Web Appendix: Supplementary Materials for "Two-fold Nested Designs: Their Analysis and Connection with Nonparametric ANCOVA" by Shu-Min Liao and Michael G. Akritas

This web appendix outlines sketch of proofs in Sections 3–5 of the paper. In this appendix we will use the following notations:

$$\begin{aligned} \mathsf{U}_c \approx \mathsf{V}_c \quad &\Leftrightarrow \quad \sqrt{c}(\mathsf{U}_c - \mathsf{V}_c) \xrightarrow{P} 0, \quad \text{as } c \to \infty, \\ \mathsf{a}_c \approx \mathsf{b}_c \quad &\Leftrightarrow \quad \sqrt{c}(\mathsf{a}_c - \mathsf{b}_c) \to 0, \quad \text{as } c \to \infty, \end{aligned}$$

where U_c and V_c are two sequences of random vectors, while a_c and b_c are two sequences of constant vectors.

Proof of Theorem 3.1

Define

$$\begin{gathered} U_{ij}^{\delta} = n_{ij}(\bar{e}_{ij\cdot} + \frac{\delta_{ij}}{\sigma})^2, \quad \bar{U}_{ic}^{\delta} = \frac{1}{c_i}\sum_{j=1}^{c_i} U_{ij}^{\delta}, \quad \bar{U}_C^{\delta} = \frac{1}{C}\sum_{i=1}^{r}\sigma^2 c_i \bar{U}_{ic}^{\delta}, \\ W_{ij} = \frac{\sum_{k=1}^{n_{ij}}(e_{ijk} - \bar{e}_{ij\cdot})^2}{\bar{n} - 1}, \quad \bar{W}_{ic} = \frac{1}{c_i}\sum_{j=1}^{c_i} W_{ij}, \quad \bar{W}_C = \frac{1}{C}\sum_{i=1}^{r}\sigma^2 c_i \bar{W}_{ic}, \qquad (1) \\ \mathsf{V}_{ij}^{\delta} = \begin{pmatrix} U_{ij}^{\delta} \\ W_{ij} \end{pmatrix}, \quad \bar{\mathsf{V}}_{ic}^{\delta} = \begin{pmatrix} \bar{U}_{ic}^{\delta} \\ \bar{W}_{ic} \end{pmatrix}, \quad \bar{\mathsf{V}}_C^{\delta} = \begin{pmatrix} \bar{U}_C^{\delta} \\ \bar{W}_C \end{pmatrix}. \end{gathered}$$

Note that

$$\begin{aligned} \bar{U}_C^{\delta} &= MS\delta + \left[\frac{1}{C-r}\sum_{i=1}^{r}\sigma^2 n_{i\cdot}\bar{e}_{i\cdot\cdot}^2 - \frac{r}{C-r}\bar{U}_C^{\delta}\right], \quad \text{and} \\ \bar{W}_C &= MSE + \left[\sum_{i=1}^{r}\left(\frac{c_i}{C} - \frac{c_i(\bar{n}-1)}{N_C - C}\right)\sigma^2 \bar{W}_{ic}\right]. \end{aligned}$$

It can be easily verified that, as $\min(c_i) \to \infty$ and r, n_{ij} remain fixed,

$$\begin{gathered} \sqrt{C}\ \frac{1}{C-r}\sum_{i=1}^{r} n_{i\cdot}\bar{e}_{i\cdot\cdot}^2 \xrightarrow{P} 0, \quad \sqrt{C}\ \frac{r}{C-r}\bar{U}_C^{\delta} \xrightarrow{P} 0, \quad \text{and} \qquad (2) \\ \sqrt{C}\ \sum_{i=1}^{r}\left(\frac{c_i}{C} - \frac{c_i(\bar{n}-1)}{N_C - C}\right)\sigma^2 \bar{W}_{ic} = \left(1 - \frac{C(\bar{n}-1)}{N_C - C}\right)\sqrt{C}\bar{W}_C \xrightarrow{P} 0. \end{gathered}$$

Combining the above we have that, as $\min(c_i) \to \infty$ and $r,\ n_{ij}$ remain fixed,

$$\bar{\mathsf{V}}_C^\delta \approx \mathsf{M}_C^\delta \equiv \begin{pmatrix} MS\delta \\ MSE \end{pmatrix}. \tag{3}$$

Hence, the asymptotic joint distribution of $MS\delta$ and MSE is the same as the asymptotic joint distribution of $\bar{U}_C^\delta$ and $\bar{W}_C$.

