Statistica Sinica: Supplement

Estimation of the Error Autocorrelation Matrix in Semiparametric Model for fMRI Data

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Supplementary Material

Summary

The supplementary material includes the Appendix of the paper. It includes

I. the conditions for theoretical analysis,

II. some notation used in the proofs,

III. proofs of Lemmas 1–9,

IV. proofs of the main theoretical results (Theorems 1 and 2, Propositions 1–4) in the paper.

Appendix: Conditions and Proofs

In this part, we will give the conditions and proofs of the main results in this paper.

Condition A.

A1. In model (2.1), $\epsilon(t_i) = \sum_{j=-\infty}^{\infty} \phi_{n;j} w_{i-j}$, where $\phi_{n;0} = 1$; for any $n \ge 1$, there exists $1 \le g_n \le n-1$ and $\alpha_n > 4$, such that $|\phi_{n;j}| \le C$ for $|j| \le g_n/2$ and $|\phi_{n;j}| \le C|2j|^{-\alpha_n}$ for $|j| > g_n/2$, with a constant $C > 0$; $\{w_i\}$ is a sequence of independent white noises with $E(w_i) = 0$, $E(w_i^2) = \sigma_w^2$, and $\sup_i E(w_i^4) < \infty$.

A2. Suppose $\lambda_{\min}(\Sigma_1) > C > 0$, where Σ_1 is an $m\ell \times m\ell$ matrix consisting of $\ell \times \ell$ blocks, i.e., $\Sigma_1 = (\Sigma_{i,j})_{i,j=1}^{\ell}$. $\Sigma_{i,j}$ is an $m \times m$ matrix defined as $\Sigma_{i,j}(u,v) = \operatorname{cov}\{\boldsymbol{z}_{v,n-1}^T(\mathbf{D}_1\mathbf{S}_i)\boldsymbol{z}_{1,m}, \boldsymbol{z}_{u,n-1}^T(\mathbf{D}_1\mathbf{S}_j)\boldsymbol{z}_{1,m}\}$, for $1 \le u,v \le m$, $1 \le i,j \le \ell$, and $\boldsymbol{z}_{p,q}$ is the pth column of the $q \times q$ identity matrix.

A3. The second derivative of the drift function $d(t)$ is continuous and bounded, i.e., $|d''(t)| \le C$.

A4. In model (2.1), $\{s_i(\cdot)\}$, $i = 1, \ldots, \ell$, are independent of $\{\epsilon(\cdot)\}$. For each $1 \le i \le \ell$, $\{s_i(\cdot)\}$ is a stationary g_s-dependent time series, where $g_s > 0$ is a fixed integer, and $E\{s_i^4(t)\} \le C < \infty$. Furthermore, $\{s_1(t_u), \ldots, s_\ell(t_u)\}$ and $\{s_1(t_v), \ldots, s_\ell(t_v)\}$ are independent if $|u-v| > g_s$. When $|u-v| \le g_s$, $E\{s_i(t_u)s_j(t_v)\}$ depends on u and v only through $u-v$, for any $1 \le i,j \le \ell$.

A5. Suppose $t_i = i/n$, $i = 1, \ldots, n$.

A6. Assume $0 < c \le \lambda_{\min}(\mathbf{R}) \le \lambda_{\max}(\mathbf{R}) \le C$, where c and C are constants.

Condition B.

B1. In model (2.1), $\boldsymbol{\epsilon}$ is a stationary g_n-dependent process with $E\{\epsilon(t_i)\} = 0$, $c \le \operatorname{var}\{\epsilon(t_i)\} \le Cg_n$ and $[E\{\epsilon(t_i)^4\}]^{1/2} = O(\gamma_e(0))$, where $1 \le g_n \le n-1$.

Notation. Now, we will give some notation that will be used in the proofs.

1. Define $\boldsymbol{\epsilon}_1 = \mathbf{D}_1\mathbf{d} + \mathbf{D}_1\boldsymbol{\epsilon}$. Then, $\epsilon_1(t_i) = \epsilon(t_i) - \epsilon(t_{i-1}) + d(t_i) - d(t_{i-1})$, where $\epsilon_1(t_i) = \boldsymbol{z}_{i-1,n-1}^T\boldsymbol{\epsilon}_1$, for $i = 2, \ldots, n$.

2. Define $\mathbf{e}_0 = \mathbf{D}_2\boldsymbol{\epsilon}$, $\mathbf{d}_0 = \mathbf{D}_2\mathbf{d}$ and $\boldsymbol{\delta} = \mathbf{D}_2\mathbf{S}(\mathbf{h} - \widehat{\mathbf{h}}_{\text{DBE}})$. Then, $\mathbf{e} = \mathbf{e}_0 + \mathbf{d}_0$ and $\widehat{\mathbf{e}} = \mathbf{e} + \boldsymbol{\delta}$. Also, $e_0(t_i) = \epsilon(t_i) - 2\epsilon(t_{i-1}) + \epsilon(t_{i-2})$, $d_0(t_i) = d(t_i) - 2d(t_{i-1}) + d(t_{i-2})$ and $e(t_i) = e_0(t_i) + d_0(t_i)$, where $e_0(t_i) = \boldsymbol{z}_{i-2,n-2}^T\mathbf{e}_0$, $d_0(t_i) = \boldsymbol{z}_{i-2,n-2}^T\mathbf{d}_0$ and $\delta(t_i) = \boldsymbol{z}_{i-2,n-2}^T\boldsymbol{\delta}$, for $i = 3, \ldots, n$.

3. For a matrix Z, denote by $Z(i,j)$ the entry of Z in the ith row and jth column.

Proof. We will present and prove Lemmas 1–9, which will be needed in the proofs of the main results.

Lemma 1 *For the* $(k+1)\times(k+1)$ *matrix* A_k *and* $(k+1)\times 2$ *matrix* B_k *in* (2.5) *with* $k \geq 1$,

$$
\begin{array}{ll}
\text{(i)} & \|A_k^{-1}\|_1 \leq Ck^4, \\
\text{(ii)} & \|\boldsymbol{z}_{1,k+1}^T A_k^{-1}\|_\infty \leq Ck^3, \\
\text{(iii)} & \|A_k^{-1}B_k\|_1 \leq Ck^2,\ \|\boldsymbol{z}_{1,k+1}^T A_k^{-1}B_k\|_\infty \leq Ck,
\end{array}
$$

where $\boldsymbol{z}_{p,q}$ *is the pth column of the* $q\times q$ *identity matrix.*

Proof: Let G, E, H and K be $(k+1)\times(k+1)$ matrices defined as follows:

$$
G = \begin{pmatrix}
6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6
\end{pmatrix}_{(k+1)\times(k+1)},
$$

$$
E = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},\
H = \begin{pmatrix}
-3 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},\
K = \begin{pmatrix}
2 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
$$

Then, $A_k = K(G+E+H)$. To prove part (i), it suffices to show that $G+E+H$ is positive definite for any $k \geq 1$, and $\|(G+E+H)^{-1}\|_1 = O(k^4)$, since $\|A_k^{-1}\|_1 \leq \|(G+E+H)^{-1}\|_1\|K^{-1}\|_1 = \|(G+E+H)^{-1}\|_1$.

First, we will show that $G+E+H$ is positive definite for any $k \geq 2$, as the result is obvious for $k=1$. From Theorem 2 of Hoskins and Ponzo (1972), G is positive definite. Since E is positive semidefinite, $G+E$ is positive definite. By the particular form of matrix H, $\det(G+E+H) > 0$ is a necessary and sufficient condition for $G+E+H$ to be positive definite. We can express $G+E+H$ as a block matrix in the following way:

$$
G+E+H = \begin{pmatrix} J_1 & J_2 \\ J_2^T & J_3 \end{pmatrix},
$$

where

$$
J_1 = \begin{pmatrix} 3 & -4 \\ -4 & 7 \end{pmatrix},\quad J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -4 & 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{2\times(k-1)},
$$

$$J_3 = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 \end{pmatrix}_{(k-1)\times(k-1)}.$$

Then,

$$G+E+H = \begin{pmatrix} \mathbf{I}_{2\times 2} & J_2 \\ \mathbf{0}_{(k-1)\times 2} & J_3 \end{pmatrix} \begin{pmatrix} J_1 - J_2 J_3^{-1} J_2^T & \mathbf{0}_{2\times(k-1)} \\ J_3^{-1} J_2^T & \mathbf{I}_{(k-1)\times(k-1)} \end{pmatrix},$$

so $\det(G+E+H) = \det(J_3)\det(J_1 - J_2 J_3^{-1} J_2^T)$. Due to Theorem 2 in Hoskins and Ponzo (1972), $\det(J_3) > 0$. Now, we only need to show $\det(J_1 - J_2 J_3^{-1} J_2^T) > 0$.

Let $x_{i,j} = \boldsymbol{z}_{i,k-1}^T J_3^{-1} \boldsymbol{z}_{j,k-1}$. From Theorem 5 of Hoskins and Ponzo (1972), $x_{1,1} = (k-1)k/\{(k+1)(k+2)\}$, $x_{2,1} = x_{1,2} = 2(k-1)(k-2)/\{(k+1)(k+2)\}$ and $x_{2,2} = (k-1)(k-2)(5k-6)/\{k(k+1)(k+2)\}$. By direct calculation,

$$J_1 - J_2 J_3^{-1} J_2^T = \begin{pmatrix} 3-(k-1)k/\{(k+1)(k+2)\} & -4+2(k-1)/(k+1) \\ -4+2(k-1)/(k+1) & 7-(5k+6)(k-1)/\{k(k+1)\} \end{pmatrix}.$$

Thus, $\det(J_1 - J_2 J_3^{-1} J_2^T) = 12(2k+3)/\{k(k+1)^2(k+2)\} > 0$, which implies that $\det(G+E+H) > 0$ and hence $G+E+H$ is positive definite.

Next, we will show $\|(G+E+H)^{-1}\|_1 = O(k^4)$ as $k \to \infty$.

Since the ranks of E and H are both 1, from Miller (1981), $(G+E)^{-1} = G^{-1} - \nu_1 G^{-1} E G^{-1}$ and $(G+E+H)^{-1} = (G+E)^{-1} - \nu_2 (G+E)^{-1} H (G+E)^{-1}$, where $\nu_1 = \{1+\mathrm{tr}(G^{-1}E)\}^{-1}$ and $\nu_2 = [1+\mathrm{tr}\{(G+E)^{-1}H\}]^{-1}$. Therefore,

$$\boldsymbol{z}_{i,k+1}^T (G+E)^{-1} \boldsymbol{z}_{j,k+1} = a_{i,j} - \nu_1 a_{i,2} a_{2,j}, \tag{S.1}$$

$$\begin{aligned} \boldsymbol{z}_{i,k+1}^T (G+E+H)^{-1} \boldsymbol{z}_{j,k+1} &= (a_{i,j} - \nu_1 a_{i,2} a_{2,j}) \\ &\quad + 3\nu_2 (a_{i,1} - \nu_1 a_{i,2} a_{2,1})(a_{1,j} - \nu_1 a_{1,2} a_{2,j}), \end{aligned} \tag{S.2}$$

where $a_{i,j} = \boldsymbol{z}_{i,k+1}^T G^{-1} \boldsymbol{z}_{j,k+1}$, for $1 \le i,j \le k+1$.

