GENERATING DISTRIBUTIONS BY TRANSFORMATION OF SCALE

M.C. Jones

The Open University

Supplementary Material

S1 Proof of Proposition 2

Let subscripts ℓ and r denote left- and right-hand 'halves' of the unimodal densities f and g, that is, $f_\ell(x) = f(x)I(x < x_0)$, $g_\ell(x) = g(x)I(x < 0)$, $f_r(x) = f(x)I(x > x_0)$ and $g_r(y) = g(x)I(x > 0)$; all these functions are invertible and $f_r^{-1}(y) = \Pi\{g_r^{-1}(y/2)\}$, $f_\ell^{-1}(y) = \Pi\{-g_r^{-1}(y/2)\}$. If (X, Y) is uniformly distributed over the region between the x-axis and f, then, unconditionally, $X \sim f$ (this is the basis of many methods of random variate generation), but conditionally

$$X|Y = y \sim U\{f_\ell^{-1}(y), f_r^{-1}(y)\}.$$

Define $Z_1 = g_r^{-1}(Y/2) > 0$ and $Z = Z_1$ with probability 1/2 and $Z = -Z_1$ otherwise. It follows that

$$X|Z = z_1 > 0 \sim U\{\Pi(z_1) - \Pi(-z_1)\} + \Pi(-z_1) = Uz_1 + \Pi(z_1) - z_1$$

and

$$X|Z = z_2 = -z_1 < 0 \sim U\{\Pi(-z_1) - \Pi(z_1)\} + \Pi(z_1) = \Pi(z_1) - Uz_1 = Uz_2 + \Pi(z_2) - z_2.$$

Unconditionally, the form of (17) in the text follows.

It remains to deduce the distribution of Z_1 and hence of Z. A minor modification of Lemma 2.4 in Chaubey, Mudholkar, and Jones (2010) shows that $Y \in (0, 2g(0))$ has distribution function

$$\begin{aligned}
F_Y(y) &= \int_0^y \int_{-\infty}^{\infty} I(0 \le z \le f(x))dxdz \\
&= \int_{-\infty}^{\infty} \min\{y, f(x)\}dx \\
&= \int_{-\infty}^{f_\ell^{-1}(y)} f(x)dx + \int_{f_\ell^{-1}(y)}^{f_r^{-1}(y)} ydx + + \int_{f_r^{-1}(y)}^{\infty} f(x)dx \\
&= F\{f_\ell^{-1}(y)\} + y\{f_r^{-1}(y) - f_\ell^{-1}(y)\} + 1 - F\{f_r^{-1}(y)\}
\end{aligned}$$

and hence that $Z_1 \in \mathbb{R}^+$ has distribution function

$$F_{Z_1}(z) = 1 - F_Y(2g_r(z)) = 1 - [F\{\Pi(-z)\} + 2zg_r(z) + 1 - F\{\Pi(z)\}]\,.$$

Differentiating,

$$f_{Z_1}(z) = -2\{-\pi(-z)g(z) + g(z) + zg'(z) - \pi(z)g(z)\} = -2zg'(z), \quad z > 0.$$

Similarly, $f_{-Z_1}(z) = -2zg'(z)$, $z < 0$, and the result of Proposition 2 follows.

S2 Proof of Shannon Entropy Version of Proposition 3

$$\begin{aligned}
S(f) &= -2\int_{S_f} g(\Pi^{-1}(x))\{\log 2 + \log g(\Pi^{-1}(x))\}dx \\
&= -\log 2 - 2\int_{\mathcal{D}} \pi(y)g(y)\log g(y)dy \\
&= -\log 2 - \int_{S_g} g(y)\log g(y)dy = S(g) - \log 2,
\end{aligned}$$

the third equality arising since $2\int_{\mathcal{D}} \pi(y)h(y)dy = \int_{\mathcal{D}} h(y)dy$ for any even function h. It remains to recall the standard result that $S(g_\sigma) = S(g) - \log\sigma$ when $g_\sigma(x) = \sigma g(\sigma x)$ and the Shannon entropy version of the proposition follows.