Expectation of the Limiting Distribution of the LSE of a Unit Root Process

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Supplementary Material

This supplementary material gives the detailed proofs of Lemma 1 in the paper: Expectation of the Limiting Distribution of the LSE of a Unit Root Process.

Proof of Lemma 1. Obviously, when $|j-k| = d+1$, $\omega^j + \omega^k = 0$. Define $\boldsymbol{c}_A = [c_0, c_2, c_4, \ldots, c_{2d}]'$, $\boldsymbol{c}_B = [c_1, c_3, c_5, \ldots, c_{2d+1}]'$. The matrix equation $\tilde{M}\boldsymbol{c} = 0$ can be rewritten as $\tilde{M}_1\boldsymbol{c}_A + \tilde{M}_2\boldsymbol{c}_B = 0$, where

$$\tilde{M}_1 = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2d} \\ 1 & (\omega^2)^2 & (\omega^4)^2 & \cdots & (\omega^{2d})^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (\omega^2)^d & (\omega^4)^d & \cdots & (\omega^{2d})^d \end{pmatrix}, \tilde{M}_2 = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \omega & \omega^3 & \omega^5 & \cdots & \omega^{2d+1} \\ \omega^2 & (\omega^3)^2 & (\omega^5)^2 & \cdots & (\omega^{2d+1})^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^d & (\omega^3)^d & (\omega^5)^d & \cdots & (\omega^{2d+1})^d \end{pmatrix}.$$

Thus, we obtain that $\boldsymbol{c}_A = -\tilde{M}_1^{-1}\tilde{M}_2\boldsymbol{c}_B$. Also, it is easily seen that $\tilde{M}_2 = diag(1, \omega, \ldots, \omega^d)\tilde{M}_1$. Since $\tilde{M}_1$ is a discrete Fourier matrix, the entries of its inverse $\tilde{M}_1^{-1}$ can be computed as

$$(\tilde{M}_1^{-1})_{ij} = \frac{1}{d+1}\omega^{-(i-1)(j-1)}, \quad i, j = 1, \ldots, d+1.$$

Let $Y = \tilde{M}_1^{-1}\tilde{M}_2$, direct computation gives

$$\begin{aligned} Y_{ij} &= \frac{2}{d+1} \cdot \frac{1}{1-\omega^{2j-2i+1}}, \\ (Y^{-1})_{ij} &= \frac{2}{d+1} \cdot \frac{1}{1-\omega^{2j-2i-1}}, \quad i, j = 1, \ldots, d+1. \end{aligned}$$

We formulate the quadratic forms (1) - (3) in Lemma 1 in terms of matrices and prove them as follows:

(1)

$$\sum_{\substack{j,k=0\\|j-k|\neq d+1}}^{2d+1} \frac{c_j c_k}{\omega^j+\omega^k} = \begin{bmatrix} \boldsymbol{c}_A' & \boldsymbol{c}_B' \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_A \\ \boldsymbol{c}_B \end{bmatrix}$$

$$= \boldsymbol{c}_A' A_{11} \boldsymbol{c}_A + \boldsymbol{c}_B' A_{21} \boldsymbol{c}_A + \boldsymbol{c}_A' A_{12} \boldsymbol{c}_B + \boldsymbol{c}_B' A_{22} \boldsymbol{c}_B$$

$$= \boldsymbol{c}_B' \left[Y' A_{11} Y - A_{21} Y - Y' A_{12} + A_{22} \right] \boldsymbol{c}_B$$

where matrices A_{11}, A_{12}, A_{21} and A_{22} are different depending on whether d is even or odd. When d is even,

$$(A_{11})_{ij} = \frac{1}{\omega^{2(i-1)} + \omega^{2(j-1)}},$$
$$(A_{22})_{ij} = \frac{1}{\omega^{2i-1} + \omega^{2j-1}},$$
$$(A_{12})_{ij} = \begin{cases} 0, & \text{if } |2i-2j-1| = d+1, \\ \frac{1}{\omega^{2(i-1)} + \omega^{2j-1}}, & \text{o.w.} \end{cases}$$
$$(A_{21})_{ij} = (A_{12})_{ji}$$

