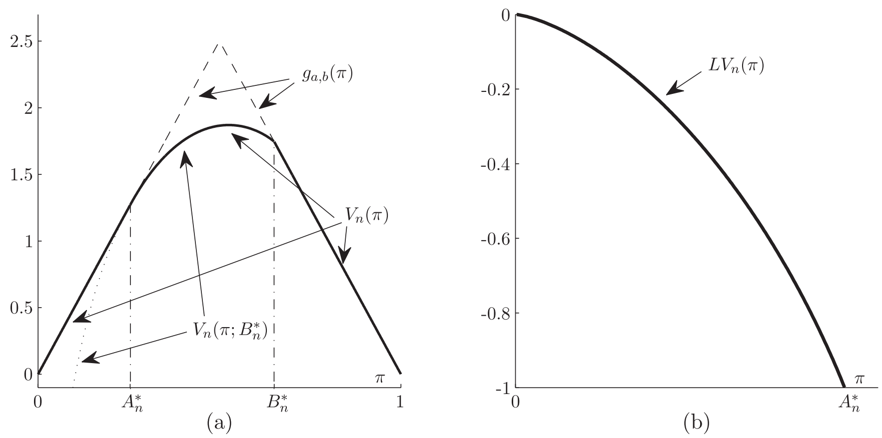

Figure 4. (a) A computer drawing of the map $\pi \mapsto V_n(\pi)$ (bold curve), as defined in (3.11), with $a = b = 5$, $\alpha_0 = 5$, $\alpha_1 = 1$ and $n = 8$ in (3.4). The set $D = [0, A_n^\star] \cup [B_n^\star, 1]$ is the stopping region, where $V_n = g_{a,b}$, while $(A_n^\star, B_n^\star) = (0.2541\cdots, 0.6510\cdots)$ is the continuation set, on which $V_n(\pi) = V_n(\pi; B_n^\star)$. We notice that $V_n(\pi)$ is differentiable at $A_n^\star$, while just continuous at $B_n^\star$, in accordance with the principle of continuous and smooth fit (2.20)−(2.22). (b) A computer drawing of the map $\pi \mapsto (\mathbb{L}V_n)(\pi)$, $\pi \in [0, A_n^\star)$, with $\mathbb{L}$ given by (2.14). The same parameters of Figure 4-a have been used. Here $\pi \mapsto (\mathbb{L}V_n)(\pi)$ is strictly decreasing on $[0, A_n^\star)$; according to Theorem 1 (I), the solution of the free-boundary problem (2.17)−(2.22) coincides with that of the optimal stopping problem (2.6).

one, respectively, P_π be the probability measure defined in (2.3), and $\pi_t = P_\pi(H_1 \text{ is true } | \mathscr{F}_t^X)$, $t \geq 0$, be the posterior probability process. To solve the optimal stopping problem (2.6) we compute the value functions and the optimal boundary points by means of our method and we compare them with the exact ones.

4.1. Sequential testing of a wiener process

Let $X = (X_t)_{t\geq 0}$ be the Wiener process with drift γ, $X_t = \gamma t + \sigma W_t$, $\sigma > 0$, and $W = (W_t)_{t\geq 0}$ a standard Wiener process. The hypotheses to sequentially test are

$$H_0 : \gamma = \gamma_0 \quad Vs \quad H_1 : \gamma = \gamma_1. \tag{4.1}$$

It is well known that π_t is given by (2.10), with Y_t replaced by

$$Y_t^\gamma = \frac{\gamma_1 - \gamma_0}{\sigma^2} \left(X_t - \frac{t}{2}(\gamma_1 + \gamma_0) \right), \tag{4.2}$$

and that the infinitesimal generator $\mathbb{L}^\gamma$ of $(\pi_t)_{t\geq 0}$ is

$$(\mathbb{L}^\gamma f)(\pi) = \frac{1}{2} \frac{(\gamma_1 - \gamma_0)^2}{\sigma^2} \pi^2 (1-\pi)^2 f''(\pi). \tag{4.3}$$

One can show that the unknown value function V from (2.6) and the unknown boundaries A and B satisfy the free-boundary problem (2.17)–(2.22) (with $\mathbb{L}^\gamma$ in place of $\mathbb{L}$), as well as the smooth fit condition at B

$$V'(B) = -b. \tag{4.4}$$

For a fixed $B > c$, let $V(\pi; B)$, $\pi \in (0, B]$, be the function solving (2.17), (2.22), (4.3), and (4.4) (see Shiryaev (1978)). V is thus expressed by (2.28) and the optimal stopping boundaries $A^\star$ and $B^\star$ are the unique solution of (2.29).

