INSTRUMENTAL-VARIABLE CALIBRATION ESTIMATION IN SURVEY SAMPLING

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Supplementary Material

S1 On the minimization of the anticipated variance in (2.6)

Let X and Z be matrices with rows $\mathbf{x}_i'$, $i \in A$ and $\mathbf{z}_i'$, $i \in A$, respectively. Also let V be the diagonal matrix with diagonal elements v_i, $i \in A$. Then it suffices to show that (a) the matrix

$$H = (Z'X)^{-1}(Z'VZ)(X'Z)^{-1} - (X'V^{-1}X)^{-1}$$

is non-negative definite and that (b) H equals the null matrix when $Z' = X'V^{-1}$. Since (b) is obvious, we consider only (a). Let $T = (Z'X)^{-1}Z' - (X'V^{-1}X)^{-1}X'V^{-1}$. Clearly, TVT' is non-negative definite. But a little algebra shows that $TVT' = H$ and this proves (a).

S2 Computational details

$\hat{\boldsymbol{\lambda}}_1$ in (3.1) is the solution of the following calibration equation

$$\sum_{j \in A} \tilde{w}_j \mathbf{x}_j = \sum_{i=1}^{N} \mathbf{x}_i, \tag{S2.1}$$

where

$$\tilde{w}_j = 1 + \tilde{w}_j^* = 1 + (N-n)\frac{(d_j - 1)\exp\left\{\boldsymbol{\lambda}_1'\mathbf{z}_{1j}/(d_j-1)\right\}}{\sum_{j \in A}(d_j - 1)\exp\left\{\boldsymbol{\lambda}_1'\mathbf{z}_{1j}/(d_j-1)\right\}}. \tag{S2.2}$$

Note that (S2.1) can be written as

$$\frac{\sum_{i \in A}(d_i - 1)\exp(\boldsymbol{\lambda}_1'\mathbf{z}_{1i}^*)\mathbf{x}_{1i}}{\sum_{i \in A}(d_i - 1)\exp(\boldsymbol{\lambda}_1'\mathbf{z}_{1i}^*)} = \frac{1}{N-n}\sum_{i \in A^c}\mathbf{x}_{1i}, \tag{S2.3}$$

where $\mathbf{z}_{1i}^* = \mathbf{z}_{1i}/(d_i - 1)$. The left side of (S2.3) is a weighted mean of $\mathbf{x}_{1i}$ among sampled units while the right side of (S2.3) is a simple mean of $\mathbf{x}_{1i}$ among non-sampled units.

To solve (S2.3), we can use a Newton-type algorithm. Let $\bar{\mathbf{x}}_{1M}$ be equal to the right side of (S2.3). We can express (S2.3) as

$$\mathbf{U}(\boldsymbol{\lambda}_1) \equiv \sum_{i \in A} w_i(\boldsymbol{\lambda}_1)\,(\mathbf{x}_{1i} - \bar{\mathbf{x}}_{1M}) = \mathbf{0} \tag{S2.4}$$

where $w_i(\boldsymbol{\lambda}_1) = (d_i - 1)\exp(\boldsymbol{\lambda}_1'\mathbf{z}_{1i}^*)$. The Newton method for solving (S2.4) can be written as

$$\hat{\boldsymbol{\lambda}}_1^{(t+1)} = \hat{\boldsymbol{\lambda}}_1^{(t)} - \left\{ M(\hat{\boldsymbol{\lambda}}_1^{(t)}) \right\}^{-1} \sum_{i \in A} w_i(\hat{\boldsymbol{\lambda}}_1^{(t)})\,(\mathbf{x}_{1i} - \bar{\mathbf{x}}_{1M}) \tag{S2.5}$$

where $M(\boldsymbol{\lambda}_1) = \sum_{i \in A} \exp(\boldsymbol{\lambda}_1'\mathbf{z}_{1i})\mathbf{z}_{1i}'\,(\mathbf{x}_{1i} - \bar{\mathbf{x}}_{1M})$. Since the partial derivative $M(\hat{\boldsymbol{\lambda}}_1^{(t)})$ in (S2.5) may be not symmetric, which can make numeric problems in computation, instead of (S2.5), we can use

