TESTING FOR CHANGE POINTS DUE TO A COVARIATE THRESHOLD IN QUANTILE REGRESSION

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Supplementary Material

This supplementary material contains two remarks and all the technical proofs.

Remark S1 Suppose that $\mathbf{X} = \mathbf{Z}$, and that U and $\mathbf{X}$ are independent, then $\mathbf{S}_{\mathbf{Z}}(u) = \mathbf{S}F_U(u)$ and $\mathbf{Q}\{\boldsymbol{\alpha}_0(\tau)\} = \{1 - F_U(u_0)\}E[\mathbf{X}\mathbf{X}^T\boldsymbol{\alpha}_0(\tau)\ f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}]$. Consequently, the covariance function in Theorem 1 is reduced to $\mathbf{W}(u,u') = \tau(1-\tau)E(\mathbf{X}\mathbf{X}^T)\left[F_U\{\min(u,u')\} - F_U(u)F_U(u')\right]$ and $\mathbf{q}\{u, \alpha_0(\tau)\}$ in Theorem 2 is reduced to $\mathbf{q}\{u, \alpha_0(\tau)\} = -F_U(u)[1-F_U(u_0)]E[\mathbf{X}\mathbf{X}^T\boldsymbol{\alpha}_0(\tau)f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}] + \{F_U(u) - F_U(u_0)\}E[\mathbf{X}\mathbf{X}^T\boldsymbol{\alpha}_0(\tau)f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}]$.

Remark S2 The results in Theorems 1 and 2 can be simplified for models with homoscedastic errors. Consider the following location-shift model under the local alternative

$$Y_i = \mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i > u_0) + \epsilon_i, \qquad \text{(S.1)}$$

where ϵ_i are $i.i.d.$ random variables with the τth quantile zero. Thus

$$T_n(\tau) \Rightarrow \sup_{u\in(0,1)} \|\boldsymbol{R}(u) + \boldsymbol{q}\{u, \boldsymbol{\alpha}_0(\tau)\}\|,$$

where $\mathbf{R}(u)$ is a mean zero Gaussian process with covariance kernel $\mathbf{W}(u,u') = \tau(1-\tau)\left[E(\mathbf{Z}_u\mathbf{Z}_{u'}^T) - E(\mathbf{Z}_u\mathbf{X}^T)\{E(\mathbf{X}\mathbf{X}^T)\}^{-1}E(\mathbf{X}\mathbf{Z}_{u'}^T)\right]$, $\boldsymbol{q}\{u, \boldsymbol{\alpha}_0(\tau)\} = f_\epsilon(0)[-E(\mathbf{Z}_u\mathbf{X}^T)\{E(\mathbf{X}\mathbf{X}^T)\}^{-1}\mathbf{Q}_1\{\boldsymbol{\alpha}_0(\tau)\} + \mathbf{P}_1\{u, \boldsymbol{\alpha}_0(\tau)\}]$, where $f_\epsilon(\cdot)$ is the density function of ϵ_i, $\mathbf{Q}_1\{\boldsymbol{\alpha}_0(\tau)\} = E\{\mathbf{X}\mathbf{Z}^T\boldsymbol{\alpha}_0(\tau)\ I(U > u_0)\}$ and $\mathbf{P}_1\{u, \boldsymbol{\alpha}_0(\tau)\} = E\{\mathbf{Z}\mathbf{Z}^T\boldsymbol{\alpha}_0(\tau)I(u_0 < U \le u)\}$. Note that in this case the limiting null distribution of $T_n(\tau)$ no longer depends on the unknown density function $f_\epsilon(\cdot)$, and this simplifies the calculation of critical values.

Throughout the paper, we use $\|x\|$ to denote the Euclidean norm for a vector x, and use the vector-induced norm, i.e. $\|A\| = \sup_{x\neq 0}\|Ax\|/\|x\|$ for a matrix A. Let $[x]$ denote the integer part of x. Denote $\boldsymbol{\theta}(\tau) = (\boldsymbol{\beta}^T(\tau), \boldsymbol{\alpha}^T(\tau))^T$, $\boldsymbol{\theta}_0(\tau) = (\boldsymbol{\beta}_0^T(\tau), \boldsymbol{\alpha}_0^T(\tau))^T$ and

$$\mathbf{R}_n\{u, \tau, \boldsymbol{\beta}(\tau)\} = n^{-1/2}\sum_{i=1}^n \psi_\tau\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}(\tau)\}\mathbf{Z}_i I(U_i \le u),$$

$$\mathbf{R}_n^c\{u,\tau,\boldsymbol{\beta}(\tau)\} = n^{-1/2}\sum_{i=1}^n [F_i\{\mathbf{X}_i^T\boldsymbol{\beta}(\tau)\} - I\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}(\tau)\le 0\}]\mathbf{Z}_i I(U_i\le u),$$

and

$$\mathbf{R}_n^d\{u,\tau,\theta_0(\tau)\} = n^{-1/2}\sum_{i=1}^n \psi_\tau\{Y_i - \mathbf{X}_i^T\beta_0(\tau) - n^{-1/2}\mathbf{Z}_i^T\alpha_0(\tau)I(U_i\ge u_0)\}\mathbf{Z}_i I(U_i\le u),$$

where $\mathbf{R}_n^c\{u,\tau,\boldsymbol{\beta}(\tau)\}$ is the statistic by re-centering $I\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}(\tau)\le 0\}$ at its expectation conditional on $\mathbf{X}_i$.

Proof of Proposition 1. The proof of Proposition 1(i) is given in the Supplementary Material in Lee et al.(2011). We only give the proof of Proposition 1(ii). Let $\boldsymbol{P}$ and $\boldsymbol{P}_n$ be the common probability measure and the empirical measure of the random sample of sample size n under the local alternative hypothesis. Also let $q\{Y,\mathbf{X};\boldsymbol{\beta}(\tau)\} = -\rho_\tau\{Y-\mathbf{X}^T\boldsymbol{\beta}(\tau)\}$ and $q\{Y,\mathbf{W};\boldsymbol{\theta}(\tau),u\} = -\rho_\tau\{Y-\mathbf{X}^T\boldsymbol{\beta}(\tau)-n^{-1/2}\mathbf{Z}^T\boldsymbol{\alpha}(\tau)I(U>u)\}$ be the objective functions under the null and the alternative hypothesis with change point $u_0 = u$. Note that

$$\begin{aligned} q\{Y,\mathbf{W};\boldsymbol{\theta}(\tau),u\} &= \{Y-\mathbf{X}^T\boldsymbol{\beta}(\tau) - n^{-1/2}\mathbf{Z}^T\boldsymbol{\alpha}(\tau)I(U>u)\} \\ &\quad \times[-\tau + I\{Y-\mathbf{X}^T\boldsymbol{\beta}(\tau) - n^{-1/2}\mathbf{Z}^T\boldsymbol{\alpha}(\tau)I(U>u)<0\}]. \end{aligned}$$

Let $\mathbf{0}_q$ be a $q-$dimensional vector of zeros. The first derivative of $q\{Y,\mathbf{W};\boldsymbol{\theta}(\tau),u\}$ with respect to $\boldsymbol{\theta}(\tau)$ evaluated at $\tilde{\boldsymbol{\theta}}_0(\tau) = (\boldsymbol{\beta}_0^T(\tau),\mathbf{0}_q^T)^T$ is as follows

$$\begin{aligned} & \frac{\partial}{\partial\boldsymbol{\theta}} q\{Y,\mathbf{W};\boldsymbol{\theta}(\tau),u\}|_{\boldsymbol{\theta}(\tau)=\tilde{\boldsymbol{\theta}}_0(\tau)} \\ = \;& -\tilde{\mathbf{X}}_u[-\tau + I\{Y-\mathbf{X}^T\boldsymbol{\beta}(\tau)-n^{-1/2}\mathbf{Z}^T\boldsymbol{\alpha}(\tau)I(U>u)<0\}]|_{\boldsymbol{\theta}(\tau)=\tilde{\boldsymbol{\theta}}_0(\tau)} \\ = \;& -\tilde{\mathbf{X}}_u[-\tau + I\{Y-\mathbf{X}^T\boldsymbol{\beta}_0(\tau)<0\}]. \end{aligned}$$