If we approximate $V(\pi; B^\star)$ by $V_n(\pi; B^\star)$, as in (3.4), the problem reduces to determining the $n+1$ coefficients of $V_n(\pi; B^\star)$, $A^\star$ and $B^\star$ from the system of $n+3$ non-linear equations

$$(\mathbb{L}^\gamma V_n)(\pi_i; B^\star) = -1, \quad i = 1, \ldots, n-1, \tag{4.5}$$
$$V_n(A^\star; B^\star) = aA^\star, \tag{4.6}$$
$$V_n'(A^\star; B^\star) = a, \tag{4.7}$$
$$V_n(B^\star; B^\star) = b(1 - B^\star), \tag{4.8}$$
$$V_n'(\pi; B^\star) = -b. \tag{4.9}$$

The expressions (4.3) and (4.5) require evaluating the second derivative of the shifted Chebyshev polynomials, see (B.14) and (B.16) in Appendix B.2. The absence of jumps in the paths of X implies that the operator (4.3) does not involve integrals and this allows us to fix the $n-1$ collocation nodes π_i in the entire interval $[0, 1]$. Once (4.5)–(4.9) has been solved, the approximated value function $V_n(\pi)$ is given by (3.11).

For a numerical application, we take the example analyzed in Buonaguidi and Muliere (2013, Figures 1 and 2), where setting $a = 15$, $b = 10$, $\sigma^2 = 1$, $\gamma_0 = -2$, and $\gamma_1 = -3$, the exact values $A^\star = 0.1593\cdots$ and $B^\star = 0.7206\cdots$ are obtained. The collocation approach (4.5)–(4.9) with $n = 8$ and $n-1$ equally spaced collocation nodes in $[0.1, 0.8]$ leads to very satisfactory results: $A^\star \approx A_n^\star = 0.1606\cdots$ and $B^\star \approx B_n^\star = 0.7206\cdots$. With

$$\|V, V_n\| = \sup_{\pi \in (0,1)} \left| \frac{V_n(\pi) - V(\pi)}{V(\pi)} \right|, \tag{4.10}$$

the maximum relative distance between the exact value function V and its approximation V_n, we get $\|V, V_n\| = 9.08 \times 10^{-4}$.

4.2. Sequential testing of a Poisson process

Let $X = (X_t)_{t \geq 0}$ be a sequentially observed Poisson process with intensity $\lambda > 0$; the aim is to test

$$H_0 : \lambda = \lambda_0 \quad Vs \quad H_1 : \lambda = \lambda_1, \quad \lambda_1 > \lambda_0. \tag{4.11}$$

The posterior probability π_t takes the expression (2.10), with Y_t substituted by

$$Y_t^\lambda = \log\left(\frac{\lambda_1}{\lambda_0}\right) X_t - t(\lambda_1 - \lambda_0); \tag{4.12}$$

the infinitesimal generator of $(\pi_t)_{t\geq 0}$ is

$$\begin{aligned}(\mathbb{L}^\lambda f)(\pi) =& -(\lambda_1 - \lambda_0) f'(\pi)\pi(1-\pi) + \big(\lambda_1\pi \\ &+ \lambda_0(1-\pi)\big)\left[f\left(\frac{\lambda_1\pi}{\lambda_1\pi + \lambda_0(1-\pi)}\right) - f(\pi)\right].\end{aligned} \tag{4.13}$$

The optimal stopping problem (2.6) can be reduced to the free-boundary problem (2.17)–(2.22) (with $\mathbb{L}^\lambda$ in place of $\mathbb{L}$): its analytical solution was derived by Peskir and Shiryaev (2000).

