TESTING FOR THE BUFFERED AUTOREGRESSIVE PROCESSES (SUPPLEMENTARY MATERIAL)

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APPENDIX: PROOFS

In this appendix, we first give the proofs of Lemmas 2.1-2.2. Denote C as a generic constant which may vary from place to place in the rest of this paper. The proofs of Lemmas 2.1-2.2 rely on the following three basic lemmas:

Lemma A.1. *Suppose that y_t is strictly stationary, ergodic and absolutely regular with mixing coefficients $\beta(m) = O(m^{-A})$ for some $A > v/(v-1)$ and $r > v > 1$; and there exists an $A_0 > 1$ such that $2A_0rv/(r-v) < A$. Then, for any $\gamma = (r_L, r_U) \in \Gamma$, we have*

$$\sum_{j=1}^{\infty}\left\{E\left[\prod_{i=1}^{j} I(r_L < y_{t-i} \leq r_U)\right]\right\}^{(r-v)/2A_0rv} < \infty.$$

Proof. First, denote $\xi_i = I(r_L < y_{t-i} \leq r_U)$. Then, ξ_i is strictly stationary, ergodic and α-mixing with mixing coefficients $\alpha(m) = O(m^{-A})$. Next, take $\iota \in \Big([2A_0rv/(r-v)+1]/(A+1), 1\Big)$, and let $p = \lfloor j^{\iota}\rfloor$ and $s = \lfloor j/j^{\iota}\rfloor$, where $\lfloor x\rfloor$ is the largest integer not greater than x. When $j \geq j_0$ is large enough, we can always find $\{\xi_{kp+1}\}_{k=0}^{s-1}$, a subsequence of $\{\xi_i\}_{i=1}^{j}$.

Furthermore, let $\mathcal{F}_m^n = \sigma(\xi_i, m \leq i \leq n)$. Then, $\xi_{kp+1} \in \mathcal{F}_{kp+1}^{kp+2}$. Note that $E\left[\xi_{kp+1}\right] < P(a \leq y_t \leq b) \triangleq \rho \in (0,1)$. Hence, by Proposition 2.6 in Fan and Yao (2003, p.72), we have that for $j \geq j_0$,

$$\begin{aligned}
E\left[\prod_{i=1}^{j}\xi_i\right] &\leq E\left[\prod_{k=0}^{s-1}\xi_{kp+1}\right] \\
&= \left\{E\left[\prod_{k=0}^{s-1}\xi_{kp+1}\right] - \prod_{k=0}^{s-1}E\left[\xi_{kp+1}\right]\right\} + \prod_{k=0}^{s-1}E\left[\xi_{kp+1}\right] \\
&\leq 16(s-1)\alpha(p) + \rho^s \\
&\leq C\lfloor j/j^{\iota}\rfloor\lfloor j^{\iota}\rfloor^{-A} + \rho^{\lfloor j/j^{\iota}\rfloor}.
\end{aligned}$$

Therefore, since $(r-v)/2A_0rv>0$, by using the inequality $(x+y)^k \le C(x^k+y^k)$ for any $x,y,k>0$, it follows that

$$\begin{aligned}\sum_{j=1}^{\infty}\left\{E\left[\prod_{i=1}^{j}\xi_i\right]\right\}^{(r-v)/2A_0rv} &\le (j_0-1)+\sum_{j=j_0}^{\infty}\left\{E\left[\prod_{i=1}^{j}\xi_i\right]\right\}^{(r-v)/2A_0rv}\\ &\le (j_0-1)+C\sum_{j=j_0}^{\infty}\left[\lfloor j/j^{\iota}\rfloor\lfloor j^{\iota}\rfloor^{-A}\right]^{(r-v)/2A_0rv}\\ &\quad + C\sum_{j=j_0}^{\infty}\rho^{\lfloor j/j^{\iota}\rfloor(r-v)/2A_0rv}. \end{aligned} \tag{A.1}$$

Since $\iota>[2A_0rv/(r-v)+1]/(A+1)$, we have $(\iota A+\iota-1)(r-v)/2A_0rv>1$, and hence $\sum_{j=1}^{\infty} j^{-(\iota A+\iota-1)(r-v)/2A_0rv}<\infty$, which implies that

$$\sum_{j=j_0}^{\infty}\left[\frac{\lfloor j/j^{\iota}\rfloor}{\lfloor j^{\iota}\rfloor^{A}}\right]^{(r-v)/2A_0rv} \le \sum_{j=1}^{\infty}\left[\frac{j}{j^{\iota}(j^{\iota}-1)^{A}}\right]^{(r-v)/2A_0rv} <\infty. \tag{A.2}$$