Thus we have

$$\begin{aligned} & n^{1/2}\boldsymbol{P}_n\frac{\partial}{\partial\boldsymbol{\theta}} q\{Y,\mathbf{W};\boldsymbol{\theta}(\tau),u\}|_{\boldsymbol{\theta}(\tau)=\tilde{\boldsymbol{\theta}}_0(\tau)} \\ = \;& n^{1/2}E(-\tilde{\mathbf{X}}_u[-\tau + F\{\mathbf{X}^T\boldsymbol{\beta}_0(\tau)|\mathbf{X}\}]) \\ = \;& n^{1/2}E(-\tilde{\mathbf{X}}_u[-F\{\mathbf{X}^T\boldsymbol{\beta}_0(\tau)+n^{-1/2}\mathbf{Z}^T\boldsymbol{\alpha}_0(\tau)I(U>u_0)|\mathbf{X}\} + F\{\mathbf{X}^T\boldsymbol{\beta}_0(\tau)|\mathbf{X}\}]) \\ = \;& n^{1/2}E(-\tilde{\mathbf{X}}_u[-n^{-1/2}f\{\alpha_{n,u_0}(\tau)|\mathbf{X}\}\mathbf{Z}^T\boldsymbol{\alpha}_0(\tau)I(U>u_0)]) \\ \to \;& (E[\mathbf{X}\mathbf{Z}^T\boldsymbol{\alpha}_0(\tau)I(U>u_0)f\{\mathbf{X}^T\boldsymbol{\beta}_0(\tau)|\mathbf{X}\}]^T,\; E[\mathbf{Z}\mathbf{Z}^T\boldsymbol{\alpha}_0(\tau)I\{U>\max(u,u_0)\} \\ & f\{\mathbf{X}^T\boldsymbol{\beta}_0(\tau)|\mathbf{X}\}]^T)^T \\ = \;& \widetilde{\boldsymbol{q}}_L\{u,\boldsymbol{\alpha}_0(\tau)\}, \end{aligned}$$

where $\alpha_{n,u_0}(\tau)$ is a value lying between $\mathbf{X}^T\boldsymbol{\beta}_0(\tau)$ and $\mathbf{X}^T\boldsymbol{\beta}_0(\tau)+n^{-1/2}\mathbf{Z}^T\boldsymbol{\alpha}_0(\tau)I(U>u_0)$. It is obvious that $\alpha_{n,u_0}(\tau)\to\mathbf{X}^T\boldsymbol{\beta}_0(\tau)$ as n goes to infinity.

Similarly, for the first derivative of $q\{Y,\mathbf{W};\boldsymbol{\beta}(\tau)\}$ we have

$$n^{1/2}\boldsymbol{P}_n\frac{\partial}{\partial\boldsymbol{\beta}}q\{Y,\mathbf{X};\boldsymbol{\beta}(\tau)\}|_{\boldsymbol{\beta}(\tau)=\boldsymbol{\beta}_0(\tau)}\ \to\ \tilde{\boldsymbol{q}}_1.$$

By Proposition 1(i), the local asymptotic limiting distribution for the local alternative hypothesis can be characterized by

$$\frac{1}{2}(\sup_u[\mathcal{G}(u)+\widetilde{\boldsymbol{q}}_L\{u,\boldsymbol{\alpha}_0(\tau)\}]^T\boldsymbol{V}(u)^{-1}[\mathcal{G}(u)+\widetilde{\boldsymbol{q}}_L\{u,\boldsymbol{\alpha}_0(\tau)\}]-(\mathcal{G}_1^T+\widetilde{\boldsymbol{q}}_1)^T\mathbf{V}_1^{-1}(\mathcal{G}_1+\widetilde{\boldsymbol{q}}_1)).$$

This completes the proof of Proposition 1.

Lemma 1 *(i) Suppose that Assumptions A1–A3 hold, we have*
(i1). $n^{-1}\sum_{i=1}^n f_i\,\{F_i^{-1}(\tau)\}\mathbf{Z}_i\mathbf{X}_i^TI(U_i\le u)\xrightarrow{p}\mathbf{S}_{\mathbf{z}}(u)$ *uniformly in* $u\in(0,1)$*;*
(i2). $n^{-1}\sum_{i=1}^n\mathbf{Z}_i\mathbf{X}_i^TI(U_i\le u)\xrightarrow{p}E(\mathbf{Z}_u\mathbf{X}^T)$ *uniformly in* $u\in(0,1)$*;*
(i3). $n^{-1}\sum_{i=1}^n f_i\{F_i^{-1}(\tau)\}\mathbf{Z}_i\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(u_0\le U_i\le u)\}\xrightarrow{p}\boldsymbol{P}\{u,\boldsymbol{\alpha}_0(\tau)\}$ *uniformly in* $u\in(0,1)$*;*
(i4). $n^{-1}\sum_{i=1}^n\mathbf{X}_i\ \mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(u_0\le U_i\le u)\}\xrightarrow{p}\boldsymbol{Q}\{\boldsymbol{\alpha}_0(\tau)\}$ *uniformly in* $u\in(0,1)$.
(ii). Suppose that Assumptions A2, A3 and A6 are satisfied, then (i1)-(i4) hold uniformly in $u\in(0,1)$ *and* $\tau\in\mathcal{T}$.

Proof of Lemma 1. (i) We only give the proof of (i1), since the proofs of (i2)-(i4) are similar. To prove (i1), we need to show that

$$\left\|n^{-1}\sum_{i=1}^n f_i\{F_i^{-1}(\tau)\}\mathbf{Z}_i\mathbf{X}_i^TI(U_i\le u)-\boldsymbol{S}_z(u)\right\|=o_p(1),$$

uniformly in $u\in(0,1)$. Let $U_1,\cdots,U_n$ be a random sample from the measurable space $(\chi,\mathcal{A})$. Hence it is sufficient to show the element-wise uniform convergence of the matrix $n^{-1}\sum_{i=1}^n f_i\{F_i^{-1}(\tau)\}\mathbf{Z}_i\mathbf{X}_i^TI(U_i\le u)$. Let $\mathbf{Z}=(Z^{(1)},\cdots,Z^{(q)})^T$, $\mathbf{X}=(X^{(1)},\cdots,X^{(p)})^T$, where $Z^{(j)}$ $(j=1,\cdots,q)$ and $X^{(k)}$ $(k=1,\cdots,p)$ could be the variable U or any other covariates. To obtain the desired results, we need to show that for any $j,k=1,\cdots,n$, we have

$$n^{-1}\sum_{i=1}^n f_i\{F_i^{-1}(\tau)\}Z_{ij}X_{ik}I(U_i\le u)\xrightarrow{p}E[f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}Z^{(j)}X^{(k)}I(U\le u)],$$

uniformly in $u\in(0,1)$. Let $f_u(Z^{(j)},X^{(k)},U)=f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}Z^{(j)}X^{(k)}I(U\le u)$, and denote the class of measurable functions $\mathcal{F}=\{f_u(Z^{(j)},Z^{(k)},U):\chi\to R\}$. It is sufficient to show that $f_u(Z^{(j)}, X^{(k)}, U)$ is *P-Glivenko-Cantelli*. That is we want to show that, for any $\epsilon > 0$, there exist finite brackets $\{(l_u^m(Z^{(j)}, X^{(k)}, U), h_u^m(Z^{(j)}, X^{(k)}, U)), m = 1, \cdots, N\}$ such that for any $f_u(Z^{(j)}, X^{(k)}, U) \in \mathcal{F}$, there exists some m such that $l_u^m(Z^{(j)}, X^{(k)}, U) \le f_u(Z^{(j)}, X^{(k)}, U) \le h_u^m(Z^{(j)}, X^{(k)}, U)$, and

$$E\{h_u^m(Z^{(j)}, X^{(k)}, U) - l_u^m(Z^{(j)}, X^{(k)}, U)\} \le \epsilon,$$

where E denotes the expectation under the variables $Z^{(j)}$, $X^{(k)}$ and U. We partition the region (0,1) into N intervals of equal length at the points $u_1, \cdots, u_{N+1}$, where $u_1 = 0$ and $u_{N+1} = 1$. For any $\epsilon > 0$, take

$$\begin{aligned} l_u^m(Z^{(j)}, X^{(k)}, U) = Z^{(j)}X^{(k)}I\{Z^{(j)}X^{(k)} > 0\}I(U \le u_{m-1}) \\ +Z^{(j)}X^{(k)}I\{Z^{(j)}X^{(k)} < 0\}I(U \le u_m), \end{aligned}$$

and

$$\begin{aligned} h_u^m(Z^{(j)}, X^{(k)}, U) = Z^{(j)}X^{(k)}I\{Z^{(j)}X^{(k)} > 0\}I(U \le u_m) \\ +Z^{(j)}X^{(k)}I\{Z^{(j)}X^{(k)} < 0\}I(U \le u_{m-1}), \end{aligned}$$

for $m = 1, \cdots, M$.

