Supplementary of "General Sliced Latin Hypercube Designs"

Proof of Theorem 1

Proof. For (i), the result follows immediately from Lemma 4, Lemma 2 in Qian (2012) and Theorem 1 in McKay et al. (1979).

For (ii), $\text{var}_{DSLH}(\hat{\eta}_i) = \text{var}_{SLH-IND}(\hat{\eta}_i)$ follows immediately from Lemma 4. The later part of (3.18) follows from the proof of Theorem 1 in Qian (2012).

For (iii), we first show $\text{var}_{DSLH}(\hat{\eta}) \leq \text{var}_{SLH-IND}(\hat{\eta})$.

$$\text{var}_{DSLH}(\hat{\eta}) = \text{var}_{DSLH}(\sum_{c_2=1}^{s_2}\sum_{c_1=1}^{s_1}\lambda_{c_2c_1}\hat{\mu}_{c_2c_1}) \tag{1}$$

$$= \sum_{c_2=1}^{s_2}\text{var}_{DSLH}(\sum_{c_1=1}^{s_1}\lambda_{c_2c_1}\hat{\mu}_{c_2c_1}) + \sum_{c_{21}=1}^{s_2}\sum_{c_{22}=1,c_{22}\neq c_{21}}^{s_2}\text{cov}_{DSLH}(\sum_{c_1=1}^{s_1}\lambda_{c_{21}c_1}\hat{\mu}_{c_{21}c_1}, \sum_{c_1=1}^{s_1}\lambda_{c_{22}c_1}\hat{\mu}_{c_{22}c_1})$$

$$= \sum_{c_2=1}^{s_2}\text{var}_{SLH-IND}(\sum_{c_1=1}^{s_1}\lambda_{c_2c_1}\hat{\mu}_{c_2c_1}) + \sum_{c_{21}=1}^{s_2}\sum_{c_{22}=1,c_{22}\neq c_{21}}^{s_2}\text{cov}_{DSLH}(\sum_{c_1=1}^{s_1}\lambda_{c_{21}c_1}\hat{\mu}_{c_{21}c_1}, \sum_{c_1=1}^{s_1}\lambda_{c_{22}c_1}\hat{\mu}_{c_{22}c_1}),$$

where the third equality follows from Lemma 4. We now proceed to show $\text{cov}_{DSLH}(\hat{\mu}_{c_{21}c_1}, \hat{\mu}_{c_{22}c_1}) \leq 0$ for $c_{21}, c_{22} = 1, \ldots, s_2, c_{21} \neq c_{22}, c_1 = 1, \ldots, s_1$. Using the notation in Section 3, we have

$$\begin{aligned}\text{cov}_{DSLH}(\hat{\mu}_{c_{21}c_1}, \hat{\mu}_{c_{22}c_1}) &= \text{cov}(m^{-1}\sum_{c=1}^{m} f_{c_{21}c_1}(\boldsymbol{d}_{c_{21}c_1}^{(c)}), m^{-1}\sum_{c=1}^{m} f_{c_{22}c_1}(\boldsymbol{d}_{c_{22}c_1}^{(c)})) \\ &= \text{cov}(f_{c_{21}c_1}(\boldsymbol{d}_{c_{21}c_1}^{(1)}), f_{c_{22}c_1}(\boldsymbol{d}_{c_{22}c_1}^{(1)})).\end{aligned} \tag{2}$$

For any dimension k of $\boldsymbol{d}_{c_{21}c_1}^{(1)}$ and $\boldsymbol{d}_{c_{21}c_1}^{(1)}$, $k = 1, \ldots, d$, the joint probability density function between $d_{c_{21}c_1}^{(1k)}$ was derived in (15). The probability density function is similar to (5) in Qian(2009) except the parameter values. Thus, similar to the proof of Theorem 1 in Qian (2009), we can prove $d_{c_{21}c_1}^{(1k)}$ and $d_{c_{22}c_1}^{(1k)}$ are *negatively quadrant dependent* (Lehmann, 1966). That is, for $0 \leq u, v \leq 1$,

$$\Pr(d_{c_{21}c_1}^{(1k)} \leq u, d_{c_{22}c_1}^{(1k)} \leq v) \leq \Pr(d_{c_{21}c_1}^{(1k)} \leq u)\Pr(d_{c_{22}c_1}^{(1k)} \leq v). \tag{3}$$