From Theorem 5 of Hoskins and Ponzo (1972), for $i = 1, \ldots, k+1$,

$$a_{i,1} = \frac{i(k+2-i)(k+3-i)}{(k+3)(k+4)}, \tag{S.3}$$

$$a_{i,2} = \frac{(k+2-i)(k+3-i)\{i(3k+4)-(k+4)\}}{(k+2)(k+3)(k+4)}. \tag{S.4}$$

Direct calculation leads to

$$\nu_1 = \{1+\mathrm{tr}(G^{-1}E)\}^{-1} = (1+a_{2,2})^{-1} \xrightarrow{k\to\infty} 1/6, \tag{S.5}$$

$$\begin{aligned} \nu_2 &= [1+\mathrm{tr}\{(G+E)^{-1}H\}]^{-1} = [1-3\{a_{1,1} - a_{1,2}^2/(1+a_{2,2})\}]^{-1} \\ &= O(k^3/4). \end{aligned} \tag{S.6}$$

From (S.3) and (S.4), for any $i = 1, \ldots, k+1$, we have $0 < a_{i,1}, a_{i,2} = O(k)$ and

$$\begin{aligned} a_{i,1} - \nu_1 a_{i,2} a_{2,1} &= \frac{(k+2-i)(k+3-i)}{(k+3)(k+4)} \frac{4k^2 i + 22ki + 24i + 2k^3 + 10k^2 + 8k}{6k^3 + 18k^2 + 30k + 24} \\ &< 2. \end{aligned} \tag{S.7}$$

Since G^{-1} is symmetric, we can immediately get $0 < a_{1,j} - \nu_1 a_{1,2} a_{2,j} < 2$ for any $j = 1, \ldots, k+1$.

By Theorem 3 of Hoskins and Ponzo (1972),

$$\begin{aligned} \sum_{i=1}^{k+1} a_{i,1} &= \sum_{i=1}^{k+1} |a_{i,1}| = \frac{(k+1)(k+2)}{12}, \\ \sum_{i=1}^{k+1} a_{i,2} &= \sum_{i=1}^{k+1} |a_{i,2}| = \frac{k(k+1)}{4}. \end{aligned} \tag{S.8}$$

Theorem 4 of Hoskins and Ponzo (1972) indicates that $\|G^{-1}\|_1 = O(k^4)$, which together with (S.1)–(S.8) implies,

$$\begin{aligned} & \|(G+E+H)^{-1}\|_1 \\ = & \max_{1 \le j \le k+1} \sum_{i=1}^{k+1} |(a_{i,j} - \nu_1 a_{i,2} a_{2,j}) + 3\nu_2 (a_{i,1} - \nu_1 a_{i,2} a_{2,1})(a_{1,j} - \nu_1 a_{1,2} a_{2,j})| \\ \le & \|G^{-1}\|_1 + \nu_1 \max_{1 \le j \le k+1} |a_{2,j}| \sum_{i=1}^{k+1} |a_{i,2}| \\ & +3\nu_2 \max_{1 \le j \le k+1} |a_{1,j} - \nu_1 a_{1,2} a_{2,j}| \sum_{i=1}^{k+1} |a_{i,1} - \nu_1 a_{i,2} a_{2,1}| \\ = & O(k^4) + O(1)O(k)O(k^2) + O(k^3)O(1)O(k) = O(k^4). \end{aligned}$$

For part (ii), by (S.1)–(S.8),

$$\begin{aligned} & \|\boldsymbol{z}_{1,k+1}^T A_k^{-1}\|_\infty \le \|\boldsymbol{z}_{1,k+1}^T (G+E+H)^{-1}\|_\infty \|K^{-1}\|_\infty = \|(G+E+H)^{-1} \boldsymbol{z}_{1,k+1}\|_\infty \\ = & \max_{1 \le i \le k+1} |(a_{i,1} - \nu_1 a_{i,2} a_{2,1}) + 3\nu_2 (a_{i,1} - \nu_1 a_{i,2} a_{2,1})(a_{1,1} - \nu_1 a_{1,2} a_{2,1})| \\ \le & \max_{1 \le i \le k+1} |a_{i,1}| + \nu_1 |a_{2,1}| \max_{1 \le i \le k+1} |a_{i,2}| + 3\nu_2 |a_{1,1} - \nu_1 a_{1,2} a_{2,1}| \max_{1 \le i \le k+1} |a_{i,1} - \nu_1 a_{i,2} a_{2,1}| \\ = & O(k) + O(1)O(1)O(k) + O(k^3)O(1)O(1) = O(k^3). \end{aligned}$$

For part (iii), we only consider $k > 3$, as the result for $k \le 3$ is obvious.

Since $A_k^{-1} \boldsymbol{z}_{k+1,k+1} = (G+E+H)^{-1} K^{-1} \boldsymbol{z}_{k+1,k+1} = (G+E+H)^{-1} \boldsymbol{z}_{k+1,k+1}$, from (S.2),

$$\begin{aligned} & \|A_k^{-1} \boldsymbol{z}_{k+1,k+1}\|_1 = \|(G+E+H)^{-1} \boldsymbol{z}_{k+1,k+1}\|_1 \\ \le & \sum_{i=1}^{k+1} |a_{i,k+1}| + \nu_1 |a_{2,k+1}| \sum_{i=1}^{k+1} |a_{i,2}| + 3\nu_2 |a_{1,k+1} - \nu_1 a_{1,2} a_{2,k+1}| \sum_{i=1}^{k+1} |a_{i,1} - \nu_1 a_{i,2} a_{2,1}| \end{aligned}$$

$$\equiv \quad \mathrm{I}+\mathrm{II}+\mathrm{III}. \tag{S.9}$$

Theorem 3 in Hoskins and Ponzo (1972) implies that $\mathrm{I} = O(k^2)$.

From Theorem 5 in Hoskins and Ponzo (1972), for $i = 1, \ldots, k+1$,

$$a_{i,k+1} = i(i+1)(k-i+2)/\{(k+3)(k+4)\}, \tag{S.10}$$

which together with (S.5) and (S.8) implies that $\mathrm{II} = O(1)O(k^{-1})O(k^2) = O(k)$.

By (S.3), (S.4), (S.5) and (S.10),

$$a_{1,k+1} - \nu_1 a_{1,2} a_{2,k+1} = \frac{2(k+1)}{(k+3)(k+4)} - \frac{a_{1,2}}{1+a_{2,2}} \frac{6k}{(k+3)(k+4)} = O(k^{-2}). \tag{S.11}$$

From (S.6), (S.7) and (S.11), $\mathrm{III} = O(k^3)O(k^{-2})O(k) = O(k^2)$. Therefore, by (S.9), $\|A_k^{-1}\boldsymbol{z}_{k+1,k+1}\|_1 = O(k^2)$. Similar arguments reveal that $\|A_k^{-1}\boldsymbol{z}_{k,k+1}\|_1 = O(k^2)$. Since, for $k > 3$, $B_k = [\boldsymbol{z}_{k,k+1} - 4\boldsymbol{z}_{k+1,k+1} \ \boldsymbol{z}_{k+1,k+1}]$,

$$\begin{aligned} & \|A_k^{-1}B_k\|_1 \le \|A_k^{-1}\boldsymbol{z}_{k,k+1} - 4A_k^{-1}\boldsymbol{z}_{k+1,k+1}\|_1 + \|A_k^{-1}\boldsymbol{z}_{k+1,k+1}\|_1 \\ \le \quad & \|A_k^{-1}\boldsymbol{z}_{k,k+1}\|_1 + 4\|A_k^{-1}\boldsymbol{z}_{k+1,k+1}\|_1 + \|A_k^{-1}\boldsymbol{z}_{k+1,k+1}\|_1 = O(k^2). \end{aligned}$$

From (S.2), (S.7) and (S.11),

$$\begin{aligned} & |\boldsymbol{z}_{1,k+1}^T A_k^{-1} \boldsymbol{z}_{k+1,k+1}| \\ = \quad & |(a_{1,k+1} - \nu_1 a_{1,2} a_{2,k+1}) + 3\nu_2 (a_{1,1} - \nu_1 a_{1,2} a_{2,1})(a_{1,k+1} - \nu_1 a_{1,2} a_{2,k+1})| \\ = \quad & O(k^{-2}) + O(k^3)O(k^{-2}) = O(k). \end{aligned}$$

Similarly, we can show $|\boldsymbol{z}_{1,k+1}^T A_k^{-1} \boldsymbol{z}_{k,k+1}| = O(k)$. Therefore,

$$\begin{aligned} & \|\boldsymbol{z}_{1,k+1}^T A_k^{-1} B_k\|_\infty \le |\boldsymbol{z}_{1,k+1}^T A_k^{-1} \boldsymbol{z}_{k,k+1} - 4\boldsymbol{z}_{1,k+1}^T A_k^{-1} \boldsymbol{z}_{k+1,k+1}| + |\boldsymbol{z}_{1,k+1}^T A_k^{-1} \boldsymbol{z}_{k+1,k+1}| \\ \le \quad & |\boldsymbol{z}_{1,k+1}^T A_k^{-1} \boldsymbol{z}_{k,k+1}| + 4|\boldsymbol{z}_{1,k+1}^T A_k^{-1} \boldsymbol{z}_{k+1,k+1}| + |\boldsymbol{z}_{1,k+1}^T A_k^{-1} \boldsymbol{z}_{k+1,k+1}| = O(k). \end{aligned}$$

Now we complete the proof. ■

Lemma 2 *Under Condition* A1 (*or* B1)*, for any* $k \in \{0, 1, \ldots, g_n\}$ *and* $\tau_{0;n} > 0$ *that satisfies* $\tau_{0;n}^{-1} = o(n/g_n^2)$ *as* $n \to \infty$,

$$\mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n-k}\{e_0(t_i)e_0(t_{i+k})\} - \gamma_e(k)\Big| \ge \tau_{0;n}\Big) \le C\frac{g_n^3}{n\tau_{0;n}^2},$$

where $e_0(t_i) = \epsilon(t_i) - 2\epsilon(t_{i-1}) + \epsilon(t_{i-2})$ *as in Notation* 2.

Proof: First, we will give the proof under Condition A1. Since $\epsilon(t_i) = \sum_{j=-\infty}^{\infty} \phi_{n;j} w_{i-j}$, we have $e_0(t_i) = \sum_{j=-\infty}^{\infty} \psi_j w_{i-j}$, where $\psi_j \equiv \psi_{n;j} = \phi_{n;j} - 2\phi_{n;j-1} + \phi_{n;j-2}$. Define $\widetilde{e}_0(t_i) = \sum_{j=-g_n-2}^{g_n+2} \psi_j w_{i-j}$ and $\widetilde{\gamma}_e(k) = \mathrm{cov}\{\widetilde{e}_0(t_i), \widetilde{e}_0(t_{i+k})\}$. Then,

$$\Big|\frac{1}{n}\sum_{i=3}^{n-k}\{e_0(t_i)e_0(t_{i+k})\} - \gamma_e(k)\Big| \le \frac{1}{n}\Big|\sum_{i=3}^{n-k}\{e_0(t_i)e_0(t_{i+k}) - \gamma_e(k)\}\Big| + (k+2)|\gamma_e(k)|/n$$

$$
\begin{aligned}
&\le \frac{1}{n}\Big|\sum_{i=3}^{n-k}\{e_0(t_i)e_0(t_{i+k})-\gamma_e(k)-\widetilde{e}_0(t_i)\widetilde{e}_0(t_{i+k})+\widetilde{\gamma}_e(k)\}\Big| \\
&\quad +\frac{1}{n}\Big|\sum_{i=3}^{n-k}\{\widetilde{e}_0(t_i)\widetilde{e}_0(t_{i+k})-\widetilde{\gamma}_e(k)\}\Big| + (k+2)|\gamma_e(k)|/n \\
&\equiv \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned} \tag{S.12}
$$

Since $\tau_{0;n}^{-1} = o(n/g_n^2)$ as $n \to \infty$ and $|\gamma_e(k)| \le \gamma_e(0) = O(g_n)$, there exists a constant L_0, such that, for any $n > L_0$ and $k \in \{0, 1, \ldots, g_n\}$, $\mathrm{III} < \tau_{0;n}/3$. From (S.12), for $n > L_0$,

$$
\mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n-k}\{e_0(t_i)e_0(t_{i+k})\}-\gamma_e(k)\Big| \ge \tau_{0;n}\Big) \le \mathrm{P}(\mathrm{I} \ge \tau_{0;n}/3) + \mathrm{P}(\mathrm{II} \ge \tau_{0;n}/3). \tag{S.13}
$$

From Lemma 3,

$$
\mathrm{P}(\mathrm{II} \ge \tau_{0;n}/3) = O(g_n^3/(n\tau_{0;n}^2)). \tag{S.14}
$$