For any m and $l_u^m(Z^{(j)}, X^{(k)}, U) \le f_u(Z^{(j)}, X^{(k)}, U) \le h_u^m(Z^{(j)}, X^{(k)}, U)$, we have

$$\begin{aligned} & E\{h_u^m(Z^{(j)}, X^{(k)}, U) - l_u^m(Z^{(j)}, X^{(k)}, U)\} \\ = & \int f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}z^{(j)}x^{(k)}I\{z^{(j)}x^{(k)} > 0\}I(u_{m-1} \le u \le u_m)dP \\ & - \int f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}z^{(j)}x^{(k)}I\{z^{(j)}x^{(k)} < 0\}I(u_{m-1} \le u \le u_m)dP \\ = & (a) + (b). \end{aligned}$$

For (a), we have

$$\begin{aligned} (a) &= \int f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}z^{(j)}x^{(k)}I\{z^{(j)}x^{(k)} > 0\}I(u_{m-1} \le u \le u_m)dP \\ &= \Big(\int [f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}z^{(j)}x^{(k)}]^2 I\{z^{(j)}x^{(k)} > 0\}dP \int I(u_{m-1} \le u \le u_m)dP\Big)^{1/2} \\ &\le \Big(E[f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}z^{(j)}x^{(k)}]^2 f_U(u^*)/N\Big)^{1/2} \\ &\le (M_1 L/N)^{1/2} \\ &= (M/N)^{1/2}, \end{aligned}$$

where M_1 and M are some positive constants, f is bounded by L, $M = M_1 L$ and u^* lies between u_{m-1} and u_m, and the third inequality follows from Cauchy-Schwarz inequality.

Similarly, for (b) we have

$$\begin{aligned}(b) &= \int f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}\{-z^{(j)}x^{(k)}\}I\{z^{(j)}x^{(k)} < 0\}I(u_{m-1} \leq u \leq u_m)dP \\ &= \Big(\int [f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}\{-z^{(j)}x^{(k)}\}]^2 I\{z^{(j)}x^{(k)} < 0\}dP \int I(u_{m-1} \leq u \leq u_m)dP\Big)^{1/2} \\ &\leq \Big(E[f\{F^{-1}(\tau|\mathbf{X})|\mathbf{X}\}z^{(j)}x^{(k)}]^2 f_U(u^*)/N\Big)^{1/2} \\ &\leq (M_1 L/N)^{1/2} \\ &= (M/N)^{1/2}.\end{aligned}$$

Hence, we obtain

$$E\{h_u^m(Z^{(j)}, X^{(k)}, U) - l_u^m(X^{(j)}, X^{(k)}, U)\} \leq 2(M/N)^{1/2} \leq \epsilon,$$

where the last inequality above follows by taking $N \geq [4M/\epsilon^2]+1$. Hence the minimum number of $\epsilon-$brackets needed to cover $\mathcal{F}$ is $[4M/\epsilon^2]+1$ and is finite. By Glivenko-Cantelli theorem, we can get the uniform convergence.

(ii). The uniform convergence in $u \in (0,1)$ and $\tau \in \mathcal{T}$ can be proven in a similar way by replacing the function $f_u(Z^{(j)}, X^{(k)}, U)$ with $f_{u,\tau}(Z^{(j)}, X^{(k)}, U)$ and thus is omitted. This completes the proof of Lemma 1.

Lemma 2 *Let* $\mathbf{D}$ *be an arbitrary compact set in* $\mathbf{R}^p$. *Under Assumptions A1-A3 and* H_1*, we have*

$$\sup_{u \in (0,1)} \sup_{\boldsymbol{\xi} \in \mathbf{D}} \|\boldsymbol{R}_n^c\{u, \tau, \boldsymbol{\beta}_0(\tau) + n^{-1/2}\boldsymbol{\xi}\} - \boldsymbol{R}_n^d\{u, \tau, \theta_0(\tau)\}\| = o_p(1).$$

Proof of Lemma 2. Without loss of generality, we assume that the components of $\mathbf{X}_i$ are nonnegative. Then $\mathbf{Z}_i I\{Y_i < \mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)+n^{-1/2}\mathbf{X}_i^T\boldsymbol{\xi}\}$ and $F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)+n^{-1/2}\mathbf{X}_i^T\boldsymbol{\xi}\}$ are nondecreasing in $\boldsymbol{\xi}$. Because $\mathbf{D}$ is compact, for any given $\delta > 0$, $\mathbf{D}$ can be partitioned into a finite number of subsets $\mathbf{D}_1, \cdots, \mathbf{D}_{n(\delta)}$, where the diameter of each subset is less than or equal to δ. For $\boldsymbol{\xi} \in \mathbf{D}_h$, $h \in \{1, \cdots, n(\delta)\}$, there exists two points $\boldsymbol{\xi}_{h,1}$ and $\boldsymbol{\xi}_{h,2}$ in $\mathbf{D}_h$ such that $\mathbf{X}_i^T\boldsymbol{\xi}_{h,1} \leq \mathbf{X}_i^T\boldsymbol{\xi} \leq \mathbf{X}_i^T\boldsymbol{\xi}_{h,2}$. By the monotonicity of

$\mathbf{Z}_i I\{Y_i < \mathbf{X}_i^T \boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{X}_i^T \boldsymbol{\xi}\}$, we have

$$\begin{aligned}
& \mathbf{R}_n^c\{u,\tau,\boldsymbol{\beta}_0(\tau)+n^{-1/2}\boldsymbol{\xi}\} - \mathbf{R}_n^d\{u,\tau,\theta_0(\tau)\} \\
\geq\ & n^{-1/2}\sum_{i=1}^n [F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)+n^{-1/2}\mathbf{X}_i^T\boldsymbol{\xi}\} \\
& -I\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) - n^{-1/2}\mathbf{X}_i^T\boldsymbol{\xi}_{h,2} < 0\}]\mathbf{Z}_i I(U_i \leq u) - \mathbf{R}_n^d\{u,\tau,\theta_0(\tau)\} \\
=\ & [\mathbf{R}_n^c\{u,\tau,\boldsymbol{\beta}_0(\tau)+n^{-1/2}\boldsymbol{\xi}_{h,2}\} - \mathbf{R}_n^d\{u,\tau,\theta_0(\tau)\}] - n^{-1/2}\sum_{i=1}^n [F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) \\
& +n^{-1/2}\mathbf{X}_i^T\xi_{h,2}\} - F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)+n^{-1/2}\mathbf{X}_i^T\boldsymbol{\xi}\}]\mathbf{Z}_i I(U_i \leq u). \qquad \text{(S.2)}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \mathbf{R}_n^c\{u,\tau,\boldsymbol{\beta}_0(\tau)+n^{-1/2}\boldsymbol{\xi}\} - \mathbf{R}_n^d\{u,\tau,\theta_0(\tau)\} \\
\leq\ & [\mathbf{R}_n^c\{u,\tau,\boldsymbol{\beta}_0(\tau)+n^{-1/2}\boldsymbol{\xi}_{h,1}\} - \mathbf{R}_n^d\{u,\tau,\theta_0(\tau)\}] - n^{-1/2}\sum_{i=1}^n [F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) \\
& +n^{-1/2}\mathbf{X}_i^T\xi_{h,1}\} - F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)+n^{-1/2}\mathbf{X}_i^T\boldsymbol{\xi}\}]\mathbf{Z}_i I(U_i \leq u). \qquad \text{(S.3)}
\end{aligned}$$