This result and Theorem 1 in Lehmann (1966) together yields that

$$\text{cov}(f_{c_{21}c_1}(\boldsymbol{d}_{c_{21}c_1}^{(1)}), f_{c_{22}c_1}(\boldsymbol{d}_{c_{22}c_1}^{(1)})) \leq 0. \tag{4}$$

Since λs are non-negative, plugging (4) into (2) proves $\text{var}_{DSLH}(\hat{\eta}) \leq \text{var}_{SLH-IND}(\hat{\eta})$, the first inequality in (3.19).

To show the second inequality in (3.19), note that

$$\text{var}_{SLH-IND}(\hat{\eta}) = \sum_{c_2=1}^{s_2}\text{var}_{SLH-IND}(\sum_{c_1=1}^{s_1}\lambda_{c_2c_1}\hat{\mu}_{c_2c_1}), \tag{5}$$

and

$$\mathrm{var}_{LH}(\hat{\eta}) = \sum_{c_2=1}^{s_2} \mathrm{var}_{LH}(\sum_{c_1=1}^{s_1} \lambda_{c_2c_1}\hat{\mu}_{c_2c_1}). \tag{6}$$

From (ii) of Theorem 1 in Qian (2012), we have, for any $c_2 = 1, \ldots, s_2$,

$$\mathrm{var}_{SLH-IND}(\sum_{c_1=1}^{s_1} \lambda_{c_2c_1}\hat{\mu}_{c_2c_1}) \leq \mathrm{var}_{LH}(\sum_{c_1=1}^{s_1} \lambda_{c_2c_1}\hat{\mu}_{c_2c_1}). \tag{7}$$

(5), (6), and (7) easily lead to the second inequality in (3.19).

The last inequality in (3.19) follows immediately from the fact that $\mathrm{var}_{\mathrm{LH}}(\hat{\mu}_{c_2c_1}) \leq \mathrm{var}_{\mathrm{IID}}(\hat{\mu}_{c_2c_1})$, $c_2 = 1, \ldots, s_2, c_1 = 1, \ldots, s_1$. □

Proof of Theorem 2

Proof. From Lemma 4, each slice of a DSLHD is statistically equivalent to an ordinary LHD. From Lemma 2 in Qian (2012), a slice of an SLHD is statistically equivalent to an ordinary LHD. Part (i) then follows immediately from Theorem 1 in Stein (1987) or Theorem 1 in Loh (1996).

From the equivalence of D_r to an SLHD in Lemma 4 and the definition of SLH-IND, (ii) follows immediately from (ii) of Theorem 2 in Qian (2012).

For (iii), Lemma 4 gives that, for $c_1 = 1, \ldots, s_1, c_2 = 1, \ldots, s_2, j_1, j_2 = 1, \ldots, m, j_1 \neq j_2$,

$$\mathrm{cov}[f_{c_2c_1}(\boldsymbol{d}_{c_2c_1}^{j_1}), f_{c_2c_1}(\boldsymbol{d}_{c_2c_1}^{j_2})] = -\frac{s_1s_2}{n}\sum_{k=1}^{d}\int_0^1 f_{c_2c_1}^{-k}(x_k)dx_k + o(m^{-1}). \tag{8}$$

The proof of Lemma 1 in Qian (2009) gives that, for $c_2 = 1, \ldots, s_2, c_{11}, c_{12} = 1, \ldots, s_1, c_{11} \neq c_{12}$, $j_1, j_2 = 1, \ldots, m$

$$\mathrm{cov}[f_{c_2c_{11}}(\boldsymbol{d}_{c_2c_{11}}^{j_1}), f_{c_2c_{12}}(\boldsymbol{d}_{c_2c_{12}}^{j_2})] = o(m^{-1}). \tag{9}$$

