Fixed-b Asymptotics for Blockwise Empirical Likelihood

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Supplementary Material

The following supplementary material contains the proof of Theorem 2.1.

S1 Technical appendix

Define the set of functions

$$Q = \{g = (g_1, g_2, \dots, g_k) \in C^{\otimes k}[0,1] : g_i's \text{ are linearly independent}\},$$

and let $G_{el}(g) = \max_{\lambda \in \mathbb{R}^k} \int_0^1 \log(1 + \lambda' g(t))dt$ be a nonlinear functional from $C^{\otimes k}[0,1]$ to the real line $\mathbb{R}$, where $\log(x) = -\infty$ for $x < 0$. We shall prove in the following that $G_{el}(g)$ is a continuous map for functions in Q under the sup norm (Seijo and Sen (2011)). For any $g \in C^{\otimes k}[0,1]$, we define $H_g = \{\lambda \in \mathbb{R}^k : \min_{t\in[0,1]}(1 + \lambda' g(t)) \geq 0\}$ and $L_g(\lambda) = -\int_0^1 \log(1 + \lambda' g(t))dt$. It is straightforward to show that $L_g(\lambda)$ is strictly convex for $g \in Q$ on the set H_g. We also note that H_g is a closed convex set, which contains a neighborhood of the origin. Let $\lambda_g = \text{argmax}_{\lambda\in\mathbb{R}^k} \int_0^1 \log(1 + \lambda' g(t))dt$ be the maximizer of $-L_g(\lambda)$.

We first show that $G_{el}(g) < \infty$ if and only if H_g is bounded. If $G_{el}(g) = \infty$, then λ_g cannot be finite, which implies that H_g is unbounded. On the other hand, suppose H_g is unbounded. Note that $H_g = \cap_{t\in[0,1]}\{\lambda \in \mathbb{R}^k : \lambda' g(t) \geq -1\}$ which is the intersection of a set of closed half-spaces. The recession cone of H_g is then given by $0^+H_g = \cap_{t\in[0,1]}\{\lambda \in \mathbb{R}^k : \lambda' g(t) \geq 0\}$ (see Section 8 of Rockafellar (1970)). By Theorem 8.4 of Rockafellar (1970), there exists a nonzero vector $\tilde{\lambda} \in 0^+H_g$, and the set $\{t \in [0,1] : \tilde{\lambda}' g(t) > 0\}$ has positive Lebesgue measure because of the linearly independence of g. We have $G_{el}(g) \geq -L_g(a\tilde{\lambda})$ for any $a > 0$, where $-L_g(a\tilde{\lambda}) \to \infty$ as $a \to \infty$. Thus we get $G_{el}(g) = \infty$.

Next, we consider the case $G_{el}(g) = \infty$. Following the discussion above, there exists $\tilde{\delta}$ such that the set $\mathcal{B} := \{t \in [0,1] : \tilde{\lambda}' g(t) > \tilde{\delta}\}$ has Lebesgue measure $\Lambda(\mathcal{B}) > 0$. For any $A_0 > 0$, we choose $\epsilon_0 \in (0,1)$ and large enough $a > 0$ so that

$$\Lambda(\mathcal{B}) \log(1 + a\tilde{\delta} - \epsilon_0) + \log(1 - \epsilon_0) > A_0.$$

For any $f \in Q$ with $||f-g|| := \sup_{t\in[0,1]}|f(t)-g(t)| \le \epsilon_0/(|\tilde{\lambda}|a)$, we have

$$\begin{aligned}\int_0^1 \log(1+a\tilde{\lambda}' f(t))dt =& \int_{\mathcal{B}} \log(1+a\tilde{\lambda}'(f(t)-g(t))+a\tilde{\lambda}' g(t))dt \\ &+ \int_{\mathcal{B}^c} \log(1+a\tilde{\lambda}'(f(t)-g(t))+a\tilde{\lambda}' g(t))dt \\ \ge& \Lambda(\mathcal{B})\log(1+a\tilde{\delta}-\epsilon_0)+\log(1-\epsilon_0) > A_0.\end{aligned}$$

In what follows, we turn to the case $G_{el}(g) < \infty$, i.e., H_g is bounded as shown before.

Case 1: we first consider the case that $\lambda_g \in \tilde{H}_g = \{\lambda \in \mathbb{R}^k : \min_{t\in[0,1]}(1+\lambda' g(t)) > 0\}$. Since $\tilde{H}_g$ is open, we can pick a positive number τ so that $\bar{B}(\lambda_g;\tau) := \{\lambda \in \mathbb{R}^k : |\lambda-\lambda_g| \le \tau\} \subseteq \tilde{H}_g$. Then we have $\min_{\lambda\in\bar{B}(\lambda_g;\tau)}\min_{t\in[0,1]}(1+\lambda' g(t)) > c > 0$. Furthermore, there exists a sufficiently small δ such that for any $f \in Q$ with $||f-g|| \le \delta$, we have $\min_{t\in[0,1]}(1+\lambda' f(t)) > c' > 0$ for any $\lambda \in \bar{B}(\lambda_g;\tau)$, i.e., $\bar{B}(\lambda_g;\tau) \subseteq \tilde{H}_f$. Notice that the constant c' only depends on g, δ and c.