Therefore, we only need to consider term I. By Markov inequality,

$$
\begin{aligned}
&\mathrm{P}(\mathrm{I} \ge \tau_{0;n}/3) \\
\le\ & \Big(\frac{3}{n\tau_{0;n}}\Big)^2 E\Big[\Big|\sum_{i=3}^{n-k}\{e_0(t_i)e_0(t_{i+k})-\gamma_e(k)-\widetilde{e}_0(t_i)\widetilde{e}_0(t_{i+k})+\widetilde{\gamma}_e(k)\}\Big|^2\Big].
\end{aligned} \tag{S.15}
$$

For any fixed $n > L_0$ and $k \in \{0, 1, \ldots, g_n\}$, let $Q_i = e_0(t_i)e_0(t_{i+k}) - \widetilde{e}_0(t_i)\widetilde{e}_0(t_{i+k})$. Define $D_{i,u} = E(Q_i \mid w_{i-u}, w_{i-u+1}, \ldots) - E(Q_i \mid w_{i-u+1}, w_{i-u+2}, \ldots)$, for $u = 0, \pm 1, \pm 2, \ldots$. It holds that $\sum_{u=-\infty}^{\infty} D_{i,u} = Q_i - E(Q_i) = e_0(t_i)e_0(t_{i+k}) - \gamma_e(k) - \widetilde{e}_0(t_i)\widetilde{e}_0(t_{i+k}) + \widetilde{\gamma}_e(k)$ almost surely. For any u, $\{D_{i,u}\}_{i=n-k}^{3}$ is a martingale difference sequence w.r.t. $\mathcal{F}_{i,u} = \sigma\{w_{i-u}, w_{i-u+1}, \ldots\}$, i.e., $E(D_{i,u} \mid \mathcal{F}_{i+1,u}) = 0$ for $i = n-k, \ldots, 3$. In the following, define $\psi_j^* = \psi_j$ for $|j| \le g_n + 2$ and $\psi_j^* = 0$ for $|j| > g_n + 2$. Direct calculation leads to,

$$
\begin{aligned}
&E(D_{i,u}^2) \\
=\ & E\Big\{\Big(\psi_{u+k}w_{i-u}\sum_{j=-\infty}^{u-1}\psi_j w_{i-j} + \psi_u w_{i-u}\sum_{j=-\infty}^{k+u-1}\psi_j w_{i+k-j} + \psi_{u+k}\psi_u w_{i-u}^2 - \psi_{u+k}\psi_u\sigma_w^2 \\
&-\psi_{u+k}^* w_{i-u}\sum_{j=-\infty}^{u-1}\psi_j^* w_{i-j} - \psi_u^* w_{i-u}\sum_{j=-\infty}^{k+u-1}\psi_j^* w_{i+k-j} - \psi_{u+k}^*\psi_u^* w_{i-u}^2 + \psi_{u+k}^*\psi_u^*\sigma_w^2\Big)^2\Big\} \\
=\ & O(g_n(\psi_u^2 + \psi_{u+k}^2)),
\end{aligned}
$$

which together with Lemma 7 of Xiao and Wu (2012) implies

$$
E\Big\{\Big(\sum_{i=3}^{n-k} D_{i,u}\Big)^2\Big\} \le \sum_{i=3}^{n-k} E(D_{i,u}^2) = O(ng_n(\psi_u^2 + \psi_{u+k}^2)). \tag{S.16}
$$

By Minkowski inequality and (S.16), for the expectation term in (S.15),

$$
E\Big[\Big|\sum_{i=3}^{n-k}\{e_0(t_i)e_0(t_{i+k})-\gamma_e(k)-\widetilde{e}_0(t_i)\widetilde{e}_0(t_{i+k})+\widetilde{\gamma}_e(k)\}\Big|^2\Big]
$$

$$= E\Big[\Big|\sum_{i=3}^{n-k}\sum_{u=-\infty}^{\infty} D_{i,u}\Big|^2\Big] \le \Big[\sum_{u=-\infty}^{\infty}\Big\{E\Big(\Big|\sum_{i=3}^{n-k} D_{i,u}\Big|^2\Big)\Big\}^{1/2}\Big]^2 = O(ng_n^3).$$

From (S.15),

$$\mathrm{P}(\mathrm{I} \ge \tau_{0;n}/3) = O(g_n^3/(n\tau_{0;n}^2)). \tag{S.17}$$

Combining (S.13), (S.14) and (S.17), we finish the proof for $n > L_0$. It is easy to see that the result is true for $n \le L_0$.

Next, we will prove the Lemma under Condition B1. For any $k = 0, 1, \ldots, g_n$,

$$\begin{aligned}
&\mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n-k}\{e_0(t_i)e_0(t_{i+k})\} - \gamma_e(k)\Big| \ge \tau_{0;n}\Big)\\
=\ &\mathrm{P}\Big(\Big|\frac{1}{4n}\Big[\sum_{i=3}^{n-k}\{e_0(t_i)+e_0(t_{i+k})\}^2 - \sum_{i=3}^{n-k}\{e_0(t_i)-e_0(t_{i+k})\}^2\Big]\\
&-\frac{1}{4}[\{2\gamma_e(0)+2\gamma_e(k)\} - \{2\gamma_e(0)-2\gamma_e(k)\}]\Big| \ge \tau_{0;n}\Big)\\
\le\ &\mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n-k}\{e_0(t_i)+e_0(t_{i+k})\}^2 - \{2\gamma_e(0)+2\gamma_e(k)\}\Big| \ge 2\tau_{0;n}\Big)\\
&+\mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n-k}\{e_0(t_i)-e_0(t_{i+k})\}^2 - \{2\gamma_e(0)-2\gamma_e(k)\}\Big| \ge 2\tau_{0;n}\Big)\\
\le\ &\mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n-k} e_1^2(t_i) - \{1+\rho_e(k)\}\Big| \ge \frac{\tau_{0;n}}{\gamma_e(0)}\Big)\\
&+\mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n-k} e_2^2(t_i) - \{1-\rho_e(k)\}\Big| \ge \frac{\tau_{0;n}}{\gamma_e(0)}\Big) \equiv \mathrm{IV} + \mathrm{V},
\end{aligned}$$

where $e_1(t_i) = \{e_0(t_i)+e_0(t_{i+k})\}/\sqrt{2\gamma_e(0)}$ and $e_2(t_i) = \{e_0(t_i)-e_0(t_{i+k})\}/\sqrt{2\gamma_e(0)}$. Then, $\{e_1(t_i)\}$ and $\{e_2(t_i)\}$ are $(2g_n+2)$-dependent time series with mean zero.

We divide $\{e_1(t_i)\}_{i=3}^{n-k}$ into consecutive blocks with length $2g_n+2$, i.e. $\{e_1(t_3),\ldots,e_1(t_{2g_n+4})\}$, $\{e_1(t_{2g_n+5}),\ldots,e_1(t_{4g_n+6})\},\ldots$. So, there are $q_n = \lceil (n-k-2)/(2g_n+2)\rceil$ blocks, where $\lceil\cdot\rceil$ denotes the ceiling function. The length of the last block is less than $2g_n+2$, if $2g_n+2$ is not a divisor of $n-k-2$. Denote the sum of $[e_1^2(t_i) - \{1+\rho_e(k)\}]$ within these blocks by $b_1,\ldots,b_{q_n}$. For example, $b_1 = \sum_{i=3}^{2g_n+4}[e_1^2(t_i) - \{1+\rho_e(k)\}]$. Then, $E(b_j) = 0$ for $j = 1,\ldots,q_n$. $\{b_1, b_3, b_5, \ldots\}$ are independent, so are $\{b_2, b_4, b_6, \ldots\}$. By Cauchy-Schwarz inequality and Condition B1, we have

$$E(b_1^2) \le (2g_n+2)\sum_{i=3}^{2g_n+4} E[\{e_1^2(t_i) - (1+\rho_e(k))\}^2] \le C(2g_n+2)^2.$$

Similarly, we can show that

$$E(b_j^2) \le C(2g_n+2)^2 \qquad \text{for any } j = 1,\ldots,q_n.$$

Since $\tau_{0;n}^{-1} = o(n/g_n^2)$, there exists a constant $L_1 > 0$, such that for any $n > L_1$ and $k \in \{0, \ldots, g_n\}$,

$$(k+2)\{1+\rho_e(k)\} \le (g_n+2)\{1+\rho_e(k)\} \le 2(g_n+2) \le \frac{n\tau_{0;n}}{2\gamma_e(0)}.$$

Then, due to Markov inequality, for $n > L_1$, we have

$$\begin{aligned}
\mathrm{IV} &\le \mathrm{P}\Big(\Big|\sum_{i=3}^{n-k}[e_1^2(t_i) - \{1+\rho_e(k)\}]\Big| + (k+2)\{1+\rho_e(k)\} \ge \frac{n\tau_{0;n}}{\gamma_e(0)}\Big) \\
&\le \mathrm{P}\Big(\Big|\sum_{i=3}^{n-k}[e_1^2(t_i) - \{1+\rho_e(k)\}]\Big| \ge \frac{n\tau_{0;n}}{2\gamma_e(0)}\Big) \\
&\le \mathrm{P}\Big(\Big|\sum_{j=1,3,5,\ldots} b_j\Big| + \Big|\sum_{j=2,4,6,\ldots} b_j\Big| \ge \frac{n\tau_{0;n}}{2\gamma_e(0)}\Big) \\
&\le \mathrm{P}\Big(\Big|\sum_{j=1,3,5,\ldots} b_j\Big| \ge \frac{n\tau_{0;n}}{4\gamma_e(0)}\Big) + P\Big(\Big|\sum_{j=2,4,6,\ldots} b_j\Big| \ge \frac{n\tau_{0;n}}{4\gamma_e(0)}\Big) \\
&\le \Big\{\frac{4\gamma_e(0)}{n\tau_{0;n}}\Big\}^2 E\Big\{\Big(\sum_{j=1,3,5,\ldots} b_j\Big)^2\Big\} + \Big\{\frac{4\gamma_e(0)}{n\tau_{0;n}}\Big\}^2 E\Big\{\Big(\sum_{j=2,4,6,\ldots} b_j\Big)^2\Big\} \\
&= \Big\{\frac{4\gamma_e(0)}{n\tau_{0;n}}\Big\}^2 \Big\{\sum_{j=1}^{g_n} E(b_j^2)\Big\} = O(g_n^3/(n\tau_{0;n}^2)).
\end{aligned}$$

Similarly we can show, $\mathrm{V} = O(g_n^3/(n\tau_{0;n}^2))$. Hence, we finish the proof. ■

Lemma 3 *Under Condition* A1*, for any positive sequence* $\tau_{0;n}$ *and* $k \in \{0, 1, \ldots, g_n\}$,

$$\mathrm{P}\Big(\frac{1}{n}\Big|\sum_{i=3}^{n-k}\{\widetilde{e}_0(t_i)\widetilde{e}_0(t_{i+k}) - \widetilde{\gamma}_e(k)\}\Big| \ge \tau_{0;n}\Big) \le C\frac{g_n^3}{n\tau_{0;n}^2},$$

where $\widetilde{e}_0(t_i) = \sum_{j=-g_n-2}^{g_n+2} \psi_j w_{i-j}$, $\psi_j \equiv \psi_{n;j} = \phi_{n;j} - 2\phi_{n;j-1} + \phi_{n;j-2}$ *and* $\widetilde{\gamma}_e(k) = \mathrm{cov}\{\widetilde{e}_0(t_i), \widetilde{e}_0(t_{i+k})\}$.