Similarly, by plugging (15) into the proof of Lemma 1 in Qian (2009), we have, for $c_{21}, c_{22} = 1, \ldots, s_2, c_{21} \neq c_{22}$, $c_{11}, c_{22} = 1, \ldots, s_1$, $j_1, j_2 = 1, \ldots, m$,

$$\mathrm{cov}[f_{c_{21}c_{11}}(\boldsymbol{d}_{c_{21}c_{11}}^{j_1}), f_{c_{22}c_{12}}(\boldsymbol{d}_{c_{22}c_{12}}^{j_2})] = o(m^{-1}). \tag{10}$$

We can then easily show (3.22) by combining (8), (9) and (10). □

Proof of Lemma 3

Proof. First note that the dimensions of H are exchangeable. To prove (i), it suffices to consider h_{111} because of the exchangeability. By symmetry, $Pr(h_{111} = u)$ takes the same value for all $u \in \boldsymbol{Z}_n$. So we have $Pr(h_{111} = u) = 1/n$. Thus, (3.13) holds.
To prove (ii), it suffices to consider h_{111} and h_{211} because of the exchangeability of H. Since the set $h(:,1,1)$ is an LHD of m levels, we have for u, v with $\lceil u/s_2s_1 \rceil = \lceil v/s_2s_1 \rceil$, $Pr(h_{111} = u, h_{211} = v) = 0$. Because there are $n(n-s_2s_1)$ pairs of (u,v) that satisfy the condition $\lceil u/s_2s_1 \rceil \neq \lceil v/s_2s_1 \rceil$, by symmetry, $Pr(h_{111} = u, h_{211} = v) = [n(n-s_2s_1)]^{-1}$ for any such (u,v). Thus (3.14) holds.

To prove (iii), it suffices to consider h_{111} and h_{112} because of the exchangeability of H. Let B_1 and B_2 be defined as in (3.10) and (3.11) respectively. Clearly B_1 and B_2 have $m(m-1)(s_2s_1)^2$ and $ms_1(s_1-1)s_2^2$ pairs, respectively. For any $(u,v) \in B_1^c \bigcap B_2^c$, where B_1^c and B_2^c are the compliments of B_1 and B_2, respectively, we have $Pr(h_{111}=u, h_{112}=v)=0$. This is obvious from the construction of DSLHD in Section 2 because $h(:,1,:)$ has to be an LHD of ms_1 levels. Therefore we only need to consider (u,v) in B_1 and B_2. For (u,v) in B_1, without loss of generality, consider $Pr(h_{111}=1, h_{112}=s_2s_1+1)$. Recall the four-step procedure for constructing general sliced Latin hypercube design in Section 2. Denote the ith row permutation in Step 2 as $\pi_{\boldsymbol{H}}^i$. Let $\boldsymbol{v}$ =SPV(s_1; m) be the generated one-layer permutation vector in Step 3 for $i=1$. We have

$$\begin{aligned} Pr(h_{111}=1, h_{112}=s_2s_1+1) &= Pr(\pi_{\boldsymbol{H}}^1(1)=1, \pi_{\boldsymbol{H}}^{s_1+1}(1)=1, \boldsymbol{v}[1]=1, \boldsymbol{v}[m+1]=s_1+1) \\ &= Pr(\pi_{\boldsymbol{H}}^1(1)=1) Pr(\pi_{\boldsymbol{H}}^{s_1+1}(1)=1) Pr(\boldsymbol{v}[1]=1, \boldsymbol{v}[m+1]=s_1+1) \\ &= \frac{1}{s_2}\frac{1}{s_2}\frac{1}{(ms_1)^2} = \frac{1}{n^2}, \end{aligned}$$