Given any $\epsilon > 0$, we shall first show that $\sup_{\lambda\in\bar{B}(\lambda_g;\tau)}|L_f(\lambda)-L_g(\lambda)| < \epsilon$ for any $f \in Q$ with $||f-g|| < \tilde{\delta}(\epsilon)$, where $0 < \tilde{\delta}(\epsilon) < \delta$. Because $G_{el}(g) < \infty$, we have $\int_0^1 \log(1+\lambda' g(t))dt < \infty$ for any $\lambda \in \bar{B}(\lambda_g;\tau)$. Simple algebra yields that

$$\begin{aligned}&\left|\int_0^1 \log(1+\lambda' f(t))dt - \int_0^1 \log(1+\lambda' g(t))dt\right| \\ \le& \max\left\{\log(1+M\tilde{\delta}(\epsilon)/c'), \log(1+M\tilde{\delta}(\epsilon)/c)\right\},\end{aligned} \tag{S1.1}$$

where $M = |\lambda_g| + \tau$. The RHS of (1) can be made arbitrarily small for sufficiently small $\tilde{\delta}(\epsilon)$. Therefore we get $\sup_{\lambda\in\bar{B}(\lambda_g;\tau)}|L_f(\lambda)-L_g(\lambda)| < \epsilon$ for small enough $\tilde{\delta}(\epsilon)$, which implies that $|G_{el}(g) - \sup_{\lambda\in\bar{B}(\lambda_g;\tau)}\int_0^1 \log(1+\lambda' f(t))dt| < \epsilon$. Next, we show that there exists a local maxima of $-L_f(\lambda)$ in $\bar{B}(\lambda_g;\tau)$. Suppose ϵ is sufficiently small and choose $0 < \xi < \tau$ such that $-L_g(\lambda_g) > \max_{\lambda\in\bar{B}(\lambda_g;\tau)\cap B^c(\lambda_g;\xi)} -L_g(\lambda) + 2\epsilon$, where $B(\lambda_g;\xi) = \{\lambda \in \mathbb{R}^k : |\lambda-\lambda_g| < \xi\}$. Thus we get

$$\begin{aligned}\max_{\lambda\in\bar{B}(\lambda_g;\tau)\cap B^c(\lambda_g;\xi)} -L_f(\lambda) \le& \max_{\lambda\in\bar{B}(\lambda_g;\tau)\cap B^c(\lambda_g;\xi)} -L_g(\lambda) + \epsilon \\ <& -L_g(\lambda_g) - \epsilon \le -L_f(\lambda_g) \le \max_{\lambda\in\bar{B}(\lambda_g;\xi)} -L_f(\lambda).\end{aligned}$$

Because $f \in Q$, $L_f(\lambda)$ is strictly convex. Hence, the local maxima is also the global maxima, which implies that $|G_{el}(g) - G_{el}(f)| < \epsilon$.

Case 2: We now consider the case $\min_{t\in[0,1]}(1+\lambda_g' g(t)) = 0$. For any $0 < \delta^* < \delta^{**} < 1$, let $H_g(\delta^*) = \{(1-\delta^*)\lambda : \lambda \in H_g\}$ and $H_f(\delta^{**}) = \{(1-\delta^{**})\lambda : \lambda \in H_f\}$. There exists a small enough $\delta > 0$ such that for any $f \in Q$ with $||f-g|| < \delta$, $H_f(\delta^{**}) \subseteq H_g(\delta^*) \in \tilde{H}_f \cap \tilde{H}_g$. By the continuity of $L_g(\lambda)$, we know for any $\epsilon > 0$, there exists a $\delta^* > 0$ such that when $|\lambda-\lambda_g| \le \delta^*|\lambda_g|$, $-L_g(\lambda_g) < -L_g(\lambda) + \epsilon/4$. By the construction of $H_g(\delta^*)$, we have

$$-L_g(\lambda_g) < -L_g((1-\delta^*)\lambda_g) + \epsilon/4 \leq \sup_{\lambda \in H_g(\delta^*)} -L_g(\lambda) + \epsilon/4.$$