When d is odd,

$$(A_{11})_{ij} = \begin{cases} 0, & \text{if } |i-j| = (d+1)/2, \\ \frac{1}{\omega^{2(i-1)} + \omega^{2(j-1)}}, & \text{o.w.} \end{cases}$$
$$(A_{22})_{ij} = \begin{cases} 0, & \text{if } |i-j| = (d+1)/2, \\ \frac{1}{\omega^{2i-1} + \omega^{2j-1}}, & \text{o.w.} \end{cases}$$
$$(A_{12})_{ij} = \frac{1}{\omega^{2(i-1)} + \omega^{2j-1}},$$
$$(A_{21})_{ij} = (A_{12})_{ji}$$

From Proposition 1 and Proposition 2 below, we have that

$$Y' A_{11} Y + A_{22} - A_{21} Y - Y' A_{12} = 0$$

and therefore $\displaystyle\sum_{\substack{j,k=0\\|j-k|\neq d+1}}^{2d+1} \frac{c_j c_k}{\omega^j+\omega^k} = 0.$

(2) The proof of (2) is similar to the proof of (1).

$$\sum_{\substack{j,k=0\\|j-k|\neq d+1}}^{2d+1} (-1)^{j+k} \frac{c_j c_k}{\omega_j+\omega_k} = [\boldsymbol{c}_A' \boldsymbol{c}_B'] \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_A \\ \boldsymbol{c}_B \end{bmatrix}$$

$$= \boldsymbol{c}_A' A_{11} \boldsymbol{c}_A - \boldsymbol{c}_B' A_{21} \boldsymbol{c}_A - \boldsymbol{c}_A' A_{12} \boldsymbol{c}_B + \boldsymbol{c}_B' A_{22} \boldsymbol{c}_B$$

$$= \boldsymbol{c}_B' \left[Y' A_{11} Y + A_{21} Y + Y' A_{12} + A_{22} \right] \boldsymbol{c}_B$$

By Proposition 1 and Proposition 2 below,

$$Y'A_{11}Y + A_{21}Y + Y'A_{12} + A_{22} = 0.$$

Hence,

$$\sum_{\substack{j,k=0 \\ |j-k|\neq d+1}}^{2d+1} (-1)^{j+k} \frac{c_j c_k}{\omega_j + \omega_k} = 0.$$

(3) Using Proposition 3, we have

$$\begin{aligned}\sum_{j,k=0}^{2d+1} (-1)^{j+k} c_j c_k &= \begin{bmatrix} \boldsymbol{c}_A' & \boldsymbol{c}_B' \end{bmatrix} \begin{bmatrix} J & -J \\ -J & J \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_A \\ \boldsymbol{c}_B \end{bmatrix} \\ &= \boldsymbol{c}_A' J \boldsymbol{c}_A - \boldsymbol{c}_B' J \boldsymbol{c}_A - \boldsymbol{c}_A' J \boldsymbol{c}_B + \boldsymbol{c}_B' J \boldsymbol{c}_B \\ &= \boldsymbol{c}_B' \left[Y'JY + JY + Y'J + J \right] \boldsymbol{c}_B \\ &= 4\boldsymbol{c}_B' J \boldsymbol{c}_B,\end{aligned}$$

where J is the $(d+1) \times (d+1)$ all one matrix.
When d is even,

$$\begin{aligned}(2d+2) \sum_{|j-k|=d+1} (-1)^{j+k} c_j c_k &= (2d+2) \begin{bmatrix} \boldsymbol{c}_A' & \boldsymbol{c}_B' \end{bmatrix} \begin{bmatrix} 0 & G \\ G' & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_A \\ \boldsymbol{c}_B \end{bmatrix} \\ &= (2d+2)(\boldsymbol{c}_B' G' \boldsymbol{c}_A + \boldsymbol{c}_A' G \boldsymbol{c}_B) \\ &= (2d+2)\boldsymbol{c}_B' \left[G'Y + Y'G \right] \boldsymbol{c}_B\end{aligned}$$

where matrix G is defined as

$$G_{ij} = \begin{cases} 1, & \text{if } |i - j - \frac{1}{2}| = \frac{1}{2}(d+1), \\ 0, & o.w \end{cases}$$