Proof: Since $\widetilde{e}_0(t_i) = \sum_{j=-g_n-2}^{g_n+2} \psi_j w_{i-j}$, $\{\widetilde{e}_0(t_i)\}$ is $(2g_n+4)$-dependent. For any $k = 0, 1, \ldots, g_n$,

$$\begin{aligned}
&\mathrm{P}\Big(\frac{1}{n}\Big|\sum_{i=3}^{n-k}\{\widetilde{e}_0(t_i)\widetilde{e}_0(t_{i+k}) - \widetilde{\gamma}_e(k)\}\Big| \ge \tau_{0;n}\Big) \\
= \; & \mathrm{P}\Big(\Big|\sum_{i=3}^{n-k}[\{\widetilde{e}_0(t_i) + \widetilde{e}_0(t_{i+k})\}^2 - \{2\widetilde{\gamma}_e(0) + 2\widetilde{\gamma}_e(k)\}] \\
&\quad - \sum_{i=3}^{n-k}[\{\widetilde{e}_0(t_i) - \widetilde{e}_0(t_{i+k})\}^2 - \{2\widetilde{\gamma}_e(0) - 2\widetilde{\gamma}_e(k)\}]\Big| \ge 4n\tau_{0;n}\Big) \\
\le \; & \mathrm{P}\Big(\Big|\sum_{i=3}^{n-k}[\widetilde{e}_1^2(t_i) - \{1+\widetilde{\rho}_e(k)\}]\Big| \ge \frac{n\tau_{0;n}}{\widetilde{\gamma}_e(0)}\Big)
\end{aligned}$$

$$+\mathrm{P}\Big(\Big|\sum_{i=3}^{n-k}[\widetilde{e}_2^2(t_i)-\{1-\widetilde{\rho}_e(k)\}]\Big|\geq\frac{n\tau_{0;n}}{\widetilde{\gamma}_e(0)}\Big),\tag{S.18}$$

where $\widetilde{\rho}_e(k)=\widetilde{\gamma}_e(k)/\widetilde{\gamma}_e(0)$, $\widetilde{e}_1(t_i)=\{\widetilde{e}_0(t_i)+\widetilde{e}_0(t_{i+k})\}/\{2\widetilde{\gamma}_e(0)\}^{1/2}$ and $\widetilde{e}_2(t_i)=\{\widetilde{e}_0(t_i)-\widetilde{e}_0(t_{i+k})\}/\{2\widetilde{\gamma}_e(0)\}^{1/2}$. Then, $\{\widetilde{e}_1(t_i)\}$ and $\{\widetilde{e}_2(t_i)\}$ are $(3g_n+4)$-dependent with mean zero.

Divide $\{\widetilde{e}_1(t_i)\}_{i=3}^{n-k}$ into non-overlapped consecutive blocks with length $3g_n+4$, i.e., $\{\widetilde{e}_1(t_3),\ldots,\widetilde{e}_1(t_{3g_n+6})\}$, $\{\widetilde{e}_1(t_{3g_n+7}),\ldots,\widetilde{e}_1(t_{6g_n+10})\},\ldots$. There are $q_n=\lceil(n-k-2)/(3g_n+4)\rceil$ blocks, where $\lceil\cdot\rceil$ denotes the ceiling function. The length of the last block is less than $3g_n+4$, if $3g_n+4$ is not a divisor of $n-k-2$. Denote by b_j the sum of $[\widetilde{e}_1^2(t_i)-\{1+\widetilde{\rho}_e(k)\}]$ within the jth block, for $j=1,\ldots,q_n$. For example, $b_1=\sum_{i=3}^{3g_n+6}[\widetilde{e}_1^2(t_i)-\{1+\widetilde{\rho}_e(k)\}]$. Then, $E(b_j)=0$ for $j=1,\ldots,q_n$. We can show that $\{b_1,b_3,b_5,\ldots\}$ are independent, and so are $\{b_2,b_4,b_6,\ldots\}$.

Since $\widetilde{\gamma}_e(0)=\sigma_w^2\sum_{j=-g_n-2}^{g_n+2}\psi_j^2=O(g_n)$ and

$$E\{\widetilde{e}_0^4(t_i)\}=\sum_{j=-g_n-2}^{g_n+2}\psi_j^4E(w_{i-j}^4)+\sum\sum_{-g_n-2\leq j\neq j'\leq g_n+2}\psi_j^2\psi_{j'}^2\sigma_w^4=O\Big(\Big(\sum_{j=-g_n-2}^{g_n+2}\psi_j^2\Big)^2\Big),$$

we have

$$\begin{aligned}&E[|\widetilde{e}_1^2(t_i)-\{1+\widetilde{\rho}_e(k)\}|^2]=E[\{\widetilde{e}_0(t_i)+\widetilde{e}_0(t_{i+k})\}^4]/\{4\widetilde{\gamma}_e^2(0)\}-\{1+\widetilde{\rho}_e(k)\}^2\\ =\;&O\Big(\Big(\sum_{j=-g_n-2}^{g_n+2}\psi_j^2\Big)^2\Big/\Big\{4\sigma_w^4\Big(\sum_{j=-g_n-2}^{g_n+2}\psi_j^2\Big)^2\Big\}\Big)-\{1+\widetilde{\rho}_e(k)\}^2=O(1).\end{aligned}$$

By Cauchy-Schwarz inequality, $E(b_1^2)\leq(3g_n+4)\sum_{i=3}^{3g_n+6}E[|\widetilde{e}_1^2(t_i)-\{1+\widetilde{\rho}_e(k)\}|^2]=O(g_n^2)$. Similarly, we can show that $E(b_j^2)=O(g_n^2)$ for $j=1,\ldots,q_n$, which together with Markov inequality implies

$$\begin{aligned}&\mathrm{P}\Big(\Big|\sum_{i=3}^{n-k}[\widetilde{e}_1^2(t_i)-\{1+\widetilde{\rho}_e(k)\}]\Big|\geq\frac{n\tau_{0;n}}{\widetilde{\gamma}_e(0)}\Big)\\ \leq\;&\mathrm{P}\Big(\Big|\sum_{j=1,3,5,\ldots}b_j\Big|+\Big|\sum_{j=2,4,6,\ldots}b_j\Big|\geq\frac{n\tau_{0;n}}{\widetilde{\gamma}_e(0)}\Big)\\ \leq\;&\mathrm{P}\Big(\Big|\sum_{j=1,3,5,\ldots}b_j\Big|\geq\frac{n\tau_{0;n}}{2\widetilde{\gamma}_e(0)}\Big)+\mathrm{P}\Big(\Big|\sum_{j=2,4,6,\ldots}b_j\Big|\geq\frac{n\tau_{0;n}}{2\widetilde{\gamma}_e(0)}\Big)\\ \leq\;&\Big\{\frac{2\widetilde{\gamma}_e(0)}{n\tau_{0;n}}\Big\}^2E\Big\{\Big(\sum_{j=1,3,5,\ldots}b_j\Big)^2\Big\}+\Big\{\frac{2\widetilde{\gamma}_e(0)}{n\tau_{0;n}}\Big\}^2E\Big\{\Big(\sum_{j=2,4,6,\ldots}b_j\Big)^2\Big\}\\ =\;&\Big\{\frac{2\widetilde{\gamma}_e(0)}{n\tau_{0;n}}\Big\}^2\Big\{\sum_{j=1}^{q_n}E(b_j^2)\Big\}=O\Big(\frac{g_n^2}{n^2\tau_{0;n}^2}q_ng_n^2\Big)=O\Big(\frac{g_n^3}{n\tau_{0;n}^2}\Big).\end{aligned}\tag{S.19}$$

Similar arguments lead to

$$\mathrm{P}\Big(\Big|\sum_{i=3}^{n-k}[\widetilde{e}_2^2(t_i)-\{1-\widetilde{\rho}_e(k)\}]\Big|\geq\frac{n\tau_{0;n}}{\widetilde{\gamma}_e(0)}\Big)=O\Big(\frac{g_n^3}{n\tau_{0;n}^2}\Big).\tag{S.20}$$

By (S.18), (S.19) and (S.20), we complete the proof. ■

Lemma 4 *Under Conditions* A2, A4 *and* A5, *for any constant* $\tau_1 > 0$ *and* $1 \le i, j \le \ell$,

$$\mathrm{P}(\|\widehat{\Sigma}_{i,j} - \Sigma_{i,j}\|_\infty > \tau_1) \le C/n,$$

where $\widehat{\Sigma}_{i,j} = (\mathbf{D}_1\mathbf{S}_i)^T(\mathbf{D}_1\mathbf{S}_j)/n$ *and* $\Sigma_{i,j}$ *is defined in Condition* A2.

Proof: For notational simplicity, define $s_{r;i} = s_i(r/n)$, for $r = 0, \ldots, n-1$.

Let J be an $m \times m$ matrix defined as

$$\mathrm{J}(u,v) = \begin{cases} n^{-1}\sum_{r=0}^{n-v+u-2}(s_{r+1;j} - s_{r;j})(s_{r+v-u+1;i} - s_{r+v-u;i}), & \text{if } 1 \le u < v \le m, \\ n^{-1}\sum_{r=0}^{n-u+v-2}(s_{r+1;i} - s_{r;i})(s_{r+u-v+1;j} - s_{r+u-v;j}), & \text{if } 1 \le v \le u \le m. \end{cases}$$

For $1 \le u < v \le m$,

$$\begin{aligned} & |\widehat{\Sigma}_{i,j}(u,v) - \mathrm{J}(u,v)| \\ = & \Big|\frac{1}{n}\sum_{r=-1}^{n-v-1}(s_{r+1;j} - s_{r;j})(s_{r+v-u+1;i} - s_{r+v-u;i}) \\ & \quad -\frac{1}{n}\sum_{r=0}^{n-v+u-2}(s_{r+1;j} - s_{r;j})(s_{r+v-u+1;i} - s_{r+v-u;i})\Big| \\ \le & \frac{1}{n}\Big\{|s_{0;j}(s_{v-u;i} - s_{v-u-1;i})| + \sum_{r=n-v-1}^{n-v+u-2}|(s_{r+1;j} - s_{r;j})(s_{r+v-u+1;i} - s_{r+v-u;i})|\Big\} \\ \equiv & \mathrm{K}(u,v), \end{aligned} \tag{S.21}$$

where $s_{-1;i} = s_{-1;j} = 0$ and K is an $m \times m$ matrix. Similarly, for $1 \le v \le u \le m$,

$$\begin{aligned} & |\widehat{\Sigma}_{i,j}(u,v) - \mathrm{J}(u,v)| \\ \le & \frac{1}{n}\Big\{|s_{0;i}(s_{u-v;j} - s_{u-v-1;j})| + \sum_{r=n-u-1}^{n-u+v-2}|(s_{r+1;i} - s_{r;i})(s_{r+u-v+1;j} - s_{r+u-v;j})|\Big\} \\ \equiv & \mathrm{K}(u,v). \end{aligned} \tag{S.22}$$

By Markov inequality and Cauchy-Schwarz inequality,

$$\begin{aligned} \mathrm{P}\Big(\sum\sum_{1\le u<v\le m}\mathrm{K}(u,v) > \frac{\tau_1}{4}\Big) \le & \frac{16}{\tau_1^2 n^2}E\Big(\Big[\sum\sum_{1\le u<v\le m}\Big\{|s_{0;j}(s_{v-u;i} - s_{v-u-1;i})| \\ & + \sum_{r=n-v-1}^{n-v+u-2}|(s_{r+1;j} - s_{r;j})(s_{r+v-u+1;i} - s_{r+v-u;i})|\Big\}\Big]^2\Big) \\ = & O(1/n^2), \end{aligned}$$

and similarly, we can show $\mathrm{P}(\sum\sum_{1\le v\le u\le m}\mathrm{K}(u,v) > \tau_1/4) = O(1/n^2)$.