By the inequality $|y| \leq \max(|x|,|z|)$ for $x \leq y \leq z$ and combining (S.2) and (S.3), we obtain

$$\begin{aligned}
& \sup_{\xi\in D}\sup_{u\in(0,1)} \|\mathbf{R}_n^c\{u,\tau,\boldsymbol{\beta}_0(\tau)+n^{-1/2}\boldsymbol{\xi}\} - \mathbf{R}_n^d\{u,\tau,\theta_0(\tau)\}\| \\
\leq\ & \max_{1\leq h\leq n(\delta)}\max_{k=1,2}\sup_{u\in(0,1)} \|\mathbf{R}_n^c\{u,\tau,\boldsymbol{\beta}_0(\tau)+n^{-1/2}\boldsymbol{\xi}_{h,k}\} - \mathbf{R}_n^d\{u,\tau,\boldsymbol{\theta}_0(\tau)\}\| \\
& + \max_{1\leq h\leq n(\delta)}\max_{k=1,2}\sup_{u\in(0,1)} \Bigg\| n^{-1/2}\sum_{i=1}^n [F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)+n^{-1/2}\mathbf{X}_i^T\xi_{h,k}\} \\
& -F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)+n^{-1/2}\mathbf{X}_i^T\boldsymbol{\xi}\}]\mathbf{Z}_i I(U_i \leq u)\Bigg\| \\
=\ & (c)+(d), \qquad \text{(S.4)}
\end{aligned}$$

where

$$(c) = \max_{1\leq h\leq n(\delta)}\max_{k=1,2}\sup_{u\in(0,1)} \Bigg\| n^{-1/2}\sum_{i=1}^n \psi_i(h,k,u)\Bigg\|,$$

and $\psi_i(h,k,u) = [F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)+n^{-1/2}\mathbf{X}_i^T\boldsymbol{\xi}_{h,k}\} - I\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) - n^{-1/2}\mathbf{X}_i^T\boldsymbol{\xi}_{h,k} < 0\} - F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)+n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i \geq u_0)\} + I\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) - n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i \geq u_0) < 0\}]\mathbf{Z}_i I(U_i \leq u)$. Because the distance between $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_{h,k}$ is less than δ, by mean value theorem and Assumption A3(a) and Lemma 1, it is easy to show that $(d) = \delta O_p(1)$, which can be arbitrarily small by choosing a small δ.

Let $U_{(i)}$ be the i-th order statistic of $\{U_i; i = 1, \cdots, n\}$, and $\{(Y_{(i)}, \mathbf{X}_{(i)}, \mathbf{Z}_{(i)}); i = 1, \cdots, n\}$ be the observation and $\{F_{(i)}; i = 1, \cdots, n\}$ be the distribution functions corresponding to $U_{(i)}$. For notational simplicity, we omit h, k, u in the expression $\psi_i(h, k, u)$ when no confusion is made. Let $\psi_{(i)}$ be the score function of the subject associated with $U_{(i)}$. Then

$$\begin{aligned}\psi_{(i)} = \Big(& F_{(i)}\{\mathbf{X}_{(i)}^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{X}_{(i)}^T\boldsymbol{\xi}_{h,k}\} - I\{Y_{(i)} - \mathbf{X}_{(i)}^T\boldsymbol{\beta}_0(\tau) - n^{-1/2}\mathbf{X}_{(i)}^T\boldsymbol{\xi}_{h,k} < 0\} \\ & - F_{(i)}[\mathbf{X}_{(i)}^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{Z}_{(i)}^T\boldsymbol{\alpha}_0(\tau)I\{U_{(i)} \geq u_0\}] \\ & + I[Y_{(i)} - \mathbf{X}_{(i)}^T\boldsymbol{\beta}_0(\tau) - n^{-1/2}\mathbf{Z}_{(i)}^T\boldsymbol{\alpha}_0(\tau)I\{U_{(i)} \geq u_0\} < 0]\Big)\mathbf{Z}_{(i)}I\{U_{(i)} \leq u\}.\end{aligned}$$

Then for an arbitrarily small number $\epsilon > 0$, we have

$$\begin{aligned} & P\Big\{\max_{h=1,\cdots,n(\delta)}\max_{k=1,2}\sup_{u\in(0,1)}\Big\|\sum_{i=1}^n \psi_i\Big\| > \epsilon\Big\} \\ = \; & P\Big\{\max_{h=1,\cdots,n(\delta)}\max_{k=1,2}\sup_{u\in(0,1)}\Big\|\sum_{i=1}^n \psi_{(i)}\Big\| > \epsilon\Big\} \\ < \; & n(\delta)\max_{h=1,\cdots,n(\delta)}\max_{k=1,2} P\Big\{\max_{j=1,\cdots,n}\Big\|\sum_{i=1}^j \psi_{(i)}\Big\| > \epsilon\Big\}. \qquad \text{(S.5)}\end{aligned}$$

Let the σ-fields $\mathscr{F}_{(i)} = \sigma\{\psi_{(1)}, \cdots, \psi_{(i)}\}$ for $i = 1, \cdots, n$. Because of the equality $E\{\psi_{(i)}|\mathbf{X}_{(i)}\} = 0$, then $\{\psi_{(i)}, \mathscr{F}_{(i)}\}$ is an array of martingale difference. For every $\gamma > 1$, applying Doob inequality gives

$$\begin{aligned} P\left\{\max_{j=1,\cdots,n}\left\|n^{-1/2}\sum_{i=1}^j \psi_{(i)}\right\| > \epsilon\right\} & \leq M_2\epsilon^{-2\gamma}E\left\|n^{-1/2}\sum_{i=1}^n \psi_{(i)}\right\|^{2\gamma} \\ & = M_2\epsilon^{-2\gamma}E\left\|n^{-1/2}\sum_{i=1}^n \psi_i\right\|^{2\gamma}, \qquad \text{(S.6)}\end{aligned}$$

where M_2 is a constant depending on γ. Because $\{\psi_i, \mathscr{F}_i\}$ is an array of martingale difference, by the Rosenthal inequality (Hall and Heyde, 1980), we have

$$E\left\|n^{-1/2}\sum_{i=1}^n \psi_i\right\|^{2\gamma} \leq M_3 n^{-\gamma}E\left\{\sum_{i=1}^n E(\|\psi_i\|^2|\mathscr{F}_{i-1})\right\}^{\gamma} + M_3 n^{-\gamma}\sum_{i=1}^n E\|\psi_i\|^{2\gamma},$$

where M_3 is a constant. Note that

$$\begin{aligned} & E(\|\psi_i\|^2|\mathscr{F}_{i-1}) \\ \leq \; & \|\mathbf{Z}_i\|^2|F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{X}_i^T\boldsymbol{\xi}_{h,k}\} - F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i \geq u_0)\}| \\ = \; & \|\mathbf{Z}_i\|^2|f_i(\zeta)|n^{-1/2}|\mathbf{X}_i^T\boldsymbol{\xi}_{h,k} - \mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i \geq u_0)| \\ \leq \; & Ln^{-1/2}\{\|\mathbf{Z}_i\|^2\|\|\mathbf{X}_i\|\|\boldsymbol{\xi}_{h,k}\| + \|\mathbf{Z}_i\|^3\|\boldsymbol{\alpha}_0(\tau)I(U_i \geq u_0)\|\},\end{aligned}$$

where ζ is between $\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)+n^{-1/2}\mathbf{Z}_i^T\alpha_0(\tau)I(U_i\geq u_0)$ and $\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)+n^{-1/2}\mathbf{X}_i^T\boldsymbol{\xi}_{h,k}$. By Hölder's inequality,

$$\begin{aligned} E\|\psi_i\|^{2\gamma} &= E\{E(\|\psi_i\|^{2\gamma}|\mathscr{F}_{i-1})\}\leq E\{E(\|\psi_i\|^2|\mathscr{F}_{i-1})\}^{\gamma} \\ &\leq L^{\gamma}n^{-\gamma/2}E\{\|\mathbf{Z}_i\|^2\|\|\mathbf{X}_i\|\|\boldsymbol{\xi}_{h,k}\|+\|\mathbf{Z}_i\|^3\|\alpha_0(\tau)I(U_i\geq u_0)\|\}^{\gamma}. \end{aligned}$$