Matrix multiplication of Y' and G yields,

$$(Y'G)_{ij} = \frac{2}{d+1} \cdot \frac{1}{1 - \omega^{2i-2j+d+1}} = \frac{2}{d+1} \cdot \frac{1}{1 + \omega^{2i-2j}}$$

and

$$(G'Y)_{ij} = \frac{2}{d+1} \cdot \frac{1}{1 + \omega^{2j-2i}}.$$

For any i and j, $(Y'G)_{ij}$ and $(G'Y)_{ij}$ are conjugate of each other and their real parts are both $\frac{1}{d+1}$. Thus, $(2d+2)(G'Y + Y'G) = 4J$.
When d is odd,

$$\begin{aligned}(2d+2) \sum_{|j-k|=d+1} (-1)^{j+k} c_j c_k &= (2d+2) \begin{bmatrix} \boldsymbol{c}_A' & \boldsymbol{c}_B' \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_A \\ \boldsymbol{c}_B \end{bmatrix} \\ &= (2d+2)(\boldsymbol{c}_A' H \boldsymbol{c}_A + \boldsymbol{c}_B' H \boldsymbol{c}_B) \\ &= (2d+2)\boldsymbol{c}_B' \left[Y'HY + H \right] \boldsymbol{c}_B\end{aligned}$$

where H is defined as

$$H_{ij} = \begin{cases} 1, & \text{if } |i-j| = \frac{1}{2}(d+1), \\ 0, & o.w \end{cases}$$

By simple computation

$$(Y'H)_{ij} = \frac{2}{d+1}\cdot\frac{1}{1-\omega^{2i-2j-d}} = \frac{2}{d+1}\cdot\frac{1}{1+\omega^{2i-2j+1}}$$
$$(HY^{-1})_{ij} = \frac{2}{d+1}\cdot\frac{1}{1-\omega^{2j-2i-d-2}} = \frac{2}{d+1}\cdot\frac{1}{1+\omega^{2j-2i-1}}$$

Since $(Y'H)_{ij}$ and $(HY^{-1})_{ij}$ are conjugate of each other and their real parts are both $\frac{1}{d+1}$, $(Y'H)_{ij} + (HY^{-1})_{ij} = \frac{2}{d+1}$. Thus, $(2d+2)(Y'H + HY^{-1}) = 4J$ and by Proposition 3, $(2d+2)(Y'HY + H) = (2d+2)(Y'H + HY^{-1})Y = 4JY = 4J$. Combining the cases of d being even and odd, we have proved that

$$\sum_{j,k=0}^{2d+1}(-1)^{j+k}c_jc_k - (2d+2)\sum_{|j-k|=d+1}(-1)^{j+k}c_jc_k = 0.$$

for all positive integer d.

Proposition 1 $A_{21}Y + Y'A_{12} = 0$.

Proof. Since the matrices A_{21} and A_{12} are different depending on d being even or odd, we divide our proof into two cases. When d is even, the ij-th entry of $A_{21}Y + Y'A_{12}$ is

$$\sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\frac{1}{\omega^{2j-1}+\omega^{2(k-1)}}\cdot\frac{1}{1-\omega^{2i-2k+1}} + \sum_{\substack{k=1,\\|2i-2k+1|\neq d+1}}^{d+1}\frac{1}{\omega^{2i-1}+\omega^{2(k-1)}}\cdot\frac{1}{1-\omega^{2j-2k+1}}$$

Multiplying each summand by $\omega^{2i-1}+\omega^{2j-1}$, and decomposing each of them using partial fraction, we obtain

$$\begin{aligned}
&(\omega^{2i-1}+\omega^{2j-1})\sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\frac{1}{\omega^{2j-1}+\omega^{2(k-1)}}\cdot\frac{1}{1-\omega^{2i-2k+1}}\\
&= \sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\frac{\omega^{2k-1}(\omega^{2i}+\omega^{2j})}{(\omega^{2k-1}+\omega^{2j})(\omega^{2k-1}-\omega^{2i})}\\
&= \sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\left(\frac{\omega^{2k-1}}{\omega^{2k-1}-\omega^{2i}} - \frac{\omega^{2k-1}}{\omega^{2k-1}+\omega^{2j}}\right)\\
&= \sum_{\substack{k=1,\\|2j-2k+1|\neq d+1}}^{d+1}\left(\frac{1}{1-\omega^{2i-2k+1}} - \frac{1}{1+\omega^{2j-2k+1}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&(\omega^{2i-1}+\omega^{2j-1}) \sum_{\substack{k=1,\\ |2i-2k+1|\neq d+1}}^{d+1} \frac{1}{\omega^{2i-1}+\omega^{2(k-1)}} \cdot \frac{1}{1-\omega^{2j-2k+1}} \\
=& \sum_{\substack{k=1,\\ |2i-2k+1|\neq d+1}}^{d+1} \frac{\omega^{2k-1}(\omega^{2i}+\omega^{2j})}{(\omega^{2k-1}-\omega^{2j})(\omega^{2k-1}+\omega^{2i})} \\
=& \sum_{\substack{k=1,\\ |2i-2k+1|\neq d+1}}^{d+1} \left(\frac{\omega^{2k-1}}{\omega^{2k-1}-\omega^{2j}} - \frac{\omega^{2k-1}}{\omega^{2k-1}+\omega^{2i}}\right) \\
=& \sum_{\substack{k=1,\\ |2i-2k+1|\neq d+1}}^{d+1} \left(\frac{1}{1-\omega^{2j-2k+1}} - \frac{1}{1+\omega^{2i-2k+1}}\right)
\end{aligned}$$