Since, by Condition A4, $\Sigma_{i,j}$ is a Toeplitz matrix and so is J, we have $\|\mathrm{J}-\Sigma_{i,j}\|_\infty \le \|\boldsymbol{z}_{1,m}^T(\mathrm{J}-\Sigma_{i,j})\|_1 + \|\boldsymbol{z}_{m,m}^T(\mathrm{J}-\Sigma_{i,j})\|_1 = \sum_{v=1}^m |\mathrm{J}(1,v)-\Sigma_{i,j}(1,v)| + \sum_{v=1}^m |\mathrm{J}(m,v)-\Sigma_{i,j}(m,v)|$. From (S.21) and (S.22),

$$\begin{aligned}
& \mathrm{P}(\|\widehat{\Sigma}_{i,j}-\Sigma_{i,j}\|_\infty > \tau_1) \le \mathrm{P}(\|\mathrm{J}-\Sigma_{i,j}\|_\infty + \|\widehat{\Sigma}_{i,j}-\mathrm{J}\|_\infty > \tau_1) \\
\le\ & \mathrm{P}(\|\mathrm{J}-\Sigma_{i,j}\|_\infty > \tau_1/2) + \mathrm{P}(\|\widehat{\Sigma}_{i,j}-\mathrm{J}\|_\infty > \tau_1/2) \\
\le\ & \mathrm{P}\Big(\sum_{v=1}^m |\mathrm{J}(1,v)-\Sigma_{i,j}(1,v)| + \sum_{v=1}^m |\mathrm{J}(m,v)-\Sigma_{i,j}(m,v)| > \frac{\tau_1}{2}\Big) \\
& \quad + \mathrm{P}\Big(\sum\sum_{1\le u,v\le m} \mathrm{K}(u,v) > \frac{\tau_1}{2}\Big) \\
\le\ & \sum_{v=1}^m \mathrm{P}(|\mathrm{J}(1,v)-\Sigma_{i,j}(1,v)| > \tau_1/(4m)) \\
& \quad + \sum_{v=1}^m \mathrm{P}(|\mathrm{J}(m,v)-\Sigma_{i,j}(m,v)| > \tau_1/(4m)) + O(1/n^2). \qquad \text{(S.23)}
\end{aligned}$$

Following basically the same method in the proof of Lemma 3, we can show $\mathrm{P}(|\mathrm{J}(1,v)-\Sigma_{i,j}(1,v)| > \tau_1/(4m)) = O(1/n)$ and $\mathrm{P}(|\mathrm{J}(m,v)-\Sigma_{i,j}(m,v)| > \tau_1/(4m)) = O(1/n)$. By (S.23) we complete the proof. ■

Lemma 5 *Under Conditions* A2, A4 *and* A5*, for any constant* $\tau_1 > 0$,

$$\mathrm{P}(\|\widehat{\Sigma}_1-\Sigma_1\| > \tau_1) \le C/n,$$

where $\widehat{\Sigma}_1 = (\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})/n$ *and* Σ_1 *is defined in Condition* A2.

Proof: $\widehat{\Sigma}_1$ could be expressed as a block matrix, i.e., $\widehat{\Sigma}_1 = (\widehat{\Sigma}_{i,j})_{i,j=1}^{\ell}$ where $\widehat{\Sigma}_{i,j} = (\mathbf{D}_1\mathbf{S}_i)^T(\mathbf{D}_1\mathbf{S}_j)/n$. Since $\widehat{\Sigma}_1$ and Σ_1 are symmetric matrices,

$$\begin{aligned}
& \mathrm{P}(\|\widehat{\Sigma}_1-\Sigma_1\| > \tau_1) \le \mathrm{P}(\|\widehat{\Sigma}_1-\Sigma_1\|_\infty > \tau_1) \le \mathrm{P}\Big(\sum\sum_{1\le i,j\le \ell} \|\widehat{\Sigma}_{i,j}-\Sigma_{i,j}\|_\infty > \tau_1\Big) \\
\le\ & \sum\sum_{1\le i,j\le \ell} \mathrm{P}(\|\widehat{\Sigma}_{i,j}-\Sigma_{i,j}\|_\infty > \tau_1/\ell^2) = O(1/n).
\end{aligned}$$

The last inequality is derived from Lemma 4. ■

Lemma 6 *Under Conditions* A1 (*or* B1) *and* A2–A5*, for any* $\tau_2 \equiv \tau_{2;n} > 0$,

$$\mathrm{P}(\|(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2 \ge \tau_2) \le Cng_n^2/\tau_2,$$

where $\boldsymbol{\epsilon}_1 = \mathbf{D}_1\mathbf{d} + \mathbf{D}_1\boldsymbol{\epsilon}$ *as in Notation* 1.

Proof: First, we will show the result under Conditions A1–A5. Let $(\eta_1,\ldots,\eta_{\ell m})^T = (\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1$ and $\vartheta_{i,j} = \boldsymbol{z}_{i,n-1}^T(\mathbf{D}_1\mathbf{S})\boldsymbol{z}_{j,\ell m}$. For $j = 1,\ldots,\ell m$, $\eta_j = \sum_{i=1}^{n-1}\epsilon_1(t_{i+1})\vartheta_{i,j} = \sum_{i=1}^{n-1}\{\epsilon(t_{i+1})-\epsilon(t_i)\}\vartheta_{i,j} + \sum_{i=1}^{n-1}\{d(t_{i+1})-d(t_i)\}\vartheta_{i,j}$.

$$\mathrm{P}(\|(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2 \ge \tau_2) = \mathrm{P}\Big(\sum_{j=1}^{\ell m}\eta_j^2 \ge \tau_2\Big) \le \sum_{j=1}^{\ell m}\mathrm{P}(\eta_j^2 \ge \tau_2/(\ell m))$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{\ell m} \mathrm{P}\Big(\Big|\sum_{i=1}^{n-1}\{\epsilon(t_{i+1})-\epsilon(t_i)\}\vartheta_{i,j}\Big| \geq \frac{\tau_2^{1/2}}{2(\ell m)^{1/2}}\Big) \\
&\qquad + \sum_{j=1}^{\ell m} \mathrm{P}\Big(\Big|\sum_{i=1}^{n-1}\{d(t_{i+1})-d(t_i)\}\vartheta_{i,j}\Big| \geq \frac{\tau_2^{1/2}}{2(\ell m)^{1/2}}\Big) \\
&\equiv \mathrm{I}+\mathrm{II}. \qquad\qquad \text{(S.24)}
\end{aligned}
$$

For term I, define $\phi_k^* = \phi_{n;k} - \phi_{n;k-1}$, so $\epsilon(t_{i+1}) - \epsilon(t_i) = \sum_{k=-\infty}^{\infty} \phi_k^* w_{i+1-k}$. Then,

$$
\sum_{i=1}^{n-1}\{\epsilon(t_{i+1})-\epsilon(t_i)\}\vartheta_{i,j} = \sum_{i=1}^{n-1}\sum_{k=-\infty}^{\infty} \phi_k^* w_{i+1-k}\vartheta_{i,j} = \sum_{k=-\infty}^{\infty}\sum_{i=1}^{n-1} \phi_k^* w_{i+1-k}\vartheta_{i,j}. \quad \text{(S.25)}
$$

For each k and j, define $\boldsymbol{\beta}_{k,j} = (\phi_k^* w_{2-k}\vartheta_{1,j}, \ldots, \phi_k^* w_{n-k}\vartheta_{n-1,j})^T$. Divide $\boldsymbol{\beta}_{k,j}$ into blocks with length g_s+1, and hence there are $q_n = \lceil (n-1)/(g_s+1) \rceil$ blocks. The sum of the elements of $\boldsymbol{\beta}_{k,j}$ within the uth block is denoted by $\kappa_{k,j,u}$, for $u = 1, \ldots, q_n$. For example, $\kappa_{k,j,1} = \sum_{i=1}^{g_s+1} \phi_k^* w_{i+1-k}\vartheta_{i,j}$. Then, $E(\kappa_{k,j,u}) = 0$, $E(\kappa_{k,j,u}^2) \leq (g_s + 1)\sum_{i=1}^{g_s+1} E[\{\phi_k^* w_{i+1-k}\vartheta_{i,j}\}^2] = O((g_s+1)^2\phi_k^{*2})$, $\{\kappa_{k,j,1}, \kappa_{k,j,3}, \ldots\}$ are independent and so are $\{\kappa_{k,j,2}, \kappa_{k,j,4}, \ldots\}$. Then,

$$
E\Big\{\Big(\sum_{i=1}^{n-1}\phi_k^* w_{i+1-k}\vartheta_{i,j}\Big)^2\Big\} \leq 2E\Big\{\Big(\sum_{u=1,3,5,\ldots}\kappa_{k,j,u}\Big)^2\Big\} + 2E\Big\{\Big(\sum_{u=2,4,6,\ldots}\kappa_{k,j,u}\Big)^2\Big\} = O(n\phi_k^{*2}),
$$

which together with (S.25) and Minkowski inequality implies

$$
\begin{aligned}
&E\Big[\Big|\sum_{i=1}^{n-1}\{\epsilon(t_{i+1})-\epsilon(t_i)\}\vartheta_{i,j}\Big|^2\Big] = E\Big[\Big|\sum_{k=-\infty}^{\infty}\sum_{i=1}^{n-1}(\phi_k^* w_{i+1-k}\vartheta_{i,j})\Big|^2\Big] \\
&\leq \Big[\sum_{k=-\infty}^{\infty}\Big\{E\Big(\Big|\sum_{i=1}^{n-1}(\phi_k^* w_{i+1-k}\vartheta_{i,j})\Big|^2\Big)\Big\}^{1/2}\Big]^2 = O(ng_n^2). \qquad \text{(S.26)}
\end{aligned}
$$

By Markov inequality and (S.26),

$$
\mathrm{I} \leq \frac{4\ell m}{\tau_2}\sum_{j=1}^{\ell m} E\Big[\Big|\sum_{i=1}^{n-1}\{\epsilon(t_{i+1})-\epsilon(t_i)\}\vartheta_{i,j}\Big|^2\Big] = O(ng_n^2/\tau_2).
$$

Similar arguments can be applied to show that $\mathrm{II} = O(ng_n^2/\tau_2)$. From (S.24), we complete the proof.

Next, we will provide the proof under Conditions B1 and A2–A5. Since $\{\epsilon(t_{i+1}) - \epsilon(t_i)\}$ is (g_n+1)-dependent and any column of $\mathbf{D}_1\mathbf{S}$ is (g_s+1)-dependent, the vector $\boldsymbol{\alpha}_j = (\{\epsilon(t_2)-\epsilon(t_1)\}\vartheta_{1,j}, \ldots, \{\epsilon(t_n)-\epsilon(t_{n-1})\}\vartheta_{n-1,j})^T$ is (g_n+g_s+1)-dependent, $j = 1, \ldots, m$. We divide $\boldsymbol{\alpha}_j$ into blocks with length g_n+g_s+1 as we did before. So, there are $\lceil (n-1)/(g_n+g_s+1) \rceil$ blocks. The sum of elements in $\boldsymbol{\alpha}_j$ within the wth block is denoted by $f_{j,w}$, for $w = 1, \ldots, \lceil (n-1)/(g_n+g_s+1) \rceil$. By Conditions B1 and A4, it is easy to see that $E(f_{j,w}^2) \le C g_n^3$. Hence,

$$\begin{aligned}
& \mathrm{P}\Big(\Big|\sum_{i=1}^{n-1}\{\epsilon(t_{i+1})-\epsilon(t_i)\}\vartheta_{i,j}\Big| \ge \frac{\tau_2^{1/2}}{2(\ell m)^{1/2}}\Big) \\
\le\ & \mathrm{P}\Big(\Big|\sum_{w=1,3,5,\ldots} f_{j,w}\Big| \ge \frac{\tau_2^{1/2}}{4(\ell m)^{1/2}}\Big) + P\Big(\Big|\sum_{w=2,4,6,\ldots} f_{j,w}\Big| \ge \frac{\tau_2^{1/2}}{4(\ell m)^{1/2}}\Big) \\
\le\ & \frac{16\ell m}{\tau_2}\Big\{E\Big(\Big|\sum_{w=1,3,5,\ldots} f_{j,w}\Big|^2\Big) + E\Big(\Big|\sum_{w=2,4,6,\ldots} f_{j,w}\Big|^2\Big)\Big\} = O(n g_n^2/\tau_2)
\end{aligned}$$

Since (S.24) still holds under Condition B1, similar arguments can be applied to show that $\mathrm{I} = O(ng_n^2/\tau_2)$ and $\mathrm{II} = O(ng_n^2/\tau_2)$. From (S.24), we complete the proof. ■

Lemma 7 *Under Conditions* A1 (*or* B1) *and* A2–A5, *for any* $\tau_3 \equiv \tau_{3;n} > 0$,

$$\mathrm{P}\Big(\frac{1}{n}\sum_{i=3}^{n}\delta^2(t_i) \ge \tau_3\Big) \le \frac{Cg_n^2}{\tau_3 n} + \frac{C}{n},$$

where $\delta(t_i)$ *is the* $(i-2)$*th element of* $\boldsymbol{\delta} = \mathbf{D}_2\mathbf{S}(\mathbf{h} - \widehat{\mathbf{h}}_{\mathrm{DBE}})$ *as defined in Notation* 2.