where the second equality holds because of the independence between row permutations and the independence between row permutations in Step 2 and the generation of SPV in Step 3. The third equality follows from $Pr(\pi_{\boldsymbol{H}}^1(1)=1) = Pr(\pi_{\boldsymbol{H}}^{s_1+1}(1)=1) = 1/s_2$, which is straightforward because $\boldsymbol{H}$ in Step 1 has s_2 columns, and $Pr(\boldsymbol{v}[1]=1, \boldsymbol{v}[m+1] = s_1+1 = 1/(ms_1)^2$, which can be derived from (iii) of Lemma 1 in Qian (2012). Similarly we can obtain the same probability for the other pairs in B_1. Since the cardinality of B_1 is $m(m-1)(s_2s_1)^2$, we have the probability that a pair (u,v) comes from B_1 is $m(m-1)(s_2s_1)^2/n^2 = 1-1/m$. Hence the probability that a pair (u,v) comes from B_2 is $1/m$. Since the cardinality of B_2 is $ms_1(s_1-1)s_2^2$, $Pr(h_{111}=u, h_{112}=v) = 1/m(ms_1(s_1-1)s_2^2) = 1/n(n-ms_2)$ for any pair of (u,v) in B_2. This concludes the proof.

The technique used to prove (iv) is quite similar to (iii). Let B_1, B_2 and B_3 be defined in (3.10), (3.11), and (3.12) respectively. Define the permutations and SPV as those in (iii). Without loss of generality, consider the joint distribution of h_{111} and h_{121}. By a similar argument, we can show that for any (u,v) from B_1 or B_2, $Pr(h_{111}=u, h_{121}=v) = 1/n^2$. Since the cardinalities of B_1 and B_2 are $m(m-1)(s_2s_1)^2$ and $ms_1(s_1-1)s_2^2$, respectively, the probability that a pair (u,v) comes from either B_1 or B_2 is $[m(m-1)(s_2s_1)^2 + ms_1(s_1-1)s_2^2]/n^2 = 1-s_2/n$. Therefore the probability that a pair (u,v) comes from B_3 is s_2/n. Since the cardinality of B_3 is $s_2(s_2-1)s_1m$, $Pr(h_{111}=u, h_{121}=v) = s_2/n(s_2(s_2-1)s_1m) = 1/[n(n-ms_1)]$ for any (u,v) in B_3. This concludes the proof. □

Proof of Lemma 4

Proof. For $r=1,\ldots,s_2$ and $c=1,\ldots,s_1$, express the $(i,\,k)$th entry d_{ik} of D_{rc} as

$$d_{ik} = \frac{b_{ik}s_2s_1 + w_{ik}s_2 - e_{ik} - u_{ik}}{n}, i=1,\ldots,m, k=1,\ldots,q. \tag{11}$$

where $\{b_{1k},\ldots,b_{mk}\}$ constitute a uniform permutation on $\boldsymbol{Z}_m - 1$; each w_{ik} is a discrete random variable with the probability mass function $Pr(w_{ik}=f) = s_1^{-1}, f=1,\ldots,s_1$;

each e_{ik} is a discrete random variable with the probability mass function $Pr(e_{ik} = a) = s_2^{-1}, a = 0, \ldots, s_2 - 1$; where u_{ik} are independent $U[0,1)$ random variables; and b_{ik}, w_{ik} and u_{ik} are mutually independent. Let $l_{ik} = w_{ik}s_2 - e_{ik}$. From the probability mass functions and the mutual independence, it is easy to see that the probability mass function of l_{ik} is $Pr(l_{ik} = l) = (s_2 s_1)^{-1}, l = 1, 2, \ldots, s_2 s_1$. Letting $v_{ik} = l_{ik} - u_{ik}$, d_{ik} in (11) becomes $b_{ik}/m + v_{ik}/n$. Since $\{b_{1k},\ b_{2k}, \ldots, b_{mk}\}$ is a uniform permutation on $\boldsymbol{Z}_m - 1$ and b_{ik} and v_{ik} are mutually independent, it remains to verify that $\frac{v_{ik}}{n}$ is a $U(0, \frac{1}{m}]$ random variable, which is shown as follows. For $x \in (0, 1/m]$, let $x_0 = \lceil nx \rceil$ and note that

$$
\begin{aligned}
Pr(\frac{v_{ik}}{n} \le x) &= \frac{1}{s_2 s_1} \sum_{a=1}^{s_2 s_1} Pr(\frac{a - u_{ik}}{n} \le x) \qquad (12)\\
&= \frac{1}{s_2 s_1} [\sum_{a=1}^{x_0 - 1} Pr(\frac{a - u_{ik}}{n} \le x) + Pr(\frac{x_0 - u_{ik}}{n} \le x) + \sum_{a=x_0+1}^{s_2 s_1} Pr(\frac{a - u_{ik}}{n} \le x].
\end{aligned}
$$