On the other hand, since $\left(\rho^{\lfloor j/j^{\iota}\rfloor(r-v)/2A_0rv}\right)^{1/j}<1$, by Cauchy's root test, we have

$$\sum_{j=j_0}^{\infty}\rho^{\lfloor j/j^{\iota}\rfloor(r-v)/2A_0rv}<\sum_{j=1}^{\infty}\rho^{\lfloor j/j^{\iota}\rfloor(r-v)/2A_0rv}<\infty. \tag{A.3}$$

Now, the conclusion follows directly from (A.1)-(A.3). This completes the proof. $\square$

LEMMA A.2. *Suppose that the conditions in Lemma A.1 hold, and y_t has a bounded and continuous density function. Then, there exists a $B_0>1$ such that for any $\gamma_1,\gamma_2\in\Gamma$, we have*

$$\|R_t(\gamma_1)-R_t(\gamma_2)\|_{2rv/(r-v)}\le C|\gamma_1-\gamma_2|^{(r-v)/2B_0rv}.$$

PROOF. Let $\gamma_1=(r_{1L},r_{1U})$ and $\gamma_2=(r_{2L},r_{2U})$. Since $R_t(\gamma)=I(y_{t-d}\le r_L)+R_{t-1}(\gamma)I(r_L<y_{t-d}\le r_U)$, we have

$$R_t(\gamma_1)-R_t(\gamma_2)=\Delta_t(\gamma_1,\gamma_2)+I(r_{1L}<y_{t-d}\le r_{1U})\left[R_{t-1}(\gamma_1)-R_{t-1}(\gamma_2)\right],$$

where

$$\begin{aligned}\Delta_t(\gamma_1,\gamma_2)&=I(r_{2L}<y_{t-d}\le r_{1L})\\ &\quad + R_{t-1}(\gamma_2)\left[I(r_{1L}<y_{t-d}\le r_{1U})-I(r_{2L}<y_{t-d}\le r_{2U})\right].\end{aligned}$$

Thus, by iteration we can show that

$$
\begin{aligned}
&R_t(\gamma_1) - R_t(\gamma_2) \\
&\quad = \Delta_t(\gamma_1, \gamma_2) + \sum_{j=1}^{\infty} \Delta_{t-j}(\gamma_1, \gamma_2) \prod_{i=1}^{j} I(r_{1L} < y_{t-i-d} \leq r_{1U}). \qquad \text{(A.4)}
\end{aligned}
$$

Next, for brevity, we assume that $r_{2L} \leq r_{1L} \leq r_{2U} \leq r_{1U}$, because the proofs for other cases are similar. Note that for any $j \geq 0$, $R_{t-j-1}(\gamma_2) \leq 1$ and

$$
\begin{aligned}
&I(r_{1L} < y_{t-j-d} \leq r_{1U}) - I(r_{2L} < y_{t-j-d} \leq r_{2U}) \\
&\quad = I(r_{2U} < y_{t-j-d} \leq r_{1U}) - I(r_{2L} < y_{t-j-d} \leq r_{1L}).
\end{aligned}
$$

Let $f(x)$ be the density function of y_t. Since $\sup_x f(x) < \infty$ and $|\Delta_{t-j}(\gamma_1, \gamma_2)| \leq 2$, by Hölder's inequality and Taylor's expansion, it follows that for any $s \geq 1$,

$$
\begin{aligned}
E|\Delta_{t-j}(\gamma_1, \gamma_2)|^s &\leq 2^{s-1} E|\Delta_{t-j}(\gamma_1, \gamma_2)| \\
&\leq 2^{s-1} \left[2 \sup_x f(x) |r_{1L} - r_{2L}| + \sup_x f(x) |r_{1U} - r_{2U}| \right] \\
&\leq C|\gamma_1 - \gamma_2|. \qquad \text{(A.5)}
\end{aligned}
$$

Let $A_0 > 1$ be specified in Lemma A.1, and choose B_0 such that $1/A_0 + 1/B_0 = 1$. By Hölder's inequality and (A.5), we can show that