Proof: The proof is the same under either Condition A1 or B1.

Since $\boldsymbol{\epsilon}_1 = \mathbf{D}_1\mathbf{d} + \mathbf{D}_1\boldsymbol{\epsilon}$, from model (2.1),

$$\widehat{\mathbf{h}}_{\mathrm{DBE}} = \{(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})\}^{-1}(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{y}) = \mathbf{h} + \{(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})\}^{-1}(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1.$$

Thus,

$$\begin{aligned}
& \sum_{i=3}^{n}\delta^2(t_i) = \|\mathbf{D}_2\mathbf{S}(\mathbf{h} - \widehat{\mathbf{h}}_{\mathrm{DBE}})\|^2 = \|\mathbf{D}_0(\mathbf{D}_1\mathbf{S})\{(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})\}^{-1}(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2 \\
\le\ & \|\mathbf{D}_0\|^2\|(\mathbf{D}_1\mathbf{S})\{(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})\}^{-1}\|^2\|(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2 \\
\le\ & 4\|\{(\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})\}^{-1}\|\|(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2 = 4n^{-1}\|\widehat{\Sigma}_1^{-1}\| \times \|(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2, \qquad \text{(S.27)}
\end{aligned}$$

where $\widehat{\Sigma}_1 = (\mathbf{D}_1\mathbf{S})^T(\mathbf{D}_1\mathbf{S})/n$ and $\mathbf{D}_0$ is an $(n-2)\times(n-1)$ matrix defined as

$$\mathbf{D}_0 = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{(n-2)\times(n-1)},$$

such that $\mathbf{D}_2 = \mathbf{D}_0\mathbf{D}_1$. By Condition A2, when $2^{-1}\lambda_{\min}(\Sigma_1) \ge \|\widehat{\Sigma}_1 - \Sigma_1\|$,

$$\frac{C}{2} \le \frac{1}{2}\lambda_{\min}(\Sigma_1) \le \lambda_{\min}(\Sigma_1) - \|\widehat{\Sigma}_1 - \Sigma_1\| \le \lambda_{\min}(\Sigma_1) + \lambda_{\min}(\widehat{\Sigma}_1 - \Sigma_1) \le \lambda_{\min}(\widehat{\Sigma}_1),$$

and thus, $\|\widehat{\Sigma}_1^{-1}\| = 1/\lambda_{\min}(\widehat{\Sigma}_1) \le 2/C$, which together with (S.27) implies

$$\begin{aligned} & \mathrm{P}\Big(\frac{1}{n}\sum_{i=3}^{n}\delta^2(t_i) \ge \tau_3\Big) \le \mathrm{P}(\|\widehat{\Sigma}_1^{-1}\|\|(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2 \ge \tau_3 n^2/4) \\ = \; & \mathrm{P}(\|\widehat{\Sigma}_1^{-1}\|\|(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2 \ge \tau_3 n^2/4,\ \|\widehat{\Sigma}_1 - \Sigma_1\| \le \lambda_{\min}(\Sigma_1)/2) \\ & + \mathrm{P}(\|\widehat{\Sigma}_1^{-1}\|\|(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2 \ge \tau_3 n^2/4,\ \|\widehat{\Sigma}_1 - \Sigma_1\| > \lambda_{\min}(\Sigma_1)/2) \\ \le \; & \mathrm{P}(\|(\mathbf{D}_1\mathbf{S})^T\boldsymbol{\epsilon}_1\|^2 \ge C\tau_3 n^2/8) + \mathrm{P}(\|\widehat{\Sigma}_1 - \Sigma_1\| > C/2) \\ \equiv \; & \mathrm{I} + \mathrm{II}. \end{aligned} \tag{S.28}$$

By Lemma 6, if we take $\tau_{2;n} = C\tau_3 n^2/8$, $\mathrm{I} = O(g_n^2/(\tau_3 n))$. From Lemma 5, by choosing $\tau_1 = C/2$, $\mathrm{II} = O(1/n)$. From (S.28), we complete the proof. ■

Lemma 8 *Under Conditions* A1 (*or* B1) *and* A2–A5, *for* $\varepsilon_0 \equiv \varepsilon_{0;n} > 0$ *that satisfies* $\varepsilon_0 = O(g_n^6)$, $\varepsilon_0 n/g_n^{11} \to \infty$ *and* $\varepsilon_0^2 n^2/g_n^{11} \to \infty$, *and for any* $k \in \{0, \dots, g_n\}$,

$$\mathrm{P}(|\widehat{\gamma}_e(k) - \gamma_e(k)| \ge \varepsilon_0/g_n^5) \le C g_n^{13}/(n\varepsilon_0^2).$$

Proof: The proof is the same under either Condition A1 or B1.

Since $\widehat{e}(t_i) = e(t_i) + \delta(t_i)$, we have

$$\begin{aligned} & \frac{1}{n}\sum_{i=3}^{n-k}\widehat{e}(t_i)\widehat{e}(t_{i+k}) = \frac{1}{n}\sum_{i=3}^{n-k}\{e(t_i)+\delta(t_i)\}\{e(t_{i+k})+\delta(t_{i+k})\} \\ = \; & \frac{1}{n}\sum_{i=3}^{n-k}e(t_i)e(t_{i+k}) + \frac{1}{n}\sum_{i=3}^{n-k}e(t_i)\delta(t_{i+k}) + \frac{1}{n}\sum_{i=3}^{n-k}e(t_{i+k})\delta(t_i) + \frac{1}{n}\sum_{i=3}^{n-k}\delta(t_i)\delta(t_{i+k}). \end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} & \mathrm{P}\Big(|\widehat{\gamma}_e(k) - \gamma_e(k)| \ge \frac{\varepsilon_0}{g_n^5}\Big) = \mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n-k}\{\widehat{e}(t_i)\widehat{e}(t_{i+k})\} - \gamma_e(k)\Big| \ge \frac{\varepsilon_0}{g_n^5}\Big) \\ \le \; & \mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n-k}\{e(t_i)e(t_{i+k})\} - \gamma_e(k)\Big| + \frac{2}{n}\Big\{\sum_{i=3}^{n}e^2(t_i)\sum_{i=3}^{n}\delta^2(t_i)\Big\}^{1/2} + \frac{1}{n}\sum_{i=3}^{n}\delta^2(t_i) \ge \frac{\varepsilon_0}{g_n^5}\Big) \\ \le \; & \mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n-k}\{e(t_i)e(t_{i+k})\} - \gamma_e(k)\Big| \ge \frac{\varepsilon_0}{3g_n^5}\Big) + \mathrm{P}\Big(\frac{2}{n}\Big\{\sum_{i=3}^{n}e^2(t_i)\sum_{i=3}^{n}\delta^2(t_i)\Big\}^{1/2} \ge \frac{\varepsilon_0}{3g_n^5}\Big) \\ & + \mathrm{P}\Big(\frac{1}{n}\sum_{i=3}^{n}\delta^2(t_i) \ge \frac{\varepsilon_0}{3g_n^5}\Big) \\ \equiv \; & \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{S.29}$$

From Conditions A3 and A5, we have $|d_0(t_i)| \le 2C/n^2$, where $d_0(t_i) = d(t_i) - 2d(t_{i-1}) + d(t_{i-2})$. Since $e(t_i) = e_0(t_i) + d_0(t_i)$, by Cauchy-Schwarz inequality, for large n,

$$\mathrm{I} \;\le\; \mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n-k}\{e_0(t_i)e_0(t_{i+k})\} - \gamma_e(k)\Big| \ge \frac{\varepsilon_0}{4g_n^5}\Big)$$

$$
\begin{aligned}
&+\mathrm{P}\Big(\frac{2}{n}\Big\{\sum_{i=3}^{n}e_0^2(t_i)\sum_{i=3}^{n}d_0^2(t_i)\Big\}^{1/2}+\frac{1}{n}\sum_{i=3}^{n}d_0^2(t_i)\geq\frac{\varepsilon_0}{12g_n^5}\Big)\\
\leq\;& \mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n-k}\{e_0(t_i)e_0(t_{i+k})\}-\gamma_e(k)\Big|\geq\frac{\varepsilon_0}{4g_n^5}\Big)\\
&+\mathrm{P}\Big(\frac{2}{n}\Big\{\sum_{i=3}^{n}e_0^2(t_i)\sum_{i=3}^{n}d_0^2(t_i)\Big\}^{1/2}\geq\frac{\varepsilon_0}{24g_n^5}\Big)\\
\equiv\;& \mathrm{I}_1+\mathrm{I}_2. \qquad\qquad (\mathrm{S}.30)
\end{aligned}
$$

The last inequality in (S.30) is true, when n is large enough, such that $n^4/g_n^5 > 96C^2/\varepsilon_0$, which implies $n^{-1}\sum_{i=3}^{n} d_0^2(t_i) < \varepsilon_0/(24g_n^5)$.

For term I_2 in (S.30), when n is large enough,

$$
\begin{aligned}
\mathrm{I}_2 \;=\;& \mathrm{P}\Big(\frac{2}{n}\Big\{\sum_{i=3}^{n}e_0^2(t_i)\sum_{i=3}^{n}d_0^2(t_i)\Big\}^{1/2}\geq\frac{\varepsilon_0}{24g_n^5}\Big)=\mathrm{P}\Big(\sum_{i=3}^{n}e_0^2(t_i)\sum_{i=3}^{n}d_0^2(t_i)\geq\frac{\varepsilon_0^2n^2}{2304g_n^{10}}\Big)\\
\leq\;& \mathrm{P}\Big(\sum_{i=3}^{n}e_0^2(t_i)\geq\frac{\varepsilon_0^2n^5}{9216C^2g_n^{10}}\Big)\leq\mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n}e_0^2(t_i)-\gamma_e(0)\Big|\geq\frac{\varepsilon_0^2n^4}{18432C^2g_n^{10}}\Big)\\
\leq\;& \mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n}e_0^2(t_i)-\gamma_e(0)\Big|\geq\frac{\varepsilon_0}{4g_n^5}\Big). \qquad\qquad (\mathrm{S}.31)
\end{aligned}
$$

The last two inequalities in (S.31) are true, when n is large enough for the following inequalities to hold, $n^4/g_n^{10} > 18432C^2\gamma_e(0)/\varepsilon_0^2$ and $n^4/g_n^5 > 4608C^2/\varepsilon_0$, which imply $\gamma_e(0) < \varepsilon_0^2n^4/(18432C^2g_n^{10})$ and $\varepsilon_0/(4g_n^5) < \varepsilon_0^2n^4/(18432C^2g_n^{10})$ respectively.