It can be shown that, under normality, U_{ij}^δ and W_{ij} are independent, and

$$U_{ij}^\delta \sim \chi_1^2\left(\frac{n_{ij}\delta_{ij}^2}{\sigma^2}\right), \quad (\bar{n}-1)W_{ij} \sim \chi^2_{n_{ij}-1}.$$

Using known results regarding the mean and covariance of quadratic forms (cf. Theorem 1 in Akritas and Arnod (2000)) and the facts that $E(\chi_a^2(a\gamma)) = a(1+\gamma)$, $Var(\chi_a^2(a\gamma)) = a(2+4\gamma)$, we obtain

$$E(\mathsf{V}_{ij}^\delta) = \begin{pmatrix} 1 + \frac{n_{ij}\delta_{ij}^2}{\sigma^2} \\ \frac{n_{ij}-1}{\bar{n}-1} \end{pmatrix},$$

$$Cov(\mathsf{V}_{ij}^\delta) = \begin{pmatrix} 2 + 4\frac{n_{ij}\delta_{ij}^2}{\sigma^2} & 0 \\ 0 & \frac{2(n_{ij}-1)}{(\bar{n}-1)^2} \end{pmatrix} + \frac{\kappa_i - 3}{n_{ij}} \begin{pmatrix} 1 & \frac{n_{ij}-1}{\bar{n}-1} \\ \frac{n_{ij}-1}{\bar{n}-1} & \left(\frac{n_{ij}-1}{\bar{n}-1}\right)^2 \end{pmatrix}.$$

Let $\theta_{ic_i}^\delta = \frac{1}{c_i}\sum_{j=1}^{c_i} n_{ij}\frac{\delta_{ij}^2}{\sigma^2}$. Then, for each class i, as $c_i \to \infty$,

$$E(\bar{\mathsf{V}}_{ic}^\delta) = \begin{pmatrix} 1 + \theta_{ic_i}^\delta \\ \frac{\bar{n}_{ic_i}-1}{\bar{n}-1} \end{pmatrix} \approx \begin{pmatrix} 1 + \theta_i \\ \frac{\bar{n}_i-1}{\bar{n}-1} \end{pmatrix} \triangleq \boldsymbol{\mu}_i, \quad \text{and} \tag{4}$$

$$\begin{aligned} & c_i \cdot Cov(\bar{\mathsf{V}}_{ic}^\delta) \\ &= \begin{pmatrix} 2 + 4\theta_{ic}^\delta & 0 \\ 0 & \frac{2}{\bar{n}-1}\frac{\bar{n}_{ic_i}-1}{\bar{n}-1} \end{pmatrix} + \frac{\kappa_i - 3}{(\bar{n}-1)^2} \begin{pmatrix} (\bar{n}-1)^2\underline{n}_{ic_i} & (\bar{n}-1)(1-\underline{n}_{ic_i}) \\ (\bar{n}-1)(1-\underline{n}_{ic_i}) & \bar{n}_{ic_i} + \underline{n}_{ic_i} - 2 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 2 + 4\theta_i & 0 \\ 0 & \frac{2(\bar{n}_i-1)}{(\bar{n}-1)^2} \end{pmatrix} + \frac{\kappa_i - 3}{(\bar{n}-1)^2} \begin{pmatrix} (\bar{n}-1)^2\underline{n}_i & (\bar{n}-1)(1-\underline{n}_i) \\ (\bar{n}-1)(1-\underline{n}_i) & \bar{n}_i + \underline{n}_i - 2 \end{pmatrix} \triangleq \Sigma_i. \end{aligned}$$

Under the assumption that $E|e_{ijk}|^{4+2\epsilon} < \infty$ for some $\epsilon > 0$, Lindeberg-Feller's theorem together with Cramér-Wold's theorem yield

$$\sqrt{c_i}(\bar{\mathsf{V}}_{ic}^\delta - \boldsymbol{\mu}_i) \xrightarrow{d} N_2(\mathsf{0}, \Sigma_i).$$

Using the independence among $\bar{\mathsf{V}}_{ic}^\delta$ and the assumption on sample sizes and subclass levels (specified as the relation (9) in the paper), one can be show that

$$\sqrt{C}(\bar{\mathsf{V}}_C^\delta - \boldsymbol{\mu}) \xrightarrow{d} N_2(\mathsf{0}, \sigma^4\sum_{i=1}^r \lambda_i\Sigma_i), \quad \text{where } \boldsymbol{\mu} = \sigma^2\begin{pmatrix} 1+\theta \\ 1 \end{pmatrix}. \tag{5}$$

By the asymptotic equivalence between $\bar{\mathsf{V}}_C^\delta$ and M_C^δ shown in (3), we then have

$$\sqrt{C}(\mathsf{M}_C^\delta - \boldsymbol{\mu}) \xrightarrow{d} N_2(\mathsf{0}, \sigma^4 \sum_{i=1}^{r} \lambda_i \Sigma_i), \text{ as } \min(c_i) \to \infty.$$