$$
\begin{aligned}
&E\left| \Delta_{t-j}(\gamma_1, \gamma_2) \prod_{i=1}^{j} I(r_{1L} < y_{t-i-d} \leq r_{1U}) \right|^{2rv/(r-v)} \\
&\leq \left\{ E[\Delta_{t-j}(\gamma_1, \gamma_2)]^{2B_0 rv/(r-v)} \right\}^{1/B_0} \\
&\quad \times \left\{ E\left[\prod_{i=1}^{j} I(r_{1L} < y_{t-i-d} \leq r_{1U}) \right] \right\}^{1/A_0} \\
&\leq 2^{[2B_0 rv/(r-v)]-1} \left\{ E|\Delta_{t-j}(\gamma_1, \gamma_2)| \right\}^{1/B_0} \\
&\quad \times \left\{ E\left[\prod_{i=1}^{j} I(r_{1L} < y_{t-i-d} \leq r_{1U}) \right] \right\}^{1/A_0} \\
&\leq C|\gamma_1 - \gamma_2|^{1/B_0} \left\{ E\left[\prod_{i=1}^{j} I(r_{1L} < y_{t-i-d} \leq r_{1U}) \right] \right\}^{1/A_0}. \qquad \text{(A.6)}
\end{aligned}
$$

By (A.4)-(A.6), Minkowski's inequality, Lemma A.1 and the compactness of Γ, we have

$$\begin{aligned}
&\|R_t(\gamma_1) - R_t(\gamma_2)\|_{2rv/(r-v)} \\
&\quad \le C|\gamma_1 - \gamma_2|^{(r-v)/2rv} + C|\gamma_1 - \gamma_2|^{(r-v)/2B_0rv} \\
&\qquad \times \sum_{j=1}^{\infty} \left\{ E\left[\prod_{i=1}^{j} I(r_{1L} < y_{t-i-d} \le r_{1U}) \right] \right\}^{(r-v)/2A_0rv} \\
&\quad \le C|\gamma_1 - \gamma_2|^{(r-v)/2B_0rv}.
\end{aligned}$$

This completes the proof. □

LEMMA A.3. *Suppose that the conditions in Lemma A.2 hold and* $Ey_t^4 < \infty$. *Then,*

$$\sup_{\gamma\in\Gamma} \left| \frac{X_\gamma' X}{n} - \Sigma_\gamma \right| \to 0 \quad a.s. \;\; as\; n \to \infty.$$

PROOF. For brevity, we only prove the uniform convergence for $n^{-1}\sum_{t=1}^{n}\phi_t(\gamma)$, the last component of $n^{-1}X_\gamma' X$, where

$$\phi_t(\gamma) = y_{t-p}^2 R_t(\gamma).$$

First, for fix $\varepsilon > 0$, we partition Γ by $\{B_1, \cdots, B_{K_\varepsilon}\}$, where $B_k = \{(r_L, r_U); \omega_k < r_L \le \omega_{k+1}, \nu_k < r_U \le \nu_{k+1}\} \cap \Gamma$. Here, $\{\omega_k\}$ and $\{\nu_k\}$ are chosen such that

$$(\omega_{k+1} - \omega_k)^{(r-v)/2B_0rv} < C_1\varepsilon \;\text{ and }\; (\nu_{k+1} - \nu_k)^{(r-v)/2B_0rv} < C_1\varepsilon, \tag{A.7}$$

where $B_0 > 1$ is specified as in Lemma A.2, and $C_1 > 0$ will be selected later.

Next, we set

$$f_t^u(\varepsilon) = y_{t-p}^2 R_t(\omega_{k+1}, \nu_{k+1}) \quad\text{and}\quad f_t^l(\varepsilon) = y_{t-p}^2 R_t(\omega_k, \nu_k).$$

By construction, since $R_t(\gamma)$ is a nondecreasing function with respect to r_L and r_U, for any $\gamma \in \Gamma$, there is some k such that $\gamma \in B_k$ and $f_t^l(\varepsilon) \le \phi_t(\gamma) \le f_t^u(\varepsilon)$.

Furthermore, since $rv/(2rv - r + v) < 1$, we have

$$\left\|y_{t-p}^2\right\|_{2rv/(2rv-r+v)} < \left\|y_{t-p}^2\right\|_2 < \infty.$$

Thus, by Hölder's inequality, Lemma A.2 and (A.7), we have

$$\begin{aligned} &E\left[f_t^u(\varepsilon) - f_t^l(\varepsilon)\right] \\ &\quad\le \left\|y_{t-p}^2\right\|_{2rv/(2rv-r+v)} \left\|R_t(\omega_{k+1}, \nu_{k+1}) - R_t(\omega_k, \nu_k)\right\|_{2rv/(r-v)} \\ &\quad\le C\left[(\omega_{k+1} - \omega_k)^{(r-v)/2B_0rv} + (\nu_{k+1} - \nu_k)^{(r-v)/2B_0rv}\right] \\ &\quad\le 2CC_1\varepsilon. \end{aligned}$$

By setting $C_1 = (2C)^{-1}$, we have $E\left[f_t^u(\varepsilon) - f_t^l(\varepsilon)\right] \le \varepsilon$. Thus, the conclusion holds according to Theorem 2 in Pollard (1984, p.8). This completes the proof. $\square$