From the assumptions that $\varepsilon_0 n/g_n^{11}\to\infty$ and $\varepsilon_0^2n^2/g_n^{11}\to\infty$, we can always choose a constant L_1, such that, for any $n > L_1$, the following inequalities hold: $n^4/g_n^5 > 96C^2/\varepsilon_0$, $n^4/g_n^{10} > 18432C^2\gamma_e(0)/\varepsilon_0^2$ and $n^4/g_n^5 > 4608C^2/\varepsilon_0$, which imply that (S.30) and (S.31) hold. Therefore, for $n > L_1$, from (S.30), (S.31), by choosing $\tau_{0;n} = \varepsilon_0/(4g_n^5)$ in Lemma 2,

$$
\begin{aligned}
\mathrm{I} \;\leq\;& \mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n-k}\{e_0(t_i)e_0(t_{i+k})\}-\gamma_e(k)\Big|\geq\frac{\varepsilon_0}{4g_n^5}\Big)+\mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n}e_0^2(t_i)-\gamma_e(0)\Big|\geq\frac{\varepsilon_0}{4g_n^5}\Big)\\
=\;& O(g_n^{13}/(n\varepsilon_0^2)). \qquad\qquad (\mathrm{S}.32)
\end{aligned}
$$

For term II,

$$
\begin{aligned}
\mathrm{II} \;=\;& \mathrm{P}\Big(\frac{1}{n}\Big\{\frac{1}{n}\sum_{i=3}^{n}e^2(t_i)-\gamma_e(0)+\gamma_e(0)\Big\}\sum_{i=3}^{n}\delta^2(t_i)\geq\frac{\varepsilon_0^2}{36g_n^{10}}\Big)\\
\leq\;& \mathrm{P}\Big(\frac{1}{n}\Big|\frac{1}{n}\sum_{i=3}^{n}e^2(t_i)-\gamma_e(0)\Big|\sum_{i=3}^{n}\delta^2(t_i)\geq\frac{\varepsilon_0^2}{72g_n^{10}}\Big)\\
&+\mathrm{P}\Big(\frac{1}{n}\gamma_e(0)\sum_{i=3}^{n}\delta^2(t_i)\geq\frac{\varepsilon_0^2}{72g_n^{10}}\Big)\\
\leq\;& \mathrm{P}\Big(\Big|\frac{1}{n}\sum_{i=3}^{n}e^2(t_i)-\gamma_e(0)\Big|\geq\frac{\varepsilon_0}{3g_n^5}\Big)+\mathrm{P}\Big(\frac{1}{n}\sum_{i=3}^{n}\delta^2(t_i)\geq\frac{\varepsilon_0}{24g_n^5}\Big)
\end{aligned}
$$

$$
\begin{aligned}
& +\mathrm{P}\Big(\frac{1}{n}\gamma_e(0)\sum_{i=3}^{n}\delta^2(t_i) \ge \frac{\varepsilon_0^2}{72g_n^{10}}\Big) \\
\equiv\ & \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3. \qquad (\mathrm{S}.33)
\end{aligned}
$$

From the assumption $0 < \varepsilon_0 = O(g_n^6)$, we can first choose $\tau_{3;n} = \varepsilon_0^2/\{72\gamma_e(0)g_n^{10}\}$ in Lemma 7 and get $\mathrm{II}_3 = O(g_n^{13}/(n\varepsilon_0^2)) + O(1/n) = O(g_n^{13}/(n\varepsilon_0^2))$. Then, from Lemma 7, by taking $\tau_{3;n} = \varepsilon_0/(24g_n^5)$, $\mathrm{II}_2 = O(g_n^7/(n\varepsilon_0)) + O(1/n) = O(g_n^7/(n\varepsilon_0))$. Based on the proof for term I, $\mathrm{II}_1 = O(g_n^{13}/(n\varepsilon_0^2))$. By (S.33) and the assumption that $\varepsilon_0 = O(g_n^6)$,

$$
\mathrm{II} = O(g_n^{13}/(n\varepsilon_0^2)) + O(g_n^7/(n\varepsilon_0)) + O(g_n^{13}/(n\varepsilon_0^2)) = O(g_n^{13}/(n\varepsilon_0^2)).
$$

It's easy to see that

$$
\mathrm{III} \le \mathrm{II}_2 = O(g_n^7/(n\varepsilon_0)) = O(g_n^{13}/(n\varepsilon_0^2)). \qquad (\mathrm{S}.34)
$$

From (S.29) and (S.32)–(S.34), we complete the proof for $n > L_1$. The result for $n \le L_1$ is straightforward. ■

Lemma 9 *Assume Conditions* A2–A5 *and that there exists* $\varepsilon \equiv \varepsilon_n > 0$ *such that* $\varepsilon = O(1)$, $\varepsilon^{1/2}n/g_n^{11} \to \infty$, $\varepsilon n^2/g_n^{11} \to \infty$. *Under either of the following two assumptions:*

- *Condition* A1 *holds and* $(g_n + 2)^{3-\alpha_n} = o(\varepsilon^{1/2})$,
- *Condition* B1 *holds,*

we have

$$
\mathrm{P}(\|\widehat{\mathbf{R}} - M_{g_n}(\mathbf{R})\|_\infty^2 \ge \varepsilon) \le Cg_n^{14}/(\varepsilon n).
$$

Proof: The proof is the same under either Condition A1 or B1.

Since $\mathbf{R}$ and $\widehat{\mathbf{R}}$ are Toeplitz matrices,

$$
\begin{aligned}
& \mathrm{P}(\|\widehat{\mathbf{R}} - M_{g_n}(\mathbf{R})\|_\infty^2 \ge \varepsilon) \le \mathrm{P}\Big(\Big\{2\sum_{k=1}^{g_n}|\widehat{\rho}(k) - \rho(k)|\Big\}^2 \ge \varepsilon\Big) \\
=\ & \mathrm{P}\Big(\sum_{k=1}^{g_n}|\widehat{\rho}(k) - \rho(k)| \ge \varepsilon^{1/2}/2\Big) = \mathrm{P}\Big(\Big\|\frac{\widehat{\boldsymbol{\gamma}}}{\widehat{\gamma}(0)} - \frac{\boldsymbol{\gamma}}{\gamma(0)}\Big\|_1 \ge \varepsilon^{1/2}/2\Big) \\
\le\ & \mathrm{P}\Big(\Big\|\frac{\widehat{\boldsymbol{\gamma}}}{\widehat{\gamma}(0)} - \frac{\boldsymbol{\gamma}}{\gamma(0)}\Big\|_1 \ge \varepsilon^{1/2}/2, |\widehat{\gamma}(0) - \gamma(0)| < \gamma(0)/2\Big) + \mathrm{P}(|\widehat{\gamma}(0) - \gamma(0)| \ge \gamma(0)/2) \\
\equiv\ & \mathrm{I} + \mathrm{II}. \qquad (\mathrm{S}.35)
\end{aligned}
$$

Since

$$
\frac{\widehat{\boldsymbol{\gamma}}}{\widehat{\gamma}(0)} - \frac{\boldsymbol{\gamma}}{\gamma(0)} = (\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma})\Big(\frac{1}{\widehat{\gamma}(0)} - \frac{1}{\gamma(0)}\Big) + \boldsymbol{\gamma}\Big(\frac{1}{\widehat{\gamma}(0)} - \frac{1}{\gamma(0)}\Big) + (\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma})\frac{1}{\gamma(0)},
$$

then,

$$\Big\|\frac{\widehat{\boldsymbol\gamma}}{\widehat\gamma(0)}-\frac{\boldsymbol\gamma}{\gamma(0)}\Big\|_1 \leq \|\widehat{\boldsymbol\gamma}-\boldsymbol\gamma\|_1\Big|\frac{1}{\widehat\gamma(0)}-\frac{1}{\gamma(0)}\Big|+\|\boldsymbol\gamma\|_1\Big|\frac{1}{\widehat\gamma(0)}-\frac{1}{\gamma(0)}\Big|+\|\widehat{\boldsymbol\gamma}-\boldsymbol\gamma\|_1\frac{1}{\gamma(0)},$$

which implies that

$$\begin{aligned}
\mathrm{I} &\leq \mathrm{P}\Big(\|\widehat{\boldsymbol\gamma}-\boldsymbol\gamma\|_1\Big|\frac{1}{\widehat\gamma(0)}-\frac{1}{\gamma(0)}\Big|\geq\varepsilon^{1/2}/6, |\widehat\gamma(0)-\gamma(0)|<\gamma(0)/2\Big)\\
&\quad+\mathrm{P}\Big(\|\boldsymbol\gamma\|_1\Big|\frac{1}{\widehat\gamma(0)}-\frac{1}{\gamma(0)}\Big|\geq\varepsilon^{1/2}/6, |\widehat\gamma(0)-\gamma(0)|<\gamma(0)/2\Big)\\
&\quad+\mathrm{P}\Big(\|\widehat{\boldsymbol\gamma}-\boldsymbol\gamma\|_1\frac{1}{\gamma(0)}\geq\varepsilon^{1/2}/6, |\widehat\gamma(0)-\gamma(0)|<\gamma(0)/2\Big)\\
&\equiv \mathrm{III}+\mathrm{IV}+\mathrm{V}.
\end{aligned}\tag{S.36}$$

Let $\varepsilon_0=\varepsilon^{1/2}\gamma(0)/96$. By Lemma 1, Remark 1 and the assumption that $(g_n+2)^{3-\alpha_n}=o(\varepsilon^{1/2})$, we have $\|A_{g_n}^{-1}B_{g_n}\|_1\{|\gamma(g_n+1)|+|\gamma(g_n+2)|\}=O(g_n^2(g_n+2)^{1-\alpha_n})=O((g_n+2)^{3-\alpha_n})=o(\varepsilon_0)$. There exists a constant $L_1>0$, such that for $n>L_1$, $\|A_{g_n}^{-1}B_{g_n}\|_1\{|\gamma(g_n+1)|+|\gamma(g_n+2)|\}\leq 2\varepsilon_0$. For $n>L_1$, from Lemma 1,

$$\begin{aligned}
\mathrm{V} &\leq \mathrm{P}(\|\widehat{\boldsymbol\gamma}-\boldsymbol\gamma\|_1\geq\varepsilon^{1/2}\gamma(0)/6)\\
&\leq \mathrm{P}(\|A_{g_n}^{-1}(\widehat{\boldsymbol\gamma}_e-\boldsymbol\gamma_e)\|_1+\|A_{g_n}^{-1}B_{g_n}\|_1\{|\gamma(g_n+1)|+|\gamma(g_n+2)|\}\geq 4\varepsilon_0)\\
&\leq \mathrm{P}(\|A_{g_n}^{-1}\|_1\|\widehat{\boldsymbol\gamma}_e-\boldsymbol\gamma_e\|_1\geq 2\varepsilon_0)\leq \mathrm{P}(\|\widehat{\boldsymbol\gamma}_e-\boldsymbol\gamma_e\|_1\geq 2\varepsilon_0/(Cg_n^4)).
\end{aligned}\tag{S.37}$$

Similarly, we can show that, for $n>L_1$,

$$\mathrm{III}\leq \mathrm{P}(\|\widehat{\boldsymbol\gamma}-\boldsymbol\gamma\|_1\geq\varepsilon^{1/2}\gamma(0)/6).$$

By Lemma 1, Remark 1 and the assumption that $(g_n+2)^{3-\alpha_n}=o(\varepsilon^{1/2})$, there exists a constant $L_2>0$, such that, for $n>L_2$,

$$\|\boldsymbol z_{1,g_n+1}^TA_{g_n}^{-1}B_{g_n}\|_\infty\{|\gamma(g_n+1)|+|\gamma(g_n+2)|\}\leq\varepsilon^{1/2}\gamma(0)/\{24(g_n+1)\}.$$

For $n>L_2$, from Lemma 1,

$$\begin{aligned}
\mathrm{IV} &\leq \mathrm{P}\Big(\Big|\frac{1}{\widehat\gamma(0)}-\frac{1}{\gamma(0)}\Big|\geq\varepsilon^{1/2}/\{6\gamma(0)(g_n+1)\}, |\widehat\gamma(0)-\gamma(0)|<\gamma(0)/2\Big)\\
&\leq \mathrm{P}(|\widehat\gamma(0)-\gamma(0)|\geq\varepsilon^{1/2}\gamma(0)/\{12(g_n+1)\})\\
&\leq \mathrm{P}(|\boldsymbol z_{1,g_n+1}^TA_{g_n}^{-1}(\widehat{\boldsymbol\gamma}_e-\boldsymbol\gamma_e)|+\|\boldsymbol z_{1,g_n+1}^TA_{g_n}^{-1}B_{g_n}\|_\infty\{|\gamma(g_n+1)|+|\gamma(g_n+2)|\}\\
&\qquad\geq\varepsilon^{1/2}\gamma(0)/\{12(g_n+1)\})\\
&\leq \mathrm{P}(\|\boldsymbol z_{1,g_n+1}^TA_{g_n}^{-1}\|_\infty\|\widehat{\boldsymbol\gamma}_e-\boldsymbol\gamma_e\|_1\geq\varepsilon^{1/2}\gamma(0)/\{24(g_n+1)\})\\
&\leq \mathrm{P}(\|\widehat{\boldsymbol\gamma}_e-\boldsymbol\gamma_e\|_1\geq 2\varepsilon_0/(Cg_n^4)).
\end{aligned}$$