We describe how the proposed collocation approach can be applied. Let $\pi \mapsto V(\pi; B)$, $\pi \in I_B = (0, B]$, and $B > c$, be the map solving the difference-differential equation defined by (2.17), (2.18), (2.22), and (4.13). With the "step" and "distance" functions

$$S(\pi) = \frac{\lambda_1\pi}{\lambda_1\pi + \lambda_0(1-\pi)}, \quad \pi \in I_B, \tag{4.14}$$

$$d^\lambda(\pi, B) = 1 + \left\lfloor \log\left(\frac{B}{1-B}\frac{1-\pi}{\pi}\right) \Big/ \log\left(\frac{\lambda_1}{\lambda_0}\right)\right\rfloor, \quad \pi \in I_B, \tag{4.15}$$

where $\lfloor x \rfloor$ is the integer part of x, it is not difficult to see that (2.17), (2.18), (2.22), and (4.13) imply that $V(\pi; B)$ solves (2.24)–(2.25), with $\mathbb{L}_B$ replaced by

$$\begin{aligned}(\mathbb{L}_B^\lambda f)(\pi) =& -(\lambda_1 - \lambda_0) f'(\pi)\pi(1-\pi) + \big(\lambda_1\pi + \lambda_0(1-\pi)\big) \\ &\times \left\{\left[b(1 - S(\pi))\mathbf{1}_{\{d^\lambda(\pi,B)=1\}} + f(S(\pi))\mathbf{1}_{\{d^\lambda(\pi,B)>1\}}\right] - f(\pi)\right\} + 1.\end{aligned} \tag{4.16}$$

As in Subsection 3.1, approximating $V(\pi; B)$ by $V_n(\pi; B)$ from (3.1) and solving the system (3.2)–(3.3) for the operator (4.16) and different values of $B > c$ allow us to check if the necessary and sufficient condition (2.27) for the existence of a solution to the free-boundary problem (2.17)–(2.22) is satisfied, and to individuate reasonable neighbourhoods of $A^\star$ and $B^\star$. Once this operation has been accomplished, the next step is to approximate $V(\pi; B^\star)$ by $V_n(\pi; B^\star)$ from (3.4) and solve (3.5)–(3.8) for the operator (4.16); in this way, the approximating boundaries $A_n^\star$ and $B_n^\star$ and the coefficients involved in the expression of $V_n(\pi; B^\star)$ can be computed. The approximating value function V_n takes the expression (3.11).

Consider the numerical example analyzed by Peskir and Shiryaev (2000, Figures 2 and 3), where $a = b = 2$, $\lambda_0 = 1$, and $\lambda_1 = 5$. The exact values of the optimal boundaries are $A^\star = 0.2253\cdots$ and $B^\star = 0.7050\cdots$. The first part of the above procedure leads us to $A^\star \in (0.2, 0.3)$ and $B^\star \in (0.68, 0.72)$ (we fixed $n = 8$ and solved (3.2)$-$(3.3) and (4.16) for $B = 0.65, 0.68, 0.72$ and n equally spaced collocation nodes in $[0.1, B]$). Then, we solved (3.5)$-$(3.8) and (4.16) for $n = 8$ and n equally spaced collocation nodes in $[0.1, 0.68]$. We obtained the very good approximations $A^\star \approx A_n^\star = 0.2245\cdots$, $B^\star \approx B_n^\star = 0.7048\cdots$, and $\|V, V_n\| = 2.59 \times 10^{-3}$.

4.3. Sequential testing of a compound Poisson process with exponential jumps

Let $X = (X_t)_{t\geq 0}$ be a compound Poisson process with intensity $1/\eta$, $\eta > 0$, and the distribution of its jumps negative exponential of parameter $\eta > 0$. We want to test

$$H_0 : \eta = \eta_0 \quad Vs \quad H_1 : \eta = \eta_1, \quad \eta_0 > \eta_1. \tag{4.17}$$

It is straightforward to show that π_t is given by (2.10), where Y_t is replaced by

$$Y_t^\eta = (\eta_0 - \eta_1)X_t - t\left(\frac{\eta_0 - \eta_1}{\eta_0\eta_1}\right), \tag{4.18}$$

and the infinitesimal generator of $(\pi_t)_{t\geq 0}$ is

$$\begin{aligned}(\mathbb{L}^\eta f)(\pi) = &- f'(\pi)\pi(1-\pi)\frac{\eta_0 - \eta_1}{\eta_0\eta_1} - f(\pi)\left(\frac{\pi}{\eta_1} + \frac{1-\pi}{\eta_0}\right) \\ &+ \int_0^\infty f\left(\frac{\pi e^{-\eta_1 x}}{\pi e^{-\eta_1 x} + (1-\pi)e^{-\eta_0 x}}\right)\left(\pi e^{-\eta_1 x} + (1-\pi)e^{-\eta_0 x}\right)dx.\end{aligned} \tag{4.19}$$

The optimal stopping problem (2.6) can be reduced to the free-boundary problem (2.17)$-$(2.22) (with $\mathbb{L}$ replaced by $\mathbb{L}^\eta$); its solution was obtained by Gapeev (2002).