$$\hat{\boldsymbol{\lambda}}_1^{(t+1)} = \hat{\boldsymbol{\lambda}}_1^{(t)} - \left\{ M(\hat{\boldsymbol{\lambda}}_1^{(t)})' M(\hat{\boldsymbol{\lambda}}_1^{(t)}) \right\}^{-1} M(\hat{\boldsymbol{\lambda}}_1^{(t)})' \sum_{i \in A} w_i(\hat{\boldsymbol{\lambda}}_1^{(t)})\,(\mathbf{x}_{1i} - \bar{\mathbf{x}}_{1M})\,, \tag{S2.6}$$

which is equivalent to finding $\hat{\boldsymbol{\lambda}}_1$ that minimizes $\mathcal{Q}(\boldsymbol{\lambda}) = \mathbf{U}(\boldsymbol{\lambda})'\mathbf{U}(\boldsymbol{\lambda})$. We can simply use $\hat{\boldsymbol{\lambda}}_1^{(0)} = \mathbf{0}$ as the initial value of $\boldsymbol{\lambda}_1$.

S3 Proof of (3.2)

To prove (3.2), assume that a sequence of finite populations and samples is defined as in Isaki and Fuller (1982), which satisfies

$$N^{-1}\left(\sum_{j \in A} d_j(\mathbf{x}_j', y_j) - \sum_{i=1}^{N} (\mathbf{x}_i', y_i) \right) = O_p(n^{-1/2}).$$

Moreover, we assume that the calibration equation (S2.1) has one unique solution almost everywhere.

For $\hat{Y}_p$, after some calculation, we can establish that $\sum_{i \in A^c} \mathbf{x}_i'\hat{\beta}_z$ is equivalent to

$$(N-n)^{-1} \sum_{i \in A^c} \mathbf{x}_i'\hat{\beta}_z = \left\{ \tilde{y} + (\bar{\mathbf{X}}_{N-n} - \tilde{\mathbf{X}}_1)'\Gamma^{-1} \sum_{j \in A} (d_j - 1)(\mathbf{z}_{1j}^* - \tilde{\mathbf{Z}}_1^*)y_j \right\},$$

where $\tilde{y} = \sum_{j \in A}(d_j - 1)y_j / \{\sum_{j \in A}(d_j - 1)\}$, $\bar{\mathbf{X}}_{N-n} = (N-n)^{-1}\sum_{i \in A^c} \mathbf{x}_{1i}$, $\Gamma = \sum_{i \in A}(d_i - 1)(\mathbf{z}_{1i}^* - \tilde{\mathbf{Z}}_1^*)(\mathbf{x}_{1i} - \tilde{\mathbf{X}}_1)'$, $\mathbf{z}_{1i}^* = \mathbf{z}_{1i}/(d_i - 1)$, $\tilde{\mathbf{X}}_1 = \sum_{i \in A}(d_i - 1)\mathbf{x}_{1i} / \sum_{i \in A}(d_i - 1)$, and $\tilde{\mathbf{Z}}_1^* = \sum_{i \in A} \mathbf{z}_{1i} / \sum_{i \in A}(d_i - 1)$.

Write

$$\eta_j(\hat{\boldsymbol{\phi}}) = \frac{(d_j-1)\exp\left(\hat{\boldsymbol{\phi}}\mathbf{z}_{1j}^*/f_N\right)}{\sum_{j\in A}(d_j-1)\exp\left(\hat{\boldsymbol{\phi}}\mathbf{z}_{1j}^*/f_N\right)},$$

where $f_N = n/N$, $\hat{\boldsymbol{\phi}} = f_N\hat{\boldsymbol{\lambda}}_1$, and $\hat{\boldsymbol{\lambda}}_1$ is the unique solution of the calibration equation (S2.1), then $\hat{Y}_{cal,p} = \sum_{j\in A}\{1+(N-n)\eta_j(\hat{\boldsymbol{\phi}})\}y_j$. Since $\hat{\boldsymbol{\phi}}$ is the unique solution of the following equation :