By (S.36) and Lemma 8, for $n>\max\{L_1,L_2\}$,

$$\mathrm{I}\leq 3\sum_{k=0}^{g_n}\mathrm{P}(|\widehat\gamma_e(k)-\gamma_e(k)|\geq\varepsilon_0/(Cg_n^5))=O(g_n^{14}/(\varepsilon_0^2n))=O(g_n^{14}/(\varepsilon n)).$$

From the assumption that $\varepsilon = O(1)$, we have $C\varepsilon_0 \le \gamma(0)/2$ for some positive constant C. Following basically the same procedure in (S.37), there exists a constant $L_3 > 0$, such that, for $n > L_3$,

$$\text{II} \le \text{P}(\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_1 \ge \gamma(0)/2) \le \text{P}(\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\|_1 \ge C\varepsilon_0) = O(g_n^{14}/(\varepsilon n)).$$

Now by (S.35), we complete the proof for $n > \max\{L_1, L_2, L_3\}$. The result for $n \le \max\{L_1, L_2, L_3\}$ could be easily derived. ■

Proof of Theorem 1. It is easy to see that

$$\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \le 2\|\widehat{\mathbf{R}} - M_{g_n}(\mathbf{R})\|_\infty^2 + 2\|M_{g_n}(\mathbf{R}) - \mathbf{R}\|_\infty^2 \equiv \text{I} + \text{II}.$$

For term II, from Remark 1, under Condition A1,

$$\|M_{g_n}(\mathbf{R}) - \mathbf{R}\|_\infty \le 2\sum_{k=g_n+1}^{n-1} |\gamma(k)|/|\gamma(0)| = O\Big(\sum_{k=g_n+1}^{n-1} (2k - g_n)^{1-\alpha_n}\Big) = O((g_n+2)^{2-\alpha_n}).$$

Therefore,

$$\text{II} = O_P((g_n+2)^{4-2\alpha_n}).$$

From $(g_n+2)^{8+2\alpha_n}/n \to \infty$ and $g_n^{14}/n = o(1)$, $(g_n+2)^{4-2\alpha_n} = o(g_n^{14}/n)$, and hence, $\text{II} = o_P(g_n^{14}/n)$. Under Condition B1, $\text{II} = 0$.

Take $\varepsilon_n = C^* g_n^{14}/n$, where $C^* > 0$ is a constant. From Lemma 9, under either Condition A1 or B1, $\text{P}(\|\widehat{\mathbf{R}} - M_{g_n}(\mathbf{R})\|_\infty^2 \ge C^* g_n^{14}/n) \le C/C^*$. Thus,

$$\text{I} = O_P(g_n^{14}/n).$$

We complete the proof. ■

Proof of Proposition 1. From the proof of Theorem 1, $\|\widehat{\mathbf{R}}-\mathbf{R}\|_\infty^2 \le 2\|\widehat{\mathbf{R}}-M_{g_n}(\mathbf{R})\|_\infty^2 + 2\|M_{g_n}(\mathbf{R}) - \mathbf{R}\|_\infty^2$, and $\|M_{g_n}(\mathbf{R}) - \mathbf{R}\|_\infty = o(1)$. From Lemma 9, for any constant $\varepsilon > 0$ and n large enough such that $2\|M_{g_n}(\mathbf{R}) - \mathbf{R}\|_\infty^2 < \varepsilon/2$,

$$\text{P}(\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \ge \varepsilon) \le \text{P}(2\|\widehat{\mathbf{R}} - M_{g_n}(\mathbf{R})\|_\infty^2 \ge \varepsilon/2) \le C g_n^{14}/n = o(1), \tag{S.38}$$

because $\alpha_n g_n \to \infty$ implies that $(g_n+2)^{3-\alpha_n} = o(1)$. So, we complete the proof. ■

Proof of Proposition 2. The proof is the same under either Condition A1 or B1. We can show that

$$\lambda_{\min}(\widehat{\mathbf{R}}) \ge \lambda_{\min}(\mathbf{R}) + \lambda_{\min}(\widehat{\mathbf{R}} - \mathbf{R}) \ge \lambda_{\min}(\mathbf{R}) - \|\widehat{\mathbf{R}} - \mathbf{R}\| \ge \lambda_{\min}(\mathbf{R}) - \|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty.$$

From Condition A6 and Proposition 1, we can show that with probability tending to one, $\lambda_{\min}(\widehat{\mathbf{R}}) > 0$. Thus, we complete the proof. ■

Proof of Proposition 3. The proof is the same under either Condition A1 or B1. The proof is similar for $\widehat{\mathbf{R}}_Z$ and $\widehat{\mathbf{R}}_*$. In the following, we will only give the proof for $\widehat{\mathbf{R}}_*$. Since $\widehat{\mathbf{R}}_*-\mathbf{R} = (\widehat{\mathbf{R}}-\mathbf{R})\,\mathrm{I}(\widehat{\mathbf{R}} \succ 0,\ \|\widehat{\mathbf{R}}^{-1}\|_\infty \le Dn^\omega)+(\mathbf{I}_n-\mathbf{R})\{1-\mathrm{I}(\widehat{\mathbf{R}} \succ 0,\ \|\widehat{\mathbf{R}}^{-1}\|_\infty \le Dn^\omega)\}$, $\|\widehat{\mathbf{R}}_* - \mathbf{R}\|_\infty \le \|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty + \|\mathbf{I}_n - \mathbf{R}\|_\infty\{1 - \mathrm{I}(\widehat{\mathbf{R}} \succ 0,\ \|\widehat{\mathbf{R}}^{-1}\|_\infty \le Dn^\omega)\}$. From the result of Theorem 1, it suffices to show $\lim_{n\to\infty} \mathrm{P}(\widehat{\mathbf{R}} \succ 0,\ \|\widehat{\mathbf{R}}^{-1}\|_\infty \le Dn^\omega) = 1$.

By Condition A6, we can verify that there is a constant $M_1 > 0$, such that $\|\mathbf{R}^{-1}\|_\infty < M_1$. Define the event $Q = \{\|\widehat{\mathbf{R}}-\mathbf{R}\|_\infty \le \xi\}$, for some $0 < \xi < \min\{1/M_1, c, 1/C\}$, where c and C are constants in Condition A6. From Theorem 1, $\lim_{n\to\infty} \mathrm{P}(Q) = 1$. Following the proof of Theorem 6 in Cai and Zhou (2012), $\|\widehat{\mathbf{R}}^{-1}\|_\infty$ is bounded on Q. Hence, $\lim_{n\to\infty} \mathrm{P}(\|\widehat{\mathbf{R}}^{-1}\|_\infty \le Dn^\omega) \ge \lim_{n\to\infty} \mathrm{P}(\|\widehat{\mathbf{R}}^{-1}\|_\infty \le Dn^\omega,\ Q) = 1$. Together with $\lim_{n\to\infty} \mathrm{P}(\widehat{\mathbf{R}} \succ 0) = 1$ from Proposition 2, we can conclude $\lim_{n\to\infty} \mathrm{P}(\widehat{\mathbf{R}} \succ 0,\ \|\widehat{\mathbf{R}}^{-1}\|_\infty \le Dn^\omega) = 1$. ■

Proof of Theorem 2. The proof is the same under either Condition A1 or B1. By Condition A6, we can verify that there is a constant $M_1 > 0$, such that $\|\mathbf{R}^{-1}\|_\infty < M_1$. Define the event $Q = \{\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty \le \xi\}$, for some $0 < \xi < \min\{1/M_1, c, 1/C\}$. Following the proof of Theorem 6 in Cai and Zhou (2012), we can show that, for n large enough, $\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty \le C_0\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty$ on Q and $\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty \le C_0 n^\omega$ on Q^c, where $C_0 > 0$ is a constant. Thus, from similar arguments in (S.38), for n large enough,

$$\begin{aligned} E(\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty^2) &= E\{\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty^2\,\mathrm{I}(Q)\} + E\{\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty^2\,\mathrm{I}(Q^c)\} \\ &\le C_0^2 E\{\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2\,\mathrm{I}(Q)\} + C_0^2 n^{2\omega}\mathrm{P}(Q^c) \\ &= C_0^2 E\{\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2\,\mathrm{I}(Q)\} + O(g_n^{14}/n^{1-2\omega}). \end{aligned} \tag{S.39}$$

Since by (S.38), for any constant $\varepsilon > 0$, $\mathrm{P}(\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \ge \varepsilon) = O(g_n^{14}/n) \to 0$, we have $\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \xrightarrow{\mathrm{P}} 0$, which implies $\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2\,\mathrm{I}(Q) \xrightarrow{\mathrm{P}} 0$. Then,

$$\begin{aligned} & E\{\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2\,\mathrm{I}(Q)\,\mathrm{I}(\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2\,\mathrm{I}(Q) \ge \varepsilon)\} \\ \le\ & [E\{\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^4\,\mathrm{I}(Q)\}\mathrm{P}(\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \ge \varepsilon)]^{1/2} \\ \le\ & \{\xi^4\mathrm{P}(\|\widehat{\mathbf{R}} - \mathbf{R}\|_\infty^2 \ge \varepsilon)\}^{1/2} = O(g_n^7/n^{1/2}) \to 0. \end{aligned}$$

By asymptotically uniform integrability, we have $E\{\|\widehat{\mathbf{R}}-\mathbf{R}\|_\infty^2\,\mathrm{I}(Q)\} \to 0$, which together with (S.39) implies $E(\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty^2) \to 0$. ■

Proof of Proposition 4. The proof is the same under either Condition A1 or B1. Following the proof in Proposition 3, $\|\mathbf{R}^{-1}\|_\infty$ is bounded, and $\|\widehat{\mathbf{R}}^{-1}\|_\infty$ is bounded on Q (defined in Proposition 3). Since $\widehat{\mathbf{R}}^{-1} - \mathbf{R}^{-1} = \widehat{\mathbf{R}}^{-1}(\mathbf{R} - \widehat{\mathbf{R}})\mathbf{R}^{-1}$, $\|\widehat{\mathbf{R}}^{-1} - \mathbf{R}^{-1}\|_\infty \le \|\widehat{\mathbf{R}}^{-1}\|_\infty\|\mathbf{R} - \widehat{\mathbf{R}}\|_\infty\|\mathbf{R}^{-1}\|_\infty$ on Q, and hence, $\|\widehat{\mathbf{R}}^{-1} - \mathbf{R}^{-1}\|_\infty \le C\|\mathbf{R} - \widehat{\mathbf{R}}\|_\infty$. From the result in Theorem 1 and $\mathrm{P}(Q^c) = o(1)$, $\|\widehat{\mathbf{R}}^{-1} - \mathbf{R}^{-1}\|_\infty = O_P(g_n^7/n^{1/2})$.

From the result in Proposition 2 that $\lim_{n\to\infty} \mathrm{P}(\widehat{\mathbf{R}} \succ 0) = 1$, it is easy to prove that $\|\widehat{\mathbf{R}}_Z^{-1} - \mathbf{R}^{-1}\|_\infty = O_P(g_n^7/n^{1/2})$.

From $\lim_{n\to\infty} \mathrm{P}(\widehat{\mathbf{R}} \succ 0) = 1$ and that $\|\widehat{\mathbf{R}}^{-1}\|_\infty$ is bounded on Q, it's easy to show $\|\widehat{\mathbf{R}}_*^{-1} - \mathbf{R}^{-1}\|_\infty = O_P(g_n^7/n^{1/2})$. ■