$|2j-2k+1| \neq d+1$ is equivalent to $k \neq j-d/2$ and $k \neq j+d/2+1$. Since $1 \leq k \leq d+1$, then for $1 \leq j \leq d/2$, $k \neq j-d/2$, and for $d/2+1 \leq j \leq d+1$, $k \neq j+d/2+1$. Thus,

$$\begin{aligned}
&\sum_{\substack{k=1,\\ |2j-2k+1|\neq d+1}}^{d+1} \frac{1}{1-\omega^{2i-2k+1}} + \sum_{\substack{k=1,\\ |2i-2k+1|\neq d+1}}^{d+1} \frac{1}{1-\omega^{2j-2k+1}} \\
=&2\sum_{k=1}^{d+1} \frac{1}{1-\omega^{2i-2k+1}} - \frac{1_{\{1\leq j\leq d/2\}}}{1-\omega^{2i-2j+d+1}} - \frac{1_{\{d/2+1\leq j\leq d+1\}}}{1-\omega^{2i-2j-d-1}} - \frac{1_{\{1\leq i\leq d/2\}}}{1-\omega^{2j-2i+d+1}} - \frac{1_{\{d/2+1\leq i\leq d+1\}}}{1-\omega^{2j-2i-d-1}} \\
=&2\sum_{k=1}^{d+1} \frac{1}{1-\omega^{2i-2k+1}} - \frac{1_{\{1\leq j\leq d/2\}}}{1+\omega^{2i-2j}} - \frac{1_{\{d/2+1\leq j\leq d+1\}}}{1+\omega^{2i-2j}} - \frac{1_{\{1\leq i\leq d/2\}}}{1+\omega^{2j-2i}} - \frac{1_{\{d/2+1\leq i\leq d+1\}}}{1+\omega^{2j-2i}}
\end{aligned}$$

Noting the fact that

$$\Re\left(\frac{1}{1\pm\omega^k}\right) = \frac{1}{2} \quad \text{for any integer } k,$$

and ω^{2i-2j} is the conjugate of ω^{2j-2i}. By symmetry, we have

$$\begin{aligned}
&\sum_{\substack{k=1,\\ |2j-2k+1|\neq d+1}}^{d+1} \frac{1}{1-\omega^{2i-2k+1}} + \sum_{\substack{k=1,\\ |2i-2k+1|\neq d+1}}^{d+1} \frac{1}{1-\omega^{2j-2k+1}} \\
=&2\sum_{k=1}^{d+1} \frac{1}{1-\omega^{2i-2k+1}} - \frac{1_{\{1\leq j\leq d/2\}}}{1+\omega^{2i-2j}} - \frac{1_{\{d/2+1\leq j\leq d+1\}}}{1+\omega^{2i-2j}} - \frac{1_{\{1\leq i\leq d/2\}}}{1+\omega^{2j-2i}} - \frac{1_{\{d/2+1\leq i\leq d+1\}}}{1+\omega^{2j-2i}} \\
=&d+1-2\cdot\frac{1}{2} = d.
\end{aligned}$$

Similarly,

$$\sum_{\substack{k=1,\\ |2i-2k+1|\neq d+1}}^{d+1} \frac{1}{1+\omega^{2i-2k+1}} + \sum_{\substack{k=1,\\ |2j-2k+1|\neq d+1}}^{d+1} \frac{1}{1+\omega^{2j-2k+1}} = d\cdot\frac{1}{2} + d\cdot\frac{1}{2} = d.$$