Using similar arguments in the first case and the boundness of H_g, we can show that

$$\left| \sup_{\lambda \in H_g(\delta^*)} -L_g(\lambda) - \sup_{\lambda \in H_g(\delta^*)} -L_f(\lambda) \right| < \epsilon/8,$$

for sufficiently small δ. Furthermore, when $\lambda_f \in H_g(\delta^*)$, we have $-L_f(\lambda_f) = \sup_{\lambda \in H_g(\delta^*)} - L_f(\lambda)$. When $\lambda_f \notin H_g(\delta^*)$, by the convexity of $L_f(\lambda)$, we get $L_f((1-\delta^{**})\lambda_f) \leq (1-\delta^{**})L_f(\lambda_f)$, which implies that

$$\begin{aligned}
-L_f(\lambda_f) \leq & \frac{-L_f((1-\delta^{**})\lambda_f)}{1-\delta^{**}} \leq \frac{\sup_{\lambda \in H_g(\delta^*)} -L_f(\lambda)}{1-\delta^{**}} \\
\leq & \frac{\sup_{\lambda \in H_g(\delta^*)} -L_g(\lambda) + \epsilon/8}{1-\delta^{**}} \leq \sup_{\lambda \in H_g(\delta^*)} -L_g(\lambda) + \epsilon/4 \\
< & \sup_{\lambda \in H_g(\delta^*)} -L_f(\lambda) + \epsilon/2
\end{aligned}$$

for small enough δ^{**} (e.g., $\delta^{**} < \min(1/3, \frac{\epsilon}{24 G_{el}(g)})$). Thus we have

$$\begin{aligned}
|G_{el}(f) - G_{el}(g)| \leq & \left| -L_f(\lambda_f) - \sup_{\lambda \in H_g(\delta^*)} -L_f(\lambda) \right| + \left| -L_g(\lambda_g) - \sup_{\lambda \in H_g(\delta^*)} -L_g(\lambda) \right| \\
& \left| \sup_{\lambda \in H_g(\delta^*)} -L_g(\lambda) - \sup_{\lambda \in H_g(\delta^*)} -L_f(\lambda) \right| < \epsilon.
\end{aligned}$$

Combining the above arguments, we show that the map G_{el} is continuous under the sup norm.

Next, we consider the limiting process $D_k(r;b) = \int_0^1 \mathcal{K}((r-s)/b)dW_k(s)$ with $b \in (0,1)$ being fixed in the asymptotics. Because the components of $D_k(r;b)$ are mutually independent, we have $P(\alpha' D_k(r;b) = 0 \text{ for some } \alpha \in \mathbb{R}^k) = 0$ which implies that $P(D_k(r;b) \in Q) = 1$. Under the assumptions in Theorem 2.1, the set $\{\lambda : \min_{r \in [0,1]}(1 + \lambda' D_k(r;b)) \geq 0\}$ is compact and convex almost surely (note the convexity and closeness of the set follow directly from its definition). Using summation by parts, we get

$$\begin{aligned}
\sqrt{n} f_{tn}(\theta_0) = & \frac{\sqrt{n}}{S_n} \sum_{s=t-n}^{t-1} \mathcal{K}\left(\frac{s}{S_n}\right) f_{t-s}(\theta_0) = \frac{\sqrt{n}}{S_n} \sum_{s=1}^{n} \mathcal{K}\left(\frac{t-s}{S_n}\right) f_s(\theta_0) \\
= & \frac{1}{b\sqrt{n}} \mathcal{K}\left(\frac{t-n}{S_n}\right) \sum_{k=1}^{n} f_k(\theta_0) + \frac{1}{b\sqrt{n}} \sum_{s=1}^{n-1} \left\{ \mathcal{K}\left(\frac{t-s}{S_n}\right) - \mathcal{K}\left(\frac{t-s-1}{S_n}\right) \right\} \sum_{k=1}^{s} f_k(\theta_0).
\end{aligned}$$

By the continuous mapping theorem and Itô's formula, we obtain

$$\sqrt{n} f_{tn}(\theta_0) \Rightarrow^d \Lambda \left\{ \frac{1}{b} \mathcal{K}\left(\frac{r-1}{b}\right) W_k(1) + \frac{1}{b^2} \int_0^1 \mathcal{K}'\left(\frac{r-s}{b}\right) W_k(s) ds \right\} =^d \Lambda D_k(r;b)/b,$$

for $t = \lfloor nr \rfloor$ with $r \in [0,1]$. Finally, by the continuous mapping theorem, we get

$$\begin{aligned} elr(\theta_0) =& \frac{2}{b} \max_{\lambda \in \mathbb{R}^k} \sum_{t=1}^{n} \log(1 + \tilde{\lambda}' \sqrt{n} b \Lambda^{-1} f_{tn}(\theta_0))/n, \quad \tilde{\lambda} = \Lambda' \lambda / (\sqrt{n} b), \\ \rightarrow^d & U_{el,k}(b;\mathcal{K}) := \frac{2}{b} \max_{\tilde{\lambda} \in \mathbb{R}^k} \int_0^1 \log\left(1 + \tilde{\lambda}' D_k(r;b)\right) dr. \end{aligned} \tag{S1.2}$$

REMARK **S1.1**. *For ET and CUE, we have $\mathcal{I} = \mathbb{R}$. Given any $g \in Q$ with $G_{gel}(g) < \infty$, we have $H_g = \{\lambda \in \mathbb{R}^k : \lambda' g(t) \in \mathcal{I}, \text{for all } t \in [0,1]\} = \mathbb{R}^k$ and $\lambda_g < \infty$. Therefore, λ_g is an interior point of H_g and the arguments in Case 1 can be applied to show the continuity of $G_{gel}(\cdot)$ at g.*

References

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