$$\hat{\mathbf{U}}(\boldsymbol{\phi}) \equiv \sum_{j\in A}\eta_j(\boldsymbol{\phi})\mathbf{x}_{1j}' - \frac{1}{N-n}\sum_{i\in A^c}\mathbf{x}_{1i}' = 0,$$

a Taylor expansion of $\hat{\mathbf{U}}(\hat{\boldsymbol{\phi}})$ around $\hat{\boldsymbol{\phi}} = \mathbf{0}$ yields

$$0 = \hat{\mathbf{U}}(\mathbf{0}) + \dot{\mathbf{U}}(\mathbf{0})(\hat{\boldsymbol{\phi}}-0) + o_p(||\hat{\boldsymbol{\phi}}||),$$

where $\dot{\mathbf{U}}(\boldsymbol{\phi}) = \partial\hat{\mathbf{U}}(\boldsymbol{\phi})/\partial\boldsymbol{\phi}$. Note that $\dot{\mathbf{U}}(0) = f_N^{-1}\left[\sum_{j\in A}(d_j-1)\right]^{-1}\Gamma$ is bounded in probability. By Cauchy-Schwarz inequality,

$$\begin{aligned}
\sqrt{n}||\hat{\boldsymbol{\phi}}|| &\le ||\dot{\mathbf{U}}^{-1}(0)||\ ||\sqrt{n}\dot{\mathbf{U}}(0)\hat{\boldsymbol{\phi}}|| \\
&\le ||\sqrt{n}\dot{\mathbf{U}}^{-1}(0)\hat{\mathbf{U}}(0)|| + o_p(\sqrt{n}||\dot{\mathbf{U}}^{-1}(0)\hat{\boldsymbol{\phi}}||) \\
&= O_p(1) + o_p(\sqrt{n}||\hat{\boldsymbol{\phi}}||).
\end{aligned}$$

Thus $\hat{\boldsymbol{\phi}} = O_p(n^{-1/2})$ and

$$\hat{\boldsymbol{\phi}} = -\dot{\mathbf{U}}^{-1}(0)\hat{\mathbf{U}}(0) + o_p(n^{-1/2}). \tag{S3.1}$$

Now, taking a Taylor expansion of $\eta_j(\hat{\boldsymbol{\phi}})$ around $\hat{\boldsymbol{\phi}} = \mathbf{0}$ and by the continuity of the partial derivatives of $\eta_j(\boldsymbol{\phi})$, we have

$$\sum_{j\in A}\eta_j(\hat{\boldsymbol{\phi}})y_j = \sum_{j\in A}\eta_j(\mathbf{0})y_j + \sum_{j\in A}y_j\dot{\eta}_j(\mathbf{0})(\hat{\boldsymbol{\phi}}-\mathbf{0}) + o_p(n^{-1/2}),$$

where $\dot{\eta}_j(\boldsymbol{\phi}) = \partial\dot{\eta}_j(\boldsymbol{\phi})/\partial\boldsymbol{\phi}$. Using (S3.1), $\eta_j(\mathbf{0}) = \{\sum_{i\in A}(d_i-1)\}^{-1}(d_j-1)$, and $\dot{\eta}_j(\mathbf{0}) = f_N^{-1}\left[\sum_{j\in A}(d_j-1)\right]^{-1}(d_j-1)(\mathbf{z}_{1j}^*-\tilde{\mathbf{Z}}_1^*)$, we have established

$$\sum_{j\in A}\eta_j(\hat{\boldsymbol{\phi}})y_j = \tilde{y} + (\bar{\mathbf{X}}_{N-n}-\tilde{\mathbf{X}}_1)'\Gamma^{-1}\sum_{j\in A}(d_j-1)(\mathbf{z}_{1j}^*-\tilde{\mathbf{Z}}_1^*)y_j + o_p(n^{-1/2}).$$

Since $(N-n)^{-1}(\hat{Y}_{cal,p}-\hat{Y}_p) = (N-n)^{-1}\sum_{i\in A^c}\mathbf{x}_i'\hat{\beta}_z - \sum_{j\in A}\eta_j(\hat{\boldsymbol{\phi}})y_j$, we have (3.2).