Hence we have

$$\begin{aligned} & E\left\|n^{-1/2}\sum_{i=1}^n\psi_i\right\|^{2\gamma} \\ \leq\; & M_3L^{\gamma}n^{-\gamma/2}E\left[n^{-1}\sum_{i=1}^n\{\|\mathbf{X}_i\|^2\|\mathbf{Z}_i\|\boldsymbol{\xi}_{h,k}\|+\|\mathbf{X}_i\|^2\|\mathbf{Z}_i^T\boldsymbol{\alpha}_0I(U_i\geq u_0)\|\}\right]^{\gamma} \\ & +M_3L^{\gamma}n^{-3\gamma/2+1}n^{-1}\sum_{i=1}^nE\{\|\mathbf{X}_i\|^2\|\mathbf{Z}_i\|\|\boldsymbol{\xi}_{h,k}\|+\|\mathbf{Z}_i\|^3\|\alpha_0(\tau)I(U_i\geq u_0)\|\}^{\gamma} \\ \leq\; & M_3L^{\gamma}n^{-\gamma/2}E\left[n^{-1}\sum_{i=1}^n\{\|\mathbf{X}_i\|^2\|\mathbf{Z}_i\|\|\mathbf{D}\|+\|\mathbf{Z}_i\|^4\|\boldsymbol{\alpha}_0(\tau)I(U_i\geq u_0)\|\}\right]^{\gamma} \\ & +M_3L^{\gamma}n^{-3\gamma/2+1}n^{-1}\sum_{i=1}^nE\{\|\mathbf{X}_i\|^2\|\mathbf{Z}_i\|\|\mathbf{D}\|+\|\mathbf{Z}_i\|^3\|\boldsymbol{\alpha}_0(\tau)I(U_i\geq u_0)\|\}^{\gamma} \\ \leq\; & M_3M_4L^{\gamma}n^{-\gamma/2}E\left[n^{-1}\sum_{i=1}^n\{\|\mathbf{X}_i\|^2\|\mathbf{Z}_i\|+\|\mathbf{Z}_i\|^3\|\alpha_0(\tau)I(U_i\geq u_0)\|\}\right]^{\gamma} \\ & +M_3L^{\gamma}n^{-3\gamma/2+1}n^{-1}\sum_{i=1}^nE\{\|\mathbf{X}_i\|^2\|\mathbf{Z}_i\|+\|\mathbf{Z}_i\|^3\|\boldsymbol{\alpha}_0(\tau)I(U_i\geq u_0)\|\}^{\gamma} \\ \leq\; & 2MM_3M_4L^{\gamma}n^{-\gamma/2}, \end{aligned} \tag{S.7}$$

where M_3 and M_4 are some finite constants, $M_4=\max(\|\mathbf{D}\|^{\gamma},1)$. In the above, the second inequality follows because $\|\boldsymbol{\xi}_{h,k}\|\leq\|\mathbf{D}\|$, and the fourth inequality follows from the Assumption A3(c). Combining (S.4), (S.5), (S.6) and (S.7), yields that

$$\begin{aligned} & \sup_{\xi\in D}\sup_{u\in(0,1)}\|\mathbf{R}_n^c\{u,\tau,\boldsymbol{\beta}_0(\tau)+n^{-1/2}\boldsymbol{\xi}\}-\mathbf{R}_n^d\{u,\tau,\theta_0(\tau)\}\| \\ \leq\; & 4n(\delta)MM_2M_3M_4L^{\gamma}n^{-\gamma/2}\epsilon^{-2\gamma} \\ =\; & o(1). \end{aligned}$$

This completes the proof of Lemma 2.

Lemma 3 *Under Assumptions A1-A3 and H_1, we have*

$$\sup_{u\in(0,1)}\sup_{\boldsymbol{\xi}\in D}\|\boldsymbol{R}_n\{u,\tau,\boldsymbol{\beta}_0(\tau)+n^{-1/2}\boldsymbol{\xi}\}-\boldsymbol{R}_n^d\{u,\tau,\boldsymbol{\theta}_0(\tau)\}+\mathbf{S}_{\mathbf{z}}(u)\boldsymbol{\xi}-\boldsymbol{P}\{u,\boldsymbol{\alpha}_0(\tau)\}\|=o_p(1).$$

Proof of Lemma 3. Direct calculation gives that

$$\begin{aligned}
& \mathbf{R}_n\{u,\tau,\boldsymbol{\beta}_0(\tau)+n^{-1/2}\boldsymbol{\xi}\} - \mathbf{R}_n^d\{u,\tau,\boldsymbol{\theta}_0(\tau)\} + \mathbf{S}_{\mathbf{z}}(u)\boldsymbol{\xi} - \mathbf{P}\{u,\boldsymbol{\alpha}_0(\tau)\} \\
= \; & \big[\mathbf{R}_n^c\{u,\tau,\boldsymbol{\beta}_0(\tau)+n^{-1/2}\boldsymbol{\xi}\} - \mathbf{R}_n^d\{u,\tau,\boldsymbol{\theta}_0(\tau)\}\big] + \Big(n^{-1/2}\sum_{i=1}^n \big[\tau - F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) \\
& + n^{-1/2}\mathbf{X}_i^T\boldsymbol{\xi}\}\big]\mathbf{Z}_i I(U_i\le u) + \mathbf{S}_{\mathbf{z}}(u)\boldsymbol{\xi} - \mathbf{P}\{u,\boldsymbol{\alpha}_0(\tau)\}\Big) \\
= \; & (e)+(f).
\end{aligned}$$

To obtain the desired result, it is sufficient to show that (e) and (f) are $o_p(1)$ uniformly in $u\in(0,1)$ and $\boldsymbol{\xi}\in\mathbf{D}$. The uniform property of the first term (e) is obtained by Lemma 1. It remains to show that (f) is $o_p(1)$ uniformly in $u\in(0,1)$ and $\boldsymbol{\xi}\in\mathbf{D}$. Because we have

$$\begin{aligned}
\|(f)\| = & \Big\| n^{-1}\sum_{i=1}^n f_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i\ge u_0)+\eta_i\} \\
& \times\{-\mathbf{Z}_i\mathbf{X}_i^T I(U_i\le u)\boldsymbol{\xi} + \mathbf{Z}_i\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(u_0\le U_i\le u)\} + \mathbf{S}_{\mathbf{z}}(u)\boldsymbol{\xi} - \mathbf{P}\{u,\boldsymbol{\alpha}_0(\tau)\}\Big\| \\
\le & \Big\| - n^{-1}\sum_{i=1}^n f_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i\ge u_0)+\eta_i\}\mathbf{Z}_i\mathbf{X}_i^T I(U_i\le u)\boldsymbol{\xi} \\
& + \mathbf{S}_{\mathbf{z}}(u)\boldsymbol{\xi}\Big\| + \Big\| n^{-1}\sum_{i=1}^n f_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)+\eta_i\}\mathbf{Z}_i\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(u_0\le U_i\le u) - \mathbf{P}\{u,\boldsymbol{\alpha}_0(\tau)\}\Big\| \\
= & \|(g)\| + \|(h)\|,
\end{aligned}$$

where the first equality follows from the mean value theorem with η_i between 0 and $n^{-1/2}\mathbf{X}_i^T\boldsymbol{\xi} - n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i\ge u_0)$. By Assumption A3(a), we have $\max_{1\le i\le n}|\eta_i| = o_p(1)$. Then from Assumption A1, for each i, we have

$$\begin{aligned}
& f_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i\ge u_0)+\eta_i\} \\
= \; & f_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i\ge u_0)\} + o_p(1).
\end{aligned}$$

Then

$$\begin{aligned}
& -n^{-1}\sum_{i=1}^n f_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i\ge u_0)+\eta_i\}\mathbf{Z}_i\mathbf{X}_i^T I(U_i\le u)\boldsymbol{\xi} + \mathbf{S}_{\mathbf{z}}(u)\boldsymbol{\xi} \\
= \; & -n^{-1}\sum_{i=1}^n f_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i\ge u_0)\}\mathbf{Z}_i\mathbf{X}_i^T I(U_i\le u)\boldsymbol{\xi} + \mathbf{S}_{\mathbf{z}}(u)\boldsymbol{\xi} + o_p(1) \\
= \; & o_p(1).
\end{aligned}$$

By Lemma 1, uniformly in $u\in(0,1)$ and $\boldsymbol{\xi}\in\mathbf{D}$, we have $\sup_{u\in(0,1)}\sup_{\boldsymbol{\xi}\in D}\|(g)\| = o_p(1)$. Following the similar arguments, we can show that $\sup_{u\in(0,1)}\sup_{\boldsymbol{\xi}\in D}\|(h)\| = o_p(1)$. This completes the proof of Lemma 3.