Proof of Lemma 2.1. First, since $K_{\gamma\gamma}$ is positive definite by Assumption 2.1, we know that both Σ and Σ_γ are positive definite. By using the same argument as for Lemma 2.1(iv) in Chan (1990), it is not hard to show that for every $\gamma \in \Gamma$, $\Sigma_\gamma - \Sigma_\gamma \Sigma^{-1} \Sigma_\gamma'$ is positive definite. Second, by the ergodic theorem, it is easy to see that

$$\frac{X'X}{n} \to \Sigma \quad \text{a.s. as } n \to \infty. \tag{A.8}$$

Third, by Lemma A.3 we have

$$\sup_{\gamma\in\Gamma}\left|\frac{X_\gamma' X}{n} - \Sigma_\gamma\right| \to 0 \quad \text{and} \quad \sup_{\gamma\in\Gamma}\left|\frac{X_\gamma' X_\gamma}{n} - \Sigma_\gamma\right| \to 0 \quad \text{a.s.} \tag{A.9}$$

as $n \to \infty$. Note that if H_0 holds, we have

$$\begin{aligned} T_\gamma &= \frac{1}{\sqrt{n}}\left\{X_\gamma' - X_\gamma' X (X'X)^{-1} X'\right\}\varepsilon \\ &= \frac{1}{\sqrt{n}}\Big(- (X_\gamma' X)(X'X)^{-1}, I\Big) Z_\gamma' \varepsilon. \end{aligned}$$

Then, (i) and (ii) follow readily from (A.8)-(A.9). This completes the proof. $\square$

Proof of Lemma 2.2. Denote

$$G_n(\gamma) \equiv \frac{1}{\sqrt{n}} Z_\gamma' \varepsilon = \frac{1}{\sqrt{n}} \sum_{t=p}^{N} x_t(\gamma)\varepsilon_t.$$

It is straightforward to show that the finite dimensional distribution of $\{G_n(\gamma)\}$ converges to that of $\{\sigma G_\gamma\}$. By Pollard (1990, Sec.10), we only need to verify the stochastic equicontinuity of $\{G_n(\gamma)\}$. To establish it, we use Theorem 1, Application

4 in Doukhan, Massart, and Rio (1995, p.405); see also Andrews (1993) and Hansen (1996).

First, the envelop function is $\sup_\gamma |x_t(\gamma)\varepsilon_t| = \bar{x}_t|\varepsilon_t|$, where $\bar{x}_t = \sup_\gamma |x_t(\gamma)|$. By Hölder's inequality and Assumption 2.1, we know that the envelop function is L^{2v} bounded. Next, for any $\gamma_1, \gamma_2 \in \Gamma$, by Assumptions 2.1-2.3, Lemma A.2 and Hölder's inequality, we have

$$\begin{aligned}\|x_t(\gamma_1)\varepsilon_t - x_t(\gamma_2)\varepsilon_t\|_{2v} &= \|h_t(\gamma_1)\varepsilon_t - h_t(\gamma_2)\varepsilon_t\|_{2v} \\ &\le \|x_t\varepsilon_t\|_{2r}\|R_t(\gamma_1) - R_t(\gamma_2)\|_{2rv/(r-v)} \\ &\le C\|x_t\|_{4r}\|\varepsilon_t\|_{4r}\,|\gamma_1-\gamma_2|^{(r-v)/2B_0rv} \\ &\le C|\gamma_1-\gamma_2|^{(r-v)/2B_0rv}\end{aligned}$$

for some $B_0 > 1$, where the last inequality holds since $\|x_t\|_{4r}\|\varepsilon_t\|_{4r} < \infty$.

Now, following the argument in Hansen (1996, p.426), we know that $G_n(\gamma)$ is stochastically equicontinuous. This completes the proof. $\square$

Next, we give Lemmas A.4-A.6, in which Lemma A.4 is crucial for proving Lemma A.5, and Lemmas A.5 and A.6 are needed to prove Corollary 2.1 and Theorem 3.1, respectively.