Note that $Pr(\frac{a-u_{ik}}{n} \le x) = 1$ for $a = 1, \ldots, x_0 - 1$; $Pr(\frac{x_0 - u_{ik}}{n} \le x) = 1 - (x_0 - nx)$; and $Pr(\frac{a-u_{ik}}{n} \le x) = 0$ for $a = x_0 + 1, \ldots, s_2 s_1$, which simplifies (12) to mx. Thus $\frac{v_{ik}}{n}$ is a $U(0, \frac{1}{m}]$ random variable.

We have shown the statistical equivalence of D_{rc} to an ordinary LHD. Hence to prove the statistical equivalence of D_r to an SLHD, we only need to consider the joint distribution of two points from two different slices in D_r. Without loss of generality, consider the joint distribution between $D_{r1}(1,:)$ and $D_{r2}(1,:)$, and use $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ to denote them respectively. For $0 \le z_1, z_2 \le 1$ and any positive integer q, define $\delta_q(z_1, z_2)$ as follows:

$$
\delta_q(z_1, z_2) = \begin{cases} 1, & \lceil q z_1 \rceil = \lceil q z_2 \rceil, \\ 0, & otherwise; \end{cases} \qquad (13)
$$

From Lemma 3(iii), it can be shown that the joint density function for $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ is

$$
p(x_1, x_2) = \frac{n^{2d}}{n^d (n-m)^d} \prod_{k=1}^{d} \{e_0 - e_1 \delta_{ms_1}(x_1^k, x_2^k) - e_2 \delta_m(x_1^k, x_2^k)\}, \qquad (14)
$$

where $e_0 = (s_1 - 1)s_1^{-1}$, $e_1 = 1$ and $e_2 = -s_1^{-1}$, x_1^k and x_2^k are the kth argument of x_1 and x_2 respectively. Similarly, from Lemma 1(iii) in Qian (2012), it can be shown that the joint density function between any two points from two different slices in an SLHD is the same as (14). We thus prove the statistical equivalence of D_r and an SLHD. □

Proof of Lemma 5

Proof. Following the notation in (2.1), consider, without loss of generality, the covariance between D_{11} and D_{21}. Since the DSLHD is on $(0,1]$, D_{11} and D_{21} are two scalars. For ease of notation, we use X_1 and X_2 to denote D_{11} and D_{21} respectively. From Lemma 3(iv), the joint density function of X_1 and X_2 is

$$
p(x_1, x_2) = 1 + \frac{1}{s_2 - 1} \delta_{ms_1}(x_1, x_2) - \frac{s_2}{s_2 - 1} \delta_n(x_1, x_2), \qquad (15)
$$

where $\delta_{ms_1}(x_1,x_2)$ and $\delta_n(x_1,x_2)$ are similarly defined as in (13). The derivation of this joint density function is as follows. Let B_1, B_2, B_3 be defined as in (3.10), (3.11), and (3.12), respectively. From Lemma 3 (iv), $Pr((u,v)\in B_1)=1-\frac{1}{m}$, $Pr((u,v)\in B_2)=\frac{1}{ms_1}$, and $Pr((u,v)\in B_3)=\frac{1}{m}-\frac{1}{ms_1}$. Let $\delta_{ms_1}(x_1,x_2)$, $\delta_m(x_1,x_2)$ and $\delta_n(x_1,x_2)$ be similarly defined as in (13). The event $(u,v)\in B_1$ is equivalent to $\delta_m(x_1,x_2)=0$, where $x_1=\frac{u-u_1}{n}$, $x_2=\frac{v-u_2}{n}$ and u_1, u_2 are two $U[0,1)$ random variables. Let $p_1(x_1,x_2)$ denote the joint density function of X_1 and X_2 when $\delta_m(x_1,x_2)=0$. Note that $p_1(x_1,x_2)$ is a constant since u_1 and u_2 are $U[0,1)$ random variables. Let $A_1=\{(x_1,x_2)\in(0,1]^2|\delta_m(x_1,x_2)=0\}$. We have