Lemma 4 *Under Assumptions A1-A3 and H_1, $\hat{\boldsymbol{\beta}}(\tau)$ has the following Bahadur representation:*

$$\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau) = \mathbf{S}^{-1}\left\{n^{-1}\sum_{i=1}^{n}\psi_\tau(\epsilon_i)\mathbf{X}_i\right\} + n^{-1/2}\boldsymbol{S}^{-1}\boldsymbol{Q}\{\boldsymbol{\alpha}_0(\tau)\} + o_p(1). \quad \text{(S.8)}$$

Proof of Lemma 4. First, the consistency of $\hat{\boldsymbol{\beta}}(\tau)$ can be obtained from Procházka (1988). Applying Lemma 4.1 of He and Shao (1996), we get

$$\sup_{\|\boldsymbol{\beta}(\tau)-\boldsymbol{\beta}_0(\tau)\|\le\delta_n}\left\|n^{-1/2}\sum_{i=1}^{n}[\psi_\tau\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}(\tau)\} - \psi_\tau(\epsilon_i)]\mathbf{X}_i - n^{-1/2}\sum_{i=1}^{n}E[\psi_\tau\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}(\tau)\}\mathbf{X}_i|\mathbf{X}_i]\right\| = O_p\{(\delta_n + n^{-1/2})^{1/2}\log n\},$$

where $\delta_n = o(1)$ as $n \to \infty$. Note that

$$E[\psi_\tau\{Y - \mathbf{X}^T\boldsymbol{\beta}(\tau)\}|\mathbf{X}] = \tau - F\{\mathbf{X}^T\boldsymbol{\beta}(\tau)|\mathbf{X}\}.$$

Therefore,

$$n^{-1/2}\sum_{i=1}^{n}(\psi_\tau\{Y_i - \mathbf{X}_i^T\hat{\boldsymbol{\beta}}(\tau)\} - [\tau - F_i\{\mathbf{X}_i^T\hat{\boldsymbol{\beta}}(\tau)\}])\mathbf{X}_i = n^{-1/2}\sum_{i=1}^{n}\psi_\tau(\epsilon_i)\mathbf{X}_i + O_p[\{\|\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\| + n^{-1/2}\}^{1/2}\log n]. \quad \text{(S.9)}$$

By the subgradient condition of quantile regression (Koenker, 2005) and Assumption 3(a), we have

$$n^{-1/2}\sum_{i=1}^{n}\psi_\tau\{Y_i - \mathbf{X}_i^T\hat{\boldsymbol{\beta}}(\tau)\}\mathbf{X}_i = o_p(1).$$

Hence we obtain

$$\begin{aligned}
& n^{-1/2}\sum_{i=1}^{n}(\psi_\tau\{Y_i - \mathbf{X}_i^T\hat{\boldsymbol{\beta}}(\tau)\} - [\tau - F_i\{\mathbf{X}_i^T\hat{\boldsymbol{\beta}}(\tau)\}])\mathbf{X}_i \\
= \; & n^{-1/2}\sum_{i=1}^{n}[F_i\{\mathbf{X}_i^T\hat{\boldsymbol{\beta}}(\tau)\} - \tau]\mathbf{X}_i + o_p(1) \\
= \; & n^{-1/2}\sum_{i=1}^{n}f_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i \ge u_0)\}\mathbf{X}_i\mathbf{X}_i^T\{\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\} \\
& -n^{-1}\sum_{i=1}^{n}f_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i \ge u_0)\}\mathbf{X}_i\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau)I(U_i > u_0) \\
& +o_p(1) + o_p[n^{1/2}\{\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\}] \\
= \; & n^{1/2}\boldsymbol{S}\{\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\} - \boldsymbol{Q}\{\boldsymbol{\alpha}_0(\tau)\} + o_p(1) + o_p[n^{1/2}\{\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\}],
\end{aligned}$$

which together with (S.9) proves Lemma 4.

Proof of Theorem 2. Because Theorem 1 is only a special case of Theorem 2 when $\boldsymbol{\alpha}_0(\tau) = 0$, we only need to prove Theorem 2, namely $Y_i = \mathbf{X}_i^T\boldsymbol{\beta}_0(\tau) + n^{-1/2}\mathbf{Z}_i^T\boldsymbol{\alpha}_0(\tau) I(U_i > u_0) + \epsilon_i$ under the alternative hypothesis. Let $\boldsymbol{\xi} = n^{1/2}\{\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\}$, it follows from Lemmas 3 and 4 that

$$\begin{aligned}
& \mathbf{R}_n\{u, \tau, \hat{\boldsymbol{\beta}}(\tau)\} \\
= \; & \mathbf{R}_n^d\{u, \tau, \boldsymbol{\beta}_0(\tau)\} - n^{1/2}\mathbf{S}_{\mathbf{z}}(u)\{\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\} + \mathbf{P}\{u, \boldsymbol{\alpha}_0(\tau)\} + o_p(1) \\
= \; & n^{-1/2}\sum_{i=1}^n \psi_\tau(\epsilon_i)\mathbf{Z}_i I(U_i \le u) - \mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}n^{-1/2}\sum_{i=1}^n \psi_\tau(\epsilon_i)\mathbf{X}_i - \mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}\mathbf{Q}\{\boldsymbol{\alpha}_0(\tau)\} \\
& + \mathbf{P}\{u, \boldsymbol{\alpha}_0(\tau)\} + o_p(1) \\
= \; & n^{-1/2}\sum_{i=1}^n \psi_\tau(\epsilon_i)\{I(U_i \le u)\mathbf{Z}_i - \mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}\mathbf{X}_i\} - \mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}\mathbf{Q}\{\boldsymbol{\alpha}_0(\tau)\} + \mathbf{P}\{u, \boldsymbol{\alpha}_0(\tau)\} \\
& + o_p(1) \\
= \; & \mathbf{R}(u) + \mathbf{q}\{u, \boldsymbol{\alpha}_0(\tau)\} + o_p(1).
\end{aligned}$$

Following the proofs in Stute (1997), the weak convergence of $\mathbf{R}(u)$ can be obtained. This completes the proof of Theorem 2.

Proof of Corollary 1. Following the same arguments used in the proof of Theorem 2, we can show that

$$\mathbf{R}_n\{u, \tau, \hat{\boldsymbol{\beta}}(\tau)\} = n^{-1/2}\sum_{i=1}^n \psi_\tau(\epsilon_i)\{I(U_i \le u)\mathbf{Z}_i - \mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}\mathbf{X}_i\} + \mathbf{q}\{u, \boldsymbol{\alpha}_0(\tau)\} + o_P(1),$$

where $\mathbf{q}\{u, \boldsymbol{\alpha}_0(\tau)\} = a_n\{-\mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}\mathbf{Q}\{\boldsymbol{\alpha}_0(\tau)\} + \mathbf{P}\{u, \boldsymbol{\alpha}_0(\tau)\}\}$. Because $\mathbf{q}\{u, \boldsymbol{\alpha}_0(\tau)\} \to +\infty$ as n goes into infinity, this implies $T_n(\tau)$ goes into infinity. This completes the proof of Corollary 1.

The following lemma is needed to prove Theorem 3.

Lemma 5 *Under Assumptions A1-A5, we have*

$$\sup_{u \in (0,1)} \|\mathbf{S}_{z,n}(u) - \mathbf{S}_{\mathbf{z}}(u)\| = o_p(1).$$

Proof of Lemma 5. Let $g\{\boldsymbol{\beta}(\tau)\} = \mathbf{Z}_i\mathbf{X}_i^T K_{h_n}\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}(\tau)\}$. Then

$$\begin{aligned}
\mathbf{S}_{z,n}(u) - \mathbf{S}_{\mathbf{z}}(u) &= n^{-1}\sum_{i=1}^{n}[g\{\hat{\boldsymbol{\beta}}(\tau)\}I(U_i \le u) - g\{\boldsymbol{\beta}_0(\tau)\}I(U_i \le u)\}] \\
&\quad + n^{-1}\sum_{i=1}^{n}[g\{\boldsymbol{\beta}_0(\tau)\}I(U_i \le u) - \mathbf{S}_{\mathbf{z}}(u)] \\
&= (i) + (j).
\end{aligned}$$

To obtain the desired result, we need to show that (i) and (j) are $o_p(1)$ uniformly in $u \in (0,1)$.