LEMMA A.4. *Suppose that y_t is strictly stationary and ergodic. Then, (i) $n_0 = O_p(1)$; (ii) furthermore, if $E|y_t|^2 < \infty$ and $E|\varepsilon_t|^2 < \infty$, for any $a_n = o(1)$, we have*

$$\sup_{\gamma\in\Gamma}\left|a_n\sum_{t=p}^{n_0-1} x_t x_t' R_t(\gamma)\right| = O_p(1) \tag{A.10}$$

and

$$\sup_{\gamma\in\Gamma}\left|a_n\sum_{t=p}^{n_0-1} h_t(\gamma)\varepsilon_t\right| = O_p(1). \tag{A.11}$$

PROOF. First, by the ergodic theory, we have that

$$\frac{1}{M}\sum_{t=p}^{M} I(a \le y_{t-d} \le b) = P(a \le y_{t-d} \le b) \triangleq \kappa > 0 \quad \text{a.s.}$$

as $M \to \infty$. Thus, $\forall\eta > 0$, there exists an integer $M(\eta) > 0$ such that

$$P\left(\frac{1}{M}\sum_{t=p}^{M} I(a \le y_{t-d} \le b) < \frac{\kappa}{2}\right) < \eta.$$

By the definition of n_0, it follows that

$$\begin{aligned} P\left(n_0 > M\right) &= P\left(\sum_{t=p}^{M} I(a \leq y_{t-d} \leq b) = 0\right) \\ &= P\left(\frac{1}{M}\sum_{t=p}^{M} I(a \leq y_{t-d} \leq b) = 0\right) \\ &\leq P\left(\frac{1}{M}\sum_{t=p}^{M} I(a \leq y_{t-d} \leq b) < \frac{\kappa}{2}\right) \\ &< \eta, \end{aligned} \tag{A.12}$$

i.e., (i) holds. Furthermore, by taking $\tilde{M} = M^2$, from (A.12) and Markov's inequality, it follows that $\forall \eta > 0$,

$$\begin{aligned} &P\left(\sup_{\gamma\in\Gamma}\left|a_n \sum_{t=p}^{n_0-1} x_t x_t' R_t(\gamma)\right| > \tilde{M}\right) \\ &= P\left(\sup_{\gamma\in\Gamma}\left|a_n \sum_{t=p}^{n_0-1} x_t x_t' R_t(\gamma)\right| > \tilde{M}, n_0 \leq M\right) \\ &\leq P\left(\max_{p\leq k\leq M}\sup_{\gamma\in\Gamma}\left|a_n \sum_{t=p}^{k-1} x_t x_t' R_t(\gamma)\right| > \tilde{M}\right) \\ &\leq \sum_{k=p}^{M} P\left(a_n \sum_{t=p}^{k-1} |x_t|^2 > \tilde{M}\right) \\ &\leq a_n \sum_{k=p}^{M}\sum_{t=p}^{k-1} \frac{E|x_t|^2}{\tilde{M}} \\ &= O\left(\frac{a_n M^2}{\tilde{M}}\right) = O\left(a_n\right) < \eta \end{aligned} \tag{A.13}$$

as n is large enough. Thus, we know that equation (A.10) holds. Next, by Hölder's inequality and a similar argument as for (A.13), it is not hard to show that $\forall \eta > 0$,

$$P\left(\sup_{\gamma\in\Gamma}\left|a_n \sum_{t=p}^{n_0-1} h_t(\gamma)\varepsilon_t\right| > \tilde{M}\right) \leq O\left(a_n\right) < \eta$$

as n is large enough, i.e., (A.11) holds. This completes the proof. □

LEMMA A.5. *If Assumptions 2.1-2.3 hold, then it follows that under H_0 or H_{1n},*

$$(i)\ \sup_{\gamma\in\Gamma}\left|\frac{1}{n}\left(X_\gamma-\tilde{X}_\gamma\right)'X\right|=o_p(1),$$

$$(ii)\ \sup_{\gamma\in\Gamma}\left|\frac{1}{n}\left(X_\gamma'X_\gamma-\tilde{X}_\gamma'\tilde{X}_\gamma\right)\right|=o_p(1),$$

$$(iii)\ \sup_{\gamma\in\Gamma}\left|T_\gamma-\tilde{T}_\gamma\right|=o_p(1),$$

where $\tilde{X}_\gamma$ and $\tilde{T}_\gamma$ are defined in the same way as X_γ and T_γ, respectively, with $R_t(\gamma)$ being replaced by $\tilde{R}_t(\gamma)$.

PROOF. (i) Note that

$$\frac{1}{n}\left(X_\gamma-\tilde{X}_\gamma\right)'X=\frac{1}{\sqrt{n}}\left[\frac{1}{\sqrt{n}}\sum_{t=p}^{n_0-1}x_tx_t'R_t(\gamma)\right].$$

Hence, we know that (i) holds by taking $a_n=n^{-1/2}$ in equation (A.10).

(ii) By a similar argument as for (i), we can show that (ii) holds.