$$\int\int_{A_1} p_1(x_1,x_2)dx_1dx_2=\frac{n(n-s_2s_1)}{n^2}p_1(x_1,x_2)=1-\frac{1}{m}. \tag{16}$$

Hence $p_1(x_1,x_2)=1$. Similarly let $A_2=\{(x_1,x_2)\in(0,1]^2|\delta_{ms_1}(x_1,x_2)=1,\delta_n(x_1,x_2)=0\}$, and $A_3=\{(x_1,x_2)\in(0,1]^2|\delta_m(x_1,x_2)=1,\delta_{ms_1}(x_1,x_2)=0\}$. A_2 and A_3 are equivalent to B_2 and B_3 respectively with the relationship $x_1=\frac{u-u_1}{n}$ and $x_2=\frac{u-u_2}{n}$, where u_1 and u_2 are $U[0,1)$ random variables. Let $p_2(x_1,x_2)$ denote the joint density function for $(x_1,x_2)\in A_2$, $p_3(x_1,x_2)$ denote the joint density function for $(x_1,x_2)\in A_3$. We have

$$\int\int_{A_2} p_2(x_1,x_2)dx_1dx_2=\frac{n(s_2-1)}{n^2}p_2(x_1,x_2)=\frac{1}{ms_1}, \tag{17}$$

and

$$\int\int_{A_3} p_3(x_1,x_2)dx_1dx_2=\frac{n(s_2s_1-s_2)}{n^2}p_3(x_1,x_2)=\frac{1}{m}-\frac{1}{ms_1}. \tag{18}$$

Hence $p_2(x_1,x_2)=\frac{s_2}{s_2-1}$ and $p_3(x_1,x_2)=1$. From (16)-(18), (15) holds.

For the covariance between X_1 and X_2, we have

$$\begin{aligned} & \mathrm{cov}(X_1,X_2)=EX_1X_2-EX_1EX_2 \\ = & \frac{1}{s_2-1}\int_0^1\int_0^1 x_1x_2\delta_{ms_1}(x_1,x_2)dx_1dx_2 \\ & -\frac{s_2}{s_2-1}\int_0^1\int_0^1 x_1x_2\delta_n(x_1,x_2)dx_1dx_2. \end{aligned} \tag{19}$$

Here the first equation is just the definition of covariance. For the second equation, plug (15) into the calculation of EX_1X_2 and observe the fact that $\int_0^1\int_0^1 x_1x_2dx_1dx_2$ is equal to EX_1X_2 since both X_1 and X_2 are uniformly distributed, the equation then follows immediately. Now note that

$$\int_0^1\int_0^1 x_1x_2\delta_n(x_1,x_2)dx_1dx_2=\sum_{i=1}^n\int_{(i-1)/n}^{i/n}x_1dx_1\int_{(i-1)/n}^{i/n}x_2dx_2=\sum_{i=1}^n(\frac{2i-1}{2n^2})^2=\frac{1}{3n}-\frac{1}{12n^2}.$$

Similarly,

$$\int_0^1 x_1x_2\delta_{ms_1}(x_1,x_2)dx_1dx_2=\frac{1}{3ms_1}-\frac{1}{12(ms_1)^2}.$$

Thus, we have

$$\mathrm{cov}(X_1,X_2)=\frac{1}{12(s_2-1)(ms_1)^3}(\frac{1}{s_2^2}-1)<0. \tag{20}$$

The conclusion is obvious from (20). □