We see that $V(\pi; B)$, $\pi \in I_B$, the solution of (2.17), (2.18), (2.22) and (4.19), satisfies (2.24)$-$(2.25), where (2.24) is defined through the operator

$$\begin{aligned}(\mathbb{L}_B^\eta f)(\pi) = &- \frac{\eta_0 - \eta_1}{\eta_0\eta_1} f'(\pi)\pi(1-\pi) \\ &- f(\pi)\left(\frac{\pi}{\eta_1} + \frac{1-\pi}{\eta_0}\right) + \frac{b(1-\pi)}{\eta_0}\left(\frac{1-\pi}{\pi}\frac{B}{1-B}\right)^{-\eta_0/(\eta_0-\eta_1)} \\ &+ \int_0^{d^\eta(\pi,B)} f\left(S^\eta(\pi, x)\right)\left(\pi e^{-\eta_1 x} + (1-\pi)e^{-\eta_0 x}\right)dx + 1,\end{aligned} \tag{4.20}$$

with $S^{\eta}(\pi, x)$ and $d^{\eta}(\pi, B)$ given by

$$S^{\eta}(\pi, x) = \frac{\pi e^{-\eta_1 x}}{(1-\pi)e^{-\eta_0 x} + \pi e^{-\eta_1 x}}, \quad \pi \in I_B, \quad x \geq 0, \tag{4.21}$$

$$d^{\eta}(\pi; B) = \log\left(\frac{1-\pi}{\pi}\frac{B}{1-B}\right) \Big/ (\eta_0 - \eta_1), \quad \pi \in I_B. \tag{4.22}$$

The arguments of Section 3 and Subsection 4.2 can be used to derive approximations of the value function V and the boundaries $A^\star$ and $B^\star$.

For a numerical example, we set $a = b = 1$, $\eta_0 = 0.5$, and $\eta_1 = 0.1$. The exact boundaries are $A^\star = 0.1632\cdots$ and $B^\star = 0.7455\cdots$. The solutions of (3.2)$-$(3.3) and (4.20) for $B = 0.68, 0.72, 0.76$, $n = 8$ and n equally spaced collocation nodes in $[0.1, B]$ allow us to fix $A^\star \in (0.1, 0.2)$ and $B^\star \in (0.72, 0.76)$. Very good approximations are then obtained as solution of (3.5)$-$(3.8) and (4.20), for $n = 8$ and n equally spaced collocation nodes in $[0.1, 0.72]$: $A^\star \approx A_n^\star = 0.1639\cdots$, $B^\star \approx B_n^\star = 0.7456\cdots$, and $\|V, V_n\| = 4.59 \times 10^{-4}$.

4.4. Sequential testing of a negative binomial process

Let $X = (X_t)_{t\geq 0}$ be a negative binomial process with parameter $0 < p < 1$, so X has independent and stationary increments and the probability that $X_t = x$ is

$$\frac{\Gamma(x+t)}{\Gamma(x+1)\Gamma(t)} p^t (1-p)^x, \quad x = 0, 1, 2, \ldots. \tag{4.23}$$

The posterior probability π_t for the sequential testing of the two simple hypotheses

$$H_0 : p = p_0 \quad Vs \quad H_1 : p = p_1, \quad p_0 > p_1, \tag{4.24}$$

is provided by (2.10), with Y_t replaced by

$$Y_t^p = \log\left(\frac{q_1}{q_0}\right) X_t - t \log\left(\frac{p_0}{p_1}\right), \tag{4.25}$$

where $q_i = 1 - p_i$, $i = 0, 1$. The infinitesimal operator of $(\pi_t)_{t\geq 0}$ takes the form

$$\begin{aligned}\big(\mathbb{L}^p f\big)(\pi) = {} & \log\left(\frac{p_1}{p_0}\right) f'(\pi)\pi(1-\pi) + f(\pi)\big((1-\pi)\log p_0 + \pi \log p_1\big) \\ & + \sum_{x=1}^{\infty} f\left(\frac{\pi q_1^x}{\pi q_1^x + (1-\pi)q_0^x}\right) \frac{\pi q_1^x + (1-\pi)q_0^x}{x}.\end{aligned} \tag{4.26}$$

In this case, the optimal stopping problem (2.6) can be reduced to the free-boundary problem (2.17)$-$(2.22) (with $\mathbb{L}^p$ in place of $\mathbb{L}$); its explicit solution was derived by Buonaguidi and Muliere (2013).