(iii) Note that when $\lambda_0=(\phi_0',h'/\sqrt{n})'$, we have

$$\begin{aligned}
T_\gamma-\tilde{T}_\gamma=&\frac{1}{\sqrt{n}}\left(X_\gamma-\tilde{X}_\gamma\right)'\varepsilon-\frac{1}{\sqrt{n}}\left(X_\gamma-\tilde{X}_\gamma\right)'X(X'X)^{-1}X'\varepsilon\\
&-\frac{1}{n}\left(X_\gamma-\tilde{X}_\gamma\right)'X(X'X)^{-1}X'X_{\gamma_0}h\\
&-\frac{1}{n}\tilde{X}_\gamma'X(X'X)^{-1}X'\left(X_{\gamma_0}-\tilde{X}_{\gamma_0}\right)h\\
&+\frac{1}{n}\left(X_\gamma'X_{\gamma_0}-\tilde{X}_\gamma'\tilde{X}_{\gamma_0}\right)h\\
&\triangleq I_{1n}(\gamma)-I_{2n}(\gamma)-I_{3n}(\gamma)-I_{4n}(\gamma)+I_{5n}(\gamma)\ \text{ say.}
\end{aligned}$$

First, since

$$I_{1n}(\gamma)=\frac{1}{n^{1/4}}\left[\frac{1}{n^{1/4}}\sum_{t=p}^{n_0-1}h_t(\gamma)\varepsilon_t\right],$$

it follows that $\sup_\gamma|I_{1n}(\gamma)|=o_p(1)$ by taking $a_n=n^{-1/4}$ in equation (A.11). Next, since

$$I_{2n}(\gamma)=\left[\frac{1}{n}\sum_{t=p}^{n_0-1}x_tx_t'R_t(\gamma)\right]\left(\frac{X'X}{n}\right)^{-1}\frac{X'\varepsilon}{\sqrt{n}},$$

we have that $\sup_\gamma|I_{2n}(\gamma)|=o_p(1)$ from (i). Similarly, we can show that $\sup_\gamma|I_{in}(\gamma)|=o_p(1)$ for $i=3,4,5$. Hence, under H_0 (i.e., $h\equiv 0$) or H_{1n}, we know that (iii) holds. This completes the proof. □

LEMMA A.6. *If Assumptions 2.1-2.3 hold, then it follows that under H_0 or H_{1n},*

$$\sup_{\gamma\in\Gamma}\sqrt{n}|\lambda_n(\gamma)-\lambda_0| = O_p(1).$$

PROOF. First, for any $\gamma\in\Gamma$, by Taylor's expansion we have

$$\begin{aligned}
&\sum_{t=p}^{N}\left[\varepsilon_t^2(\lambda_n(\gamma),\gamma)-\varepsilon_t^2(\lambda_0,\gamma)\right]\\
&\quad= -(\lambda_n(\gamma)-\lambda_0)'\left(\sum_{t=p}^{N}2\varepsilon_t(\lambda_0,\gamma)x_t(\gamma)\right)\\
&\qquad+(\lambda_n(\gamma)-\lambda_0)'\left(\sum_{t=p}^{N}x_t(\gamma)x_t(\gamma)'\right)(\lambda_n(\gamma)-\lambda_0).
\end{aligned}\tag{A.14}$$

Next, when $\lambda_0=(\phi_0',h'/\sqrt{n})'$, we can show that

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{t=p}^{N}\varepsilon_t(\lambda_0,\gamma)x_t(\gamma) &= \frac{1}{\sqrt{n}}Z_\gamma'\varepsilon+\frac{1}{\sqrt{n}}\sum_{t=p}^{N}x_t(\gamma)[x_t(\gamma_0)-x_t(\gamma)]'\lambda_0\\
&= \frac{1}{\sqrt{n}}Z_\gamma'\varepsilon+\frac{1}{n}\sum_{t=p}^{N}\begin{pmatrix} x_tx_t'[R_t(\gamma_0)-R_t(\gamma)]\\ x_tx_t'[R_t(\gamma)R_t(\gamma_0)-R_t(\gamma)]\end{pmatrix}h\\
&\triangleq G_n^*(\gamma).
\end{aligned}\tag{A.15}$$