First note that

$$\begin{aligned}
(i) &= n^{-1}\sum_{i=1}^{n}\mathbf{Z}_i\mathbf{X}_i^T I(U_i \le u)[K_{h_n}\{Y_i - \mathbf{X}_i^T\hat{\boldsymbol{\beta}}(\tau)\} - K_{h_n}\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)\}] \\
&\le n^{-1}\sum_{i=1}^{n}\mathbf{Z}_i\mathbf{X}_i^T I(U_i \le u)\max_{1\le i\le n}[K_{h_n}\{Y_i - \mathbf{X}_i^T\hat{\boldsymbol{\beta}}(\tau)\} - K_{h_n}\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)\}].
\end{aligned}$$

By Assumptions A3(a), A4 and A5, and the fact that $\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau) = O_p(n^{-1/2})$, following mean value theorem, we have

$$\begin{aligned}
&\max_{1\le i\le n}[K_{h_n}\{Y_i - \mathbf{X}_i^T\hat{\boldsymbol{\beta}}(\tau)\} - K_{h_n}\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)\}] \\
\le\ & h_n^{-2}\max_{1\le i\le n}\|\mathbf{X}_i\|K'(\boldsymbol{\zeta})O_p(n^{-1/2}) = o_p(1),
\end{aligned}$$

where $\boldsymbol{\zeta}$ is a point between $Y_i - \mathbf{X}_i^T\hat{\boldsymbol{\beta}}(\tau)$ and $Y_i - \mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)$. Hence $(i) = o_p(1)$.

Next, by the Taylor expansion we have

$$\begin{aligned}
& n^{-1}\sum_{i=1}^{n}E[g\{\boldsymbol{\beta}_0(\tau)\}I(U_i \le u)] \\
=\ & n^{-1}\sum_{i=1}^{n}E_{\mathbf{X}_i}(\mathbf{Z}_i\mathbf{X}_i^T I(U_i \le u)E[K_{h_n}\{Y_i - \mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)\}|\mathbf{X}_i]) \\
=\ & n^{-1}\sum_{i=1}^{n}E_{\mathbf{X}_i}\left(\mathbf{Z}_i\mathbf{X}_i^T I(U_i \le u)[f_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau)\} + o(1)]\right) \\
=\ & \mathbf{S}_{\mathbf{z}}(u) + o_p(1).
\end{aligned}$$

The rest of the proof follows with the similar arguments used in Lemma 1 of Hansen (1996) and thus is omitted.

Proof of Theorem 3. Denote

$$\mathbf{R}_n^{**}(u) = n^{-1/2}\sum_{i=1}^n \omega_i\psi_\tau(e_i)\{I(U_i \le u)\mathbf{Z}_i - \mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}\mathbf{X}_i\}. \qquad \text{(S.10)}$$

To obtain the result, we need to show that

(i) $\mathbf{R}_n^*(u)$ and $\mathbf{R}_n^{**}(u)$ are uniformly asymptotically equivalent, that is

$$\begin{aligned}\sup_{u\in(0,1)}\|\mathbf{R}_n^*(u) - \mathbf{R}_n^{**}(u)\| &= \sup_{u\in(0,1)}\left\|n^{-1/2}\sum_{i=1}^n \omega_i\psi_\tau(e_i)\{\mathbf{S}_{z,n}(u)\mathbf{S}_n^{-1} - \mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}\}\mathbf{X}_i\right\| \\ &= o_p(1);\end{aligned}$$

and (ii) $\mathbf{R}_n^{**}(u)$ converges to the Gaussian process $\mathbf{R}(u)$.

The first part (i) is a direct conclusion of Lemma 4 by allowing $u = 1$. Then it follows that $\mathbf{S}_n = \mathbf{S} + o_p(1)$ and

$$\begin{aligned}&\sup_{u\in(0,1)}\left\|n^{-1/2}\sum_{i=1}^n \omega_i\psi_\tau(e_i)\{\mathbf{S}_{z,n}(u)\mathbf{S}_n^{-1} - \mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}\}\mathbf{X}_i\right\| \\ =\ &\sup_{u\in(0,1)}\left\|n^{-1/2}\sum_{i=1}^n \omega_i\psi_\tau(e_i)[\{\mathbf{S}_{z,n}(u) - \mathbf{S}_{\mathbf{z}}(u)\}\mathbf{S}_n^{-1} - \mathbf{S}_{\mathbf{z}}(u)(\mathbf{S}_n^{-1} - \mathbf{S}^{-1})]\mathbf{X}_i\right\| \\ \le\ &\sup_{u\in(0,1)}\|[\{\mathbf{S}_{z,n}(u) - \mathbf{S}_{\mathbf{z}}(u)\}\mathbf{S}_n^{-1} - \mathbf{S}_{\mathbf{z}}(u)(\mathbf{S}_n^{-1} - \mathbf{S}^{-1})]\|\left\|n^{-1/2}\sum_{i=1}^n \omega_i\psi_\tau(e_i)\mathbf{X}_i\right\| \\ =\ &o_p(1),\end{aligned}$$

where the second equation follows by Lemma 4, and $n^{-1/2}\sum_{i=1}^n \omega_i\psi_\tau(e_i)\mathbf{X}_i = O_p(1)$.

For (ii), we first show that the covariance function of $\mathbf{R}_n^{**}(u)$ converges to that of $\mathbf{R}(u)$. For each $u \in R$ and $u' \in R$, we have

$$\begin{aligned}&\mathrm{Cov}\{\mathbf{R}_n^{**}(u), \mathbf{R}_n^{**}(u')\} \\ =\ &\mathrm{Cov}\Big[n^{-1/2}\sum_{i=1}^n \omega_i\psi_\tau(e_i)\left\{I(U_i \le u)\mathbf{Z}_i - \mathbf{S}_{z,n}(u)\mathbf{S}_n^{-1}\mathbf{X}_i\right\}, \\ &\quad n^{-1/2}\sum_{i=1}^n \omega_i\psi_\tau(e_i)\left\{I(U_i \le u')\mathbf{Z}_i - \mathbf{S}_{z,n}(u')\mathbf{S}_n^{-1}\mathbf{X}_i\right\}\Big] \\ =\ &n^{-1}\sum_{i=1}^n \mathrm{Cov}[\omega_i\psi_\tau(e_i)\{I(U_i \le u)\mathbf{Z}_i - \mathbf{S}_{z,n}(u)\mathbf{S}_n^{-1}\mathbf{X}_i\}, \\ &\quad \omega_i\psi_\tau(e_i)\{I(U_i \le u')\mathbf{Z}_i - \mathbf{S}_{z,n}(u')\mathbf{S}_n^{-1}\mathbf{X}_i\}]\end{aligned}$$

$$
\begin{aligned}
&= n^{-1}\sum_{i=1}^{n} E\{\omega_i\psi_\tau(e_i)\}^2\{I(U_i\le u)\mathbf{Z}_i - \mathbf{S}_{z,n}(u)\mathbf{S}_n^{-1}\mathbf{X}_i\} \\
&\quad \times\{I(U_i\le u')\mathbf{Z}_i - \mathbf{S}_{z,n}(u')\mathbf{S}_n^{-1}\mathbf{X}_i\}^T \\
&\to \tau(1-\tau)\{E(\mathbf{Z}_u\mathbf{Z}_{u'}) - E(\mathbf{Z}_u\mathbf{X}^T)\mathbf{S}^{-1}\mathbf{S}_z^T(u') - \mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}E(\mathbf{X}\mathbf{Z}_{u'}^T) \\
&\quad +\mathbf{S}_{\mathbf{z}}(u)\mathbf{S}^{-1}E(\mathbf{X}\mathbf{X}^T)\mathbf{S}^{-1}\mathbf{S}_z^T(u')\}, \text{ almost surely,}
\end{aligned}
$$

which is the same as the covariance of $\mathbf{R}_n^{**}(u)$ in Theorem 1.

Next, by the Cramer-Wold device, the finite dimensional convergence of the process $\mathbf{R}_n^{**}(u)$ can be obtained.

Finally, because the class of functions $\mathcal{F}_n = [\psi_\tau(\cdot)\{I(U\le u)\mathbf{Z} - \mathbf{S}_{z,n}(u)\mathbf{S}_n^{-1}\mathbf{X} : u\in R\}]$ is a Vapnik-Chervonenskis class of functions. Applying Lemma 15 in Pollard (1984), we can show that $R_n^{**}(u)$ is uniformly tight. This completes the proof of Theorem 3.

To prove Theorem 4, we need to show that the processes $\mathbf{R}_n\{u,\tau,\boldsymbol{\beta}_0(\tau)+n^{-1/2}\boldsymbol{\xi}\}$ and $\mathbf{R}_n^d\{u,\tau,\boldsymbol{\theta}_0(\tau)\} - \mathbf{S}_{\mathbf{z}}(u)\boldsymbol{\xi} + \mathbf{P}\{u,\boldsymbol{\alpha}_0(\tau)\}$ are uniformly asymptotically equivalent in $(u,\tau)\in(0,1)\times\mathcal{T}$.