Thus,

$$\begin{aligned}
&\sum_{\substack{k=1,\\ |2j-2k+1|\neq d+1}}^{d+1} \frac{1}{\omega^{2j-1}+\omega^{2(k-1)}}\cdot\frac{1}{1-\omega^{2i-2k+1}} + \sum_{\substack{k=1,\\ |2i-2k+1|\neq d+1}}^{d+1} \frac{1}{\omega^{2i-1}+\omega^{2(k-1)}}\cdot\frac{1}{1-\omega^{2j-2k+1}}\\
=&\frac{1}{\omega^{2i-1}+\omega^{2j-1}}\left(\sum_{\substack{k=1,\\ |2j-2k+1|\neq d+1}}^{d+1}\left(\frac{1}{1-\omega^{2i-2k+1}}-\frac{1}{1+\omega^{2j-2k+1}}\right)\right.\\
&\qquad\left.+\sum_{\substack{k=1,\\ |2i-2k+1|\neq d+1}}^{d+1}\left(\frac{1}{1-\omega^{2j-2k+1}}-\frac{1}{1+\omega^{2i-2k+1}}\right)\right)=0.
\end{aligned}$$

When d is odd, the ij-th entry of $A_{21}Y+Y'A_{12}$ is

$$\sum_{k=1}^{d+1}\frac{1}{\omega^{2i-1}+\omega^{2(k-1)}}\cdot\frac{1}{1-\omega^{2j-2k+1}}+\sum_{k=1}^{d+1}\frac{1}{\omega^{2(k-1)}+\omega^{2j-1}}\cdot\frac{1}{1-\omega^{2i-2k+1}}.$$

By symmetry,

$$\sum_{k=1}^{d+1}\left(\frac{1}{1-\omega^{2i-2k+1}}+\frac{1}{1-\omega^{2j-2k+1}}\right)=\sum_{k=1}^{d+1}\left(\frac{1}{1+\omega^{2i-2k+1}}+\frac{1}{1+\omega^{2j-2k+1}}\right)=d+1.$$

Then

$$\begin{aligned}
&\sum_{k=1}^{d+1}\frac{1}{\omega^{2i-1}+\omega^{2(k-1)}}\cdot\frac{1}{1-\omega^{2j-2k+1}}+\sum_{k=1}^{d+1}\frac{1}{\omega^{2(k-1)}+\omega^{2j-1}}\cdot\frac{1}{1-\omega^{2i-2k+1}}\\
=&\frac{1}{\omega^{2i-1}+\omega^{2j-1}}\left(\sum_{k=1}^{d+1}\left(\frac{1}{1-\omega^{2j-2k+1}}-\frac{1}{1+\omega^{2i-2k+1}}\right)+\sum_{k=1}^{d+1}\left(\frac{1}{1-\omega^{2i-2k+1}}-\frac{1}{1+\omega^{2j-2k+1}}\right)\right)\\
=&\frac{1}{\omega^{2i-1}+\omega^{2j-1}}\left(\sum_{k=1}^{d+1}\left(\frac{1}{1-\omega^{2j-2k+1}}+\frac{1}{1-\omega^{2i-2k+1}}\right)-\sum_{k=1}^{d+1}\left(\frac{1}{1+\omega^{2i-2k+1}}+\frac{1}{1+\omega^{2j-2k+1}}\right)\right)\\
=&0.
\end{aligned}$$

Proposition 2 $Y'A_{11}+A_{22}Y^{-1}=0.$

Proof. When d is even, the ij-th entry of $Y'A_{11} + A_{22}Y^{-1}$ is

$$\sum_{k=1}^{d+1} \frac{1}{\omega^{2(j-1)} + \omega^{2(k-1)}} \cdot \frac{1}{1-\omega^{2i-2k+1}} + \sum_{k=1}^{d+1} \frac{1}{\omega^{2i-1} + \omega^{2k-1}} \cdot \frac{1}{1-\omega^{2j-2k-1}}$$

By symmetry,

$$\sum_{k=1}^{d+1} \left(\frac{1}{1-\omega^{2i-2k+1}} + \frac{1}{1-\omega^{2j-2k-1}} \right) = \sum_{k=1}^{d+1} \left(\frac{1}{1+\omega^{2i-2k}} + \frac{1}{1+\omega^{2j-2k}} \right) = d+1.$$