Let $\lambda_{min}(\gamma)>0$ be the minimum eigenvalue of $K_{\gamma\gamma}$. Then, by equations (A.14)-(A.15), $\forall\eta>0$, there exists a $M(\eta)>0$ such that

$$\begin{aligned}
&P\left(\sup_{\gamma\in\Gamma}\sqrt{n}|\lambda_n(\gamma)-\lambda_0|>M\right)\\
&= P\Bigg(\sqrt{n}|\lambda_n(\gamma)-\lambda_0|>M,\ \sum_{t=p}^{N}\left[\varepsilon_t^2(\lambda_n(\gamma),\gamma)-\varepsilon_t^2(\lambda_0,\gamma)\right]\le 0\\
&\qquad\text{for some }\gamma\in\Gamma\Bigg)\\
&\le P\Big(\sqrt{n}|\lambda_n(\gamma)-\lambda_0|>M,\ -2\sqrt{n}|\lambda_n(\gamma)-\lambda_0||G_n^*(\gamma)|\\
&\qquad+n|\lambda_n(\gamma)-\lambda_0|^2[\lambda_{min}(\gamma)+o_p(1)]\le 0\text{ for some }\gamma\in\Gamma\Big)\\
&\le P\Big(M<\sqrt{n}|\lambda_n(\gamma)-\lambda_0|\le 2[\lambda_{min}(\gamma)+o_p(1)]^{-1}|G_n^*(\gamma)|\\
&\qquad\text{for some }\gamma\in\Gamma\Big)\\
&\le P\left(|G_n^*(\gamma)|>M[\lambda_{min}(\gamma)+o_p(1)]/2\text{ for some }\gamma\in\Gamma\right)\\
&\le\eta,
\end{aligned}$$

where the last inequality holds because $G_n^*(\gamma) = O_p(1)$ by Lemma 2.2 and Lemma A.3. Hence, under H_0 (i.e., $h \equiv 0$) or H_{1n}, our conclusion holds. This completes the proof. □

Proof of Corollary 2.1. The conclusion follows directly from Theorems 2.1-2.2 and Lemma A.5. □

Proof of Theorem 3.1. We use the method in the proof of Theorem 2 in Hansen (1996). Let W denote the set of samples ω for which

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=p}^{N} \sup_{\gamma\in\Gamma} |x_t(\gamma)|\varepsilon_t^2 < \infty \text{ a.s.,} \tag{A.16}$$

$$\lim_{n\to\infty} \sup_{\gamma,\delta\in\Gamma} \left| \frac{1}{n}\sum_{t=p}^{N} x_t(\gamma)x_t(\delta)'\varepsilon_t^2 - \sigma^2 K_{\gamma\delta} \right| \to 0 \quad \text{a.s.} \tag{A.17}$$

Since $\sup_{\gamma\in\Gamma} |x_t(\gamma)| \leq \sqrt{2}|x_t|$ and $E|x_t|\varepsilon_t^2 < \infty$ due to Assumption 2.1, by the ergodic theorem we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=p}^{N} \sup_{\gamma\in\Gamma} |x_t(\gamma)|\varepsilon_t^2 \leq \lim_{n\to\infty} \frac{\sqrt{2}}{n}\sum_{t=p}^{N} |x_t|\varepsilon_t^2 < \infty \text{ a.s.,}$$

i.e., (A.16) holds. Furthermore, by Assumptions 2.1-2.3 and a similar argument as for Lemma A.3, it is not hard to see that

$$\lim_{n\to\infty} \sup_{\gamma,\delta\in\Gamma} \left| \frac{1}{n}\sum_{t=p}^{N} x_t(\gamma)x_t(\delta)'\varepsilon_t^2 - \sigma^2 K_{\gamma\delta} \right| \to 0 \quad \text{a.s.,}$$

i.e., (A.17) holds. Thus, $P(W) = 1$. Take any $\omega \in W$. For the remainder of the proof, all operations are conditionally on ω, and hence all of the randomness appears in the i.i.d. N(0, 1) variables $\{v_t\}$.

Define

$$Z_n^*(\gamma) = \frac{1}{\sqrt{n}}\sum_{t=p}^{N} x_t(\gamma)\varepsilon_t v_t.$$

By using the same argument as in Hansen (1996, p.426-427), we have

$$Z_n^*(\gamma) \Rightarrow \sigma G_\gamma \quad \text{a.s. as } n \to \infty. \tag{A.18}$$

Note that

$$\sup_{\gamma\in\Gamma} |\hat{Z}_n(\gamma) - Z_n^*(\gamma)| \leq \sup_{\gamma\in\Gamma} \left| \frac{1}{n}\sum_{t=p}^{N} x_t(\gamma)x_t(\gamma)'v_t \right| \sup_{\gamma\in\Gamma} \left| \sqrt{n}(\lambda_n(\gamma) - \lambda_0) \right|.$$

Using the same argument as for (A.18) (see, e.g., Hansen (1996, p.427)), we have

$$\frac{1}{n}\sum_{t=p}^{N} x_t(\gamma)x_t(\gamma)'v_t \Rightarrow 0 \text{ a.s. as } n \to \infty. \tag{A.19}$$