S4 Two-step Calibration

For the two-step calibration weights proposed in Section 4, we have

$$
\begin{aligned}
w_{2k}^{(1)} &\cong d_{2k} + \left(\sum_{A_1} d_{1k}\mathbf{x}_k - \sum_{A_2} d_{2k}\mathbf{x}_k\right)' \left(\sum_{A_2} \mathbf{z}_k \mathbf{x}_k'\right)^{-1} \mathbf{z}_k, \\
w_{2k}^{(2)} &\cong w_{2k}^{(1)} + \left(\sum_{U} \mathbf{x}_{1k} - \sum_{A_2} w_{2k}^{(1)}\mathbf{x}_{1k}\right)' \left(\sum_{A_2} \frac{d_{2k}}{d_{1k}} \mathbf{z}_{1k} \mathbf{x}_k'\right)^{-1} \frac{d_{2k}}{d_{1k}} \mathbf{z}_{1k}
\end{aligned}
$$

and the alternative estimator $\hat{Y}_{tp,new} = \sum_{i \in A_2} w_{2i}^{(2)} y_i$ has the form of

$$
\begin{aligned}
\hat{Y}_{tp,new} &= \sum_{i \in A_2} w_{2i}^{(1)} y_i + \left(\sum_{i \in U} \mathbf{x}_{1i} - \sum_{i \in A_2} w_{2i}^{(1)} \mathbf{x}_{1i}\right)' \hat{\mathbf{b}}_1, \\
&= \sum_{i \in A_2} d_{2i}\left(y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}}\right) + \sum_{i \in A_1} d_{1i}\left(\mathbf{x}_i'\hat{\boldsymbol{\beta}} - \mathbf{x}_{1i}'\hat{\mathbf{b}}_1\right) + \sum_{i \in U} \mathbf{x}_{1i}\hat{\mathbf{b}}_1,
\end{aligned}
$$

where

$$
\hat{\mathbf{b}}_1 = \left(\sum_{i \in A_2} \frac{d_{2k}}{d_{1k}} \mathbf{z}_{1i}\mathbf{x}_{1i}'\right)^{-1} \left(\sum_{i \in A_2} \frac{d_{2k}}{d_{1k}} \mathbf{z}_{1i} y_i\right).
$$

Writing

$$
\tilde{\mathbf{b}}_1 = \left(\sum_{i \in A_1} \mathbf{z}_{1i}\mathbf{x}_{1i}'\right)^{-1} \left(\sum_{i \in A_1} \mathbf{z}_{1i} y_i\right),
$$

we can establish

$$
\hat{Y}_{tp,new} \cong \sum_{i \in A_2} d_{2i}\left(y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}}\right) + \sum_{i \in A_1} d_{1i}\left(\mathbf{x}_i'\hat{\boldsymbol{\beta}} - \mathbf{x}_{1i}'\tilde{\mathbf{b}}_1\right) + \sum_{i \in U} \mathbf{x}_{1i}\tilde{\mathbf{b}}_1
$$

and

$$
p\lim \tilde{\mathbf{b}}_1 = p\lim\left(\hat{\boldsymbol{\beta}}_{1,z} + \hat{\boldsymbol{\beta}}_{x,z}\hat{\boldsymbol{\beta}}_{2,z}\right) = p\lim \hat{\mathbf{B}}_{1,z}.
$$

Thus,

$$
\hat{Y}_{tp,new} \cong \sum_{i \in A_2} d_{2i}\left(y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}}\right) + \sum_{i \in A_1} d_{1i}\left(\mathbf{x}_i'\hat{\boldsymbol{\beta}} - \mathbf{x}_{1i}'\hat{\mathbf{B}}_{1,z}\right) + \sum_{i \in U} \mathbf{x}_{1i}\hat{\mathbf{B}}_{1,z},
$$

which establishes the asymptotic equivalence between $\hat{Y}_{tp,r}$ in (4.5) and $\hat{Y}_{tp,new} = \sum_{i \in A_2} w_{2i}^{(2)} y_i$.