Then

$$\begin{aligned}
&\sum_{k=1}^{d+1} \frac{1}{\omega^{2(j-1)} + \omega^{2(k-1)}} \cdot \frac{1}{1-\omega^{2i-2k+1}} + \sum_{k=1}^{d+1} \frac{1}{\omega^{2i-1} + \omega^{2k-1}} \cdot \frac{1}{1-\omega^{2j-2k-1}} \\
=&\frac{1}{\omega^{2i-1} + \omega^{2(j-1)}} \left(\sum_{k=1}^{d+1} \left(\frac{1}{1-\omega^{2i-2k+1}} - \frac{1}{1+\omega^{2j-2k}} \right) + \sum_{k=1}^{d+1} \left(\frac{1}{1-\omega^{2j-2k-1}} - \frac{1}{1+\omega^{2i-2k}} \right) \right) \\
=&\frac{1}{\omega^{2i-1} + \omega^{2(j-1)}} \left(\sum_{k=1}^{d+1} \left(\frac{1}{1-\omega^{2i-2k+1}} + \frac{1}{1-\omega^{2j-2k-1}} \right) - \sum_{k=1}^{d+1} \left(\frac{1}{1+\omega^{2i-2k}} + \frac{1}{1+\omega^{2j-2k}} \right) \right) \\
=&0.
\end{aligned}$$

When d is odd, the ij-th entry of $Y'A_{11} + A_{22}Y^{-1}$ is

$$\sum_{\substack{k=1, \\ |k-j| \neq (d+1)/2}}^{d+1} \frac{1}{\omega^{2(k-1)} + \omega^{2(j-1)}} \cdot \frac{1}{1-\omega^{2i-2k+1}} + \sum_{\substack{k=1, \\ |i-k| \neq (d+1)/2}}^{d+1} \frac{1}{\omega^{2i-1} + \omega^{2k-1}} \cdot \frac{1}{1-\omega^{2j-2k-1}}.$$

By symmetry,

$$\sum_{\substack{k=1, \\ |k-j| \neq (d+1)/2}}^{d+1} \left(\frac{1}{1-\omega^{2i-2k+1}} + \frac{1}{1-\omega^{2j-2k-1}} \right) = \sum_{\substack{k=1, \\ |i-k| \neq (d+1)/2}}^{d+1} \left(\frac{1}{1+\omega^{2i-2k}} + \frac{1}{1+\omega^{2j-2k}} \right).$$

Then,

$$\begin{aligned}
&\sum_{\substack{k=1, \\ |k-j| \neq (d+1)/2}}^{d+1} \frac{1}{\omega^{2(k-1)} + \omega^{2(j-1)}} \cdot \frac{1}{1-\omega^{2i-2k+1}} + \sum_{\substack{k=1, \\ |i-k| \neq (d+1)/2}}^{d+1} \frac{1}{\omega^{2i-1} + \omega^{2k-1}} \cdot \frac{1}{1-\omega^{2j-2k-1}} \\
=&\frac{1}{\omega^{2i-1} + \omega^{2j-2}} \left(\sum_{\substack{k=1, \\ |k-j| \neq (d+1)/2}}^{d+1} \left(\frac{1}{1-\omega^{2i-2k+1}} - \frac{1}{1+\omega^{2j-2k)}} \right) \right. \\
&\qquad \left. + \sum_{\substack{k=1, \\ |i-k| \neq (d+1)/2}}^{d+1} \left(\frac{1}{1-\omega^{2j-2k-1}} - \frac{1}{1+\omega^{2i-2k}} \right) \right) = 0.
\end{aligned}$$

Thus, $Y'A_{11} + A_{22}Y^{-1} = 0$.

Proposition 3

$$YJ = J, Y'J = J$$

Proof.

$$(YJ)_{ij} = \frac{2}{d+1}\sum_{k=1}^{d+1}\frac{1}{1-\omega^{2k-2i+1}} = \frac{2}{d+1}\sum_{k=1}^{d+1}\frac{1}{1-\omega^{2k-1}} = 1,$$

and

$$(Y'J)_{ij} = \frac{2}{d+1}\sum_{k=1}^{d+1}\frac{1}{1-\omega^{2i-2k+1}} = \frac{2}{d+1}\sum_{k=1}^{d+1}\frac{1}{1-\omega^{2k-1}} = 1.$$