Let $V(\pi; B)$, $\pi \in I_B$, be the map solving (2.17), (2.18), (2.22) and (4.26), and take the "step" and "distance" functions

$$S^p(\pi, x) = \frac{\pi q_1^x}{\pi q_1^x + (1-\pi) q_0^x}, \quad \pi \in I_B, \quad x = 1, 2, \ldots, \tag{4.27}$$

$$d^p(\pi, B) = 1 + \left\lfloor \log\left(\frac{B}{1-B} \frac{1-\pi}{\pi} \right) \Big/ \log\left(\frac{q_1}{q_0} \right) \right\rfloor, \quad \pi \in I_B. \tag{4.28}$$

It is not difficult to verify that $V(\pi; B)$ must solve (2.24)$-$(2.25) for the operator

$$\begin{aligned}(\mathbb{L}_B^p f)(\pi) = &\log\left(\frac{p_1}{p_0}\right) f'(\pi)\pi(1-\pi) + f(\pi)\big((1-\pi)\log p_0 + \pi \log p_1\big) \\ &+ \sum_{x=1}^{d^p(\pi,B)-1} \left(f\left(S^p(\pi; x)\right) \frac{\pi q_1^x + (1-\pi) q_0^x}{x} \right) \\ &- b(1-\pi)\Big(\log p_0 + \sum_{x=1}^{d^p(\pi,B)-1} \frac{q_0^x}{x} \Big) + 1. \end{aligned} \tag{4.29}$$

Repeating step by step the procedure of Section 3 and Subsection 4.2, approximations of V, $A^\star$, and $B^\star$ can be easily computed.

For a numerical illustration, take the example in Buonaguidi and Muliere (2013, Figures 3 and 4), where $a = b = 8$, $p_0 = 0.8$, and $p_1 = 0.3$. The exact stopping boundaries are $A^\star = 0.2004\cdots$ and $B^\star = 0.7142\cdots$. Fixing $B = 0.64, 0.68, 0.72$, $n = 8$, and n equally spaced nodes in $[0.1, B]$, solving (3.2)$-$(3.3) and (4.29) shows that $A^\star \in (0.2, 0.3)$ and $B^\star \in (0.68, 0.72)$. The solution of the system (3.5)$-$(3.8) and (4.29), for $n = 8$ and n equally spaced collocation nodes in $[0.1, 0.68]$, leads to satisfactory results: $A^\star \approx A_n^\star = 0.2004\cdots$, $B^\star \approx B_n^\star = 0.7174\cdots$, and $\|V, V_n\| = 4.92 \times 10^{-3}$.

5. Conclusions

We considered the sequential testing of two simple hypotheses for a Lévy gamma process. Our study was an attempt to extend the existing literature on sequential testing to processes with infinite jump activity on finite time intervals.

We approached the problem from a probabilistic-analytic view point, showing such properties of the value function as the smoothness and/or continuity at the stopping boundaries, and we constructed the free-boundary problem that the value function and the boundaries must satisfy. Then, we verified that if the free-boundary problem admits a solution, it is unique and coincides with that of the original optimal stopping problem.

Since deriving an explicit solution of the free-boundary problem was very hard, we proposed a numerical collocation approach. The value function was approximated by a linear combination of Chebyshev polynomials; we showed that its coefficients and the two stopping boundaries can be determined as solution of a system of non-linear equations, obtained by forcing the linear combination to solve a complex integro-differential equation, at fixed and properly chosen collocation nodes, and satisfying the boundary conditions in accordance with the smooth and continuous fit principles. The performances of our approximation method were evaluated in explicitly solved sequential testing problems where we obtained very good approximations of the exact solutions.

The presented collocation approach can be adapted to other optimal stopping problems (like sequential detection and optimal prediction problems) whose solutions are difficult to determine.

Acknowledgements

The authors express their gratitude to an associate editor and the referee for constructive comments that improved the presentation of the paper.