Lemma 6 *Under Assumptions A2-A6 and H_1^*, we have*

$$
\begin{aligned}
&\sup_{\tau\in\mathcal{T}}\sup_{u\in(0,1)}\sup_{\boldsymbol{\xi}\in\mathbf{D}} \|\boldsymbol{R}_n\{u,\tau,\boldsymbol{\beta}_0(\tau)+n^{-1/2}\boldsymbol{\xi}\} - \boldsymbol{R}_n^d\{u,\tau,\boldsymbol{\theta}_0(\tau)\} + \mathbf{S}_{\mathbf{z}}(u)\boldsymbol{\xi} \\
&\qquad -\boldsymbol{P}\{u,\boldsymbol{\alpha}_0(\tau)\}\| = o_p(1).
\end{aligned}
$$

Proof of Lemma 6. We first partition $\mathcal{T}$ into $n(\varepsilon_n)$ parts with points $\omega_1=\tau_0<\tau_1<\cdots<\tau_{n(\varepsilon_n)}=\omega_2$, where $n(\varepsilon_n)=[(\omega_2-\omega_1)/\varepsilon_n]+1$, $\varepsilon_n=n^{-1/2-d}$ for some $d>0$. For $\tau_{j-1}<\tau<\tau_j$,

$$
\begin{aligned}
&\mathbf{R}_n^c\{u,\tau,\beta_0(\tau)+n^{-1/2}\boldsymbol{\xi}\} - \mathbf{R}_n^d\{u,\tau,\boldsymbol{\theta}_0(\tau)\} \\
\le\ & \mathbf{R}_n^c\{u,\tau_{j-1},\beta_0(\tau_{j-1})+n^{-1/2}\boldsymbol{\xi}\} - \mathbf{R}_n^d\{u,\tau_j,\boldsymbol{\theta}_0(\tau_j)\} \\
&+n^{-1/2}\sum_{i=1}^{n}(\tau_j-\tau_{j-1})\mathbf{Z}_i I(U_i\le u) \\
&+n^{-1/2}\sum_{i=1}^{n}[F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau_j)+n^{-1/2}\boldsymbol{\xi}\} - F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau_{j-1})+n^{-1/2}\boldsymbol{\xi}\}]\mathbf{Z}_i I(U_i\le u).
\end{aligned}
$$

Using the same argument, a reverse inequality holds when τ_{j-1} and τ_j are switched.

Then we can get

$$
\begin{aligned}
&\sup_{\tau\in\mathcal{T}}\sup_{u\in(0,1)}\sup_{\boldsymbol{\xi}\in\mathbf{D}}\|\mathbf{R}_n^c\{u,\tau,\beta_0(\tau)+n^{-1/2}\boldsymbol{\xi}\}-\mathbf{R}_n^d\{u,\tau,\boldsymbol{\theta}_0(\tau)\}\|\\
&\le \max_{1\le j\le n(\varepsilon_n)}\sup_{u\in(0,1)}\sup_{\boldsymbol{\xi}\in\mathbf{D}}\|\mathbf{R}_n^c\{u,\tau_{j-1},\beta_0(\tau_{j-1})+n^{-1/2}\boldsymbol{\xi}\}-\mathbf{R}_n^d\{u,\tau_j,\theta_0(\tau_j)\}\|\\
&\quad+\max_{1\le j\le n(\varepsilon_n)}\sup_{u\in(0,1)}\sup_{\boldsymbol{\xi}\in\mathbf{D}}\|\mathbf{R}_n^c\{u,\tau_j,\beta_0(\tau_j)+n^{-1/2}\boldsymbol{\xi}\}-\mathbf{R}_n^d\{u,\tau_{j-1},\boldsymbol{\theta}_0(\tau_{j-1})\}\|\\
&\quad+\max_{1\le j\le n(\varepsilon_n)}\sup_{u\in(0,1)}\sup_{\boldsymbol{\xi}\in\mathbf{D}}\left\|n^{-1/2}\sum_{i=1}^n(\tau_j-\tau_{j-1})\mathbf{Z}_iI(U_i\le u)\right\|\\
&\quad+\max_{1\le j\le n(\varepsilon_n)}\sup_{u\in(0,1)}\sup_{\boldsymbol{\xi}\in\mathbf{D}}\left\|n^{-1/2}\sum_{i=1}^n[F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau_j)+n^{-1/2}\boldsymbol{\xi}\}-F_i\{\mathbf{X}_i^T\boldsymbol{\beta}_0(\tau_{j-1})\right.\\
&\quad\left.+n^{-1/2}\boldsymbol{\xi}\}]\mathbf{Z}_iI(U_i\le u)\right\|\\
&=(k)+(l)+(m)+(n).
\end{aligned}
$$

For the terms (k) and (l), we have

$$
\begin{aligned}
(k)+(l)&\le 2\max_{1\le j\le n(\varepsilon_n)}\sup_{u\in(0,1)}\sup_{\boldsymbol{\xi}\in\mathbf{D}}\|\mathbf{R}_n^c\{u,\tau_j,\beta_0(\tau_j)+n^{-1/2}\boldsymbol{\xi}\}-\mathbf{R}_n^d\{u,\tau_j,\boldsymbol{\theta}_0(\tau_j)\}\|\\
&\quad+2\max_{1\le j\le n(\varepsilon_n)}\sup_{u\in(0,1)}\|\mathbf{R}_n^d\{u,\tau_j,\beta_0(\tau_j)\|-\mathbf{R}_n^d\{u,\tau_{j-1},\beta_0(\tau_{j-1})\|\\
&=(o)+(p),
\end{aligned}
$$

where τ is invariant in the first inequality while $\boldsymbol{\xi}$ is invariant in the second inequality. The term (o) can be shown to be $o_p(1)$ by using similar arguments as the proof of Lemma 1. For the term (p), let $\Phi=(0,1)\times(0,1)$ be a parameter set with metric $\rho\{(u,\tau),(u',\tau')\}=|u'-u|+|\tau'-\tau|$. For a given τ,

$$
\begin{aligned}
\mathbf{R}_n\{u,\tau,\beta_0(\tau)\}&=n^{-1/2}\sum_{i=1}^n[\tau-I\{Y_i-\mathbf{X}_i^T\beta_0(\tau)\le 0\}]\mathbf{Z}_iI(U_i\le u)\\
&=n^{-1/2}\sum_{i=1}^n[\tau-I\{F_i(Y_i)\le\tau)\}]\mathbf{Z}_iI(U_i\le u).
\end{aligned}
$$

Define $V_i=F_i(Y_i)$, then V_i has a standard uniform distribution. Hence,

$$
\mathbf{R}_n\{u,\tau,\beta_0(\tau)\}=n^{-1/2}\sum_{i=1}^n\{\tau-I(V_i\le\tau)\}\mathbf{Z}_iI(U_i\le u).
$$

Moreover, $\{(\tau-I(V_i\le\tau))\mathbf{Z}_iI(U_i\le u);i=1,\cdots,n\}$ is a sequence of vector martingale differences. Hence, following the same lines as that in Theorem A1 of Bai

(1996), we can obtain the stochastic equicontinuity of $\mathbf{R}_n\{u, \tau, \beta_0(\tau)\}$ on (Φ, ρ). That is, for any $\epsilon > 0$ and $\eta > 0$, there exists a $\phi > 0$ such that for large n, we have

$$P\Big[\sup_{\Omega} \| - \mathbf{R}_n\{u, \tau, \beta_0(\tau)\} + \mathbf{R}_n\{u', \tau', \beta_0(\tau')\}\| > \eta\Big] < \epsilon,$$

where $\Omega = \{(s_1, s_2) \in \Phi; s_1 = (u, \tau)$ and $s_2 = (u', \tau'), \rho(s_1, s_2) < \phi\}$, thus $(p) = o_p(1)$. Finally it is easy to show that the last two terms (m) and (n) are $o_p(1)$ by using the similar arguments as that of Lemma A.2 in Qu (2008), so we omit the details here.

Proof of Theorem 4. The proof is similar to that of Theorem 2 based on results of Lemmas 4 and 6 and thus is omitted.

Additional References

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