Now, by Lemma A.6 and (A.19), it follows that under H_0 or H_{1n},

$$\hat{Z}_n(\gamma) - Z_n^*(\gamma) \Rightarrow 0 \text{ in probability as } n \to \infty. \tag{A.20}$$

Thus, by (A.18) and (A.20), we know that under H_0 or H_{1n},

$$\hat{Z}_n(\gamma) \Rightarrow \sigma G_\gamma \text{ in probability as } n \to \infty. \tag{A.21}$$

Next, we consider the functional

$$L : x(\cdot) \in D_{2p+2}(\Gamma) \to \frac{1}{\sigma^2}\sup_{\gamma\in\Gamma} x(\gamma)'\Omega_\gamma x(\gamma),$$

where $D_{2p+2}(\Gamma)$ denotes the function spaces of all functions, mapping $\mathcal{R}^2(\Gamma)$ into $\mathcal{R}^{2p+2}$, that are right continuous and have right-hand limits. Clearly, $L(\cdot)$ is a continuous functional; see e.g., Chan (1990, p.1891). By the continuous mapping theory and (A.21), it follows that under H_0 or H_{1n},

$$L(\hat{Z}_n(\gamma)) \Rightarrow L(\sigma G_\gamma) \text{ in probability as } n \to \infty. \tag{A.22}$$

Furthermore, since $\sigma_n^2 \to \sigma^2$ a.s. and $(X_{1n}(\gamma), I)'[X_{2n}(\gamma)]^{-1}(X_{1n}(\gamma), I) \to \Omega_\gamma$ uniformly in γ by Lemma A.3, we have that

$$\sup_{\gamma\in\Gamma} \hat{LR}_n(\gamma) = L(\hat{Z}_n(\gamma)) + o_p(1). \tag{A.23}$$

Finally, the conclusion follows from (A.22)-(A.23). This completes the proof. □

Proof of Corollary 3.1. Conditional on the sample $\{y_0, \cdots, y_N\}$, let $\hat{F}_{n,J}$ and $\hat{F}_n$ be the conditional empirical c.d.f. and c.d.f. of $\hat{LR}_n$, respectively. Then,

$$\begin{aligned}
&P\left(LR_n \geq c_{n,\alpha}^J\right) \\
&= E\left[P\left(LR_n \geq c_{n,\alpha}^J | y_0, \cdots, y_N\right)\right] \\
&= E\left[P\left(\hat{F}_{n,J}(LR_n) \geq 1-\alpha | y_0, \cdots, y_N\right)\right].
\end{aligned}$$

By the Glivenko-Cantelli Theorem and Theorem 3.1, it follows that under H_0 or H_{1n},

$$\begin{aligned} & \lim_{n\to\infty}\lim_{J\to\infty} P\left(LR_n \geq c_{n,\alpha}^J\right) \\ & = \lim_{n\to\infty} E\left[P\left(\hat{F}_n(LR_n) \geq 1-\alpha | y_0, \cdots, y_N\right)\right] \\ & = \lim_{n\to\infty} E\left[P\left(F_0(LR_n) \geq 1-\alpha | y_0, \cdots, y_N\right)\right] \\ & = \lim_{n\to\infty} P\left(F_0(LR_n) \geq 1-\alpha\right), \end{aligned} \tag{A.24}$$

where F_0 is the c.d.f. of $\sup_{\gamma\in\Gamma} G_\gamma'\Omega_\gamma G_\gamma$. Thus, by (A.24) and Theorem 2.1, under H_0 we have

$$\lim_{n\to\infty}\lim_{J\to\infty} P\left(LR_n \geq c_{n,\alpha}^J\right) = P\left(\sup_{\gamma\in\Gamma} G_\gamma'\Omega_\gamma G_\gamma \geq F_0^{-1}(1-\alpha)\right) = \alpha,$$

i.e., (i) holds. Furthermore, by (A.24) and Theorem 2.2, under H_{1n} we have

$$\lim_{h\to\infty}\lim_{n\to\infty}\lim_{J\to\infty} P\left(LR_n \geq c_{n,\alpha}^J\right) = \lim_{h\to\infty} P\left(B_h \geq F_0^{-1}(1-\alpha)\right) = 1,$$

where $B_h \triangleq \sup_{\gamma\in\Gamma}\left\{G_\gamma'\Omega_\gamma G_\gamma + h'\mu_{\gamma\gamma_0}h\right\}$ and the last equation holds since $B_h \to \infty$ in probability as $h \to \infty$. Thus, (ii) holds. This completes the proof. $\square$

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