References

Boyd, J. P. and Petschek, R. (2014). The relationships between Chebyshev, Legendre and Jacobi polynomials: the generic superiority of Chebyshev polynomials and three important exceptions. *J. Scientific Computing* **59**, 1-27.

Brunner, H. (2004). *Collocation Methods for Volterra Integral and Related Functional Equations.* Cambridge University Press, Cambridge.

Buonaguidi, B. and Muliere, P. (2013). Sequential testing problems for Lévy processes. *Sequential Anal.* **32**, 47-70.

Dayanik, S., Poor, H. and Sezer, S. O. (2008). Sequential multi-hypothesis testing for compound Poisson processes. *Stochastics* **80**, 19-50.

Dayanik, S. and Sezer, S. O. (2006). Sequential testing of simple hypotheses about compound Poisson processes. *Stochastic Process. Appl.* **116**, 1892-1919.

Dayanik, S. and Sezer, S. O. (2012). Multisource Bayesian sequential binary hypothesis testing problem. *Ann. Oper. Res.* **201**, 99-130.

Dufresne, F., Gerber, H. U. and Shiu, E. S. W. (1991). Risk theory with the Gamma process. *Astin Bull.* **21**, 177-192.

Dvoretzky, A., Kiefer, J. and Wolfowitz, J. (1953). Sequential decision problems for processes with continuous time parameter. Testing hypotheses. *Ann. Math. Statist.* **24**, 254-264.

Gapeev, P. V. (2002). Problems of the sequential discrimination of hypotheses for a compound Poisson process with exponential jumps. *Uspekhi Matematicheskikh Nauk* **57**, 171-172.

Gapeev, P. V. and Peskir, G. (2004). The Wiener sequential testing problem with finite horizon. *Stochastics and Stochastic Reports* **76**, 59-75.

Gapeev, P. V. and Shiryaev, A. N. (2011). On the sequential testing problem for some diffusion processes. *Stochastics* **83**, 519-535.

Hamming, R. W. (1986). *Numerical Methods for Scientist and Engineers.* Dover, New York.

James, L. F., Roynette, B. and Yor, M. (2008). Generalized gamma convolutions, Dirichlet means, Thorin measures, with explicit examples. *Probab. Surveys* **5**, 346-415.

Kress, R. (1998). *Numerical Analysis*. Springer-Verlag, New York.

Kyprianou, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, Berlin.

Lanczos, C. (1988). *Applied Analysis*. Dover, New York.

Lawless, J. and Crowder, M. (2004). Covariates and random effects in a Gamma process model with application to degradation and failure. *Lifetime Data Anal.* **10**, 213-227.

Linz, P. (1985). *Analytical and Numerical Methods for Volterra Equations*. SIAM, Philadelphia.

Ludkovski, M. and Sezer, S. O. (2012). Finite horizon decision timing with partially observable Poisson processes. *Stochastic Models* **28**, 207-247.

Park, C. and Padgett, W. J. (2005). Accelerated degradation models for failure based on geometric Brownian motion and gamma processes. *Lifetime Data Anal.* **11**, 511-527.

Peskir, G. and Shiryaev, A. N. (2000). Sequential testing problems for Poisson processes. *Ann. Statist.* **28**, 837-859.

Peskir, G. and Shiryaev, A. N. (2006). *Optimal Stopping and Free-Boundary Problems*. Birkhäuser Verlag, Basel.

Sato, K. I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge.

Shiryaev, A. N. (1978). *Optimal Stopping Rules*. Springer-Verlag, New York.

Shiryaev, A. N. and Zhitlukhin, M. (2011). A Bayesian sequential testing problem of three hypotheses for Brownian motion. *Statistics and Risk Modeling* **28**, 227-249.

Van Noortwijk, J. M. (2009). A survey of the application of Gamma processes in maintenance. *Reliability Engineering* & *System Safety* **94**, 2-21.

Yor, M. (2007). Some Remarkable Properties of the Gamma Process. In *Advances in Mathematical Finance* (Edited by M. Fu, R. Jarrow, and R. Elliott), 37-47. Birkhäuser, Boston.

Department of Decision Sciences, Bocconi University, Via Roentgen 1, 20136 Milan, Italy.

E-mail: bruno.buonaguidi@phd.unibocconi.it

Department of Decision Sciences, Bocconi University, Via Roentgen 1, 20136 Milan, Italy.

E-mail: pietro.muliere@unibocconi.it

(Received June 2013; accepted October 2014)