Note that if $\mathsf{s}' = (1, -(1+\theta))/\sigma^2$, $\sqrt{C}\,\mathsf{s}'(\mathsf{M}_C^\delta - \boldsymbol{\mu}) = \sqrt{C}[MS\delta - (1+\theta)MSE]/\sigma^2$ which, by Slutsky's theorem, is asymptotically equivalent to $\sqrt{C}\left(F_C^\delta - (1+\theta)\right)$. Thus, by the Δ-method, as $\min(c_i) \to \infty$,

$$\sqrt{C}\left(F_C^\delta - (1+\theta)\right) \xrightarrow{d} N\left(0, \sigma^4 \sum_{i=1}^{r} \lambda_i \mathsf{s}'\Sigma_i \mathsf{s}\right) = N\left(\mathsf{0}, \Sigma_s\right),$$

where Σ_s is as defined in Theorem 3.1.

Proof of Corollary 3.2

It can be easily verified that for C large enough, the approximate distribution of the classical F-test under $H_0 : \delta_{ij} = 0$, and under the normality assumption is:

$$\sqrt{C}\left(F_C^\delta - 1\right) \dot{\sim} N\left(0, 2\left(1 + \frac{C}{N_C - C}\right)\right), \tag{6}$$

where $\dot{\sim}$ means "approximately distributed". The relation (6) is obviously not equivalent to the asymptotic null distribution specified in Theorem 3.1 (shown as the relation (11) in the paper), unless $n_{ij} = n$ for all i and j so that

$$\bar{n}_{ic_i} = n = \bar{n}_i, \quad \underline{n}_{ic_i} = \frac{1}{n} = \underline{n}_i \longrightarrow \bar{n}_i \underline{n}_i - 1 = 0,$$

and hence both of the asymptotic null distribution and the relation (6) would become

$$\sqrt{C}\left(F_C^\delta - 1\right) \xrightarrow{d} N\left(0, 2\left(1 + \frac{1}{n-1}\right)\right). \tag{7}$$

Proof of Theorem 4.1

Define new quantities U_{ij}^δ, $\bar{U}_C^\delta$, $\bar{W}_C$ to be as the corresponding quantities in (1) but with σ_i^2 replacing σ^2, and the new quantity W_{ij} to be as the corresponding quantity in (1) but with $\bar{n}_i$ replacing $\bar{n}$. Finally, let $\bar{U}_{ic}^\delta$, $\bar{W}_{ic}$, V_{ij}^δ, $\bar{\mathsf{V}}_{ic}^\delta$, and $\bar{\mathsf{V}}_C^\delta$ be as defined in (1) but using the above new quantities. It can then be shown that $\bar{U}_C^\delta$, $\bar{W}_C$ are related to $MS\delta$, MSE^* via

$$\begin{aligned} \bar{U}_C^\delta &= MS\delta + \left[\frac{1}{C-r}\sum_{i=1}^{r} \sigma_i^2 n_{i\cdot} \bar{e}_{i\cdot\cdot}^2 - \frac{r}{C-r}\bar{U}_C^\delta\right], \\ \bar{W}_C &= MSE^* + \left[\sum_{i=1}^{r}\left(\frac{c_i}{C} - \frac{(c_i-1)c_i(\bar{n}_i - 1)}{(C-r)(n_{i\cdot} - c_i)}\right)\sigma_i^2 \bar{W}_{ic}\right]. \end{aligned}$$

Using (2), and the fact that, as $\min(c_i) \to \infty$ and $r,\ n_{ij}$ remain fixed,

$$\sqrt{C}\ \sum_{i=1}^{r} \left(\frac{c_i}{C} - \frac{(c_i - 1)c_i(\bar{n}_i - 1)}{(C-r)(n_{i\cdot} - c_i)} \right) \sigma_i^2 \bar{W}_{ic} \xrightarrow{P} 0, \tag{8}$$

we have that, as $\min(c_i) \to \infty$ and $r,\ n_{ij}$ remain fixed,

$$\bar{\mathsf{V}}_C^\delta \approx \mathsf{M}_C^* \equiv \begin{pmatrix} MS\delta \\ MSE^* \end{pmatrix}. \tag{9}$$

Following the same derivation in the proof of Theorem 3.1, one can easily get

$$\sqrt{c_i}(\bar{\mathsf{V}}_{ic}^\delta - \boldsymbol{\mu}_i^*) \xrightarrow{d} N_2(\mathbf{0}, \Sigma_i^*), \quad \text{where } \boldsymbol{\mu}_i^* \text{ and } \Sigma_i^* \text{ are defined by} \tag{10}$$

$$E(\bar{\mathsf{V}}_{ic}^\delta) = \begin{pmatrix} 1 + \theta_{ic_i}^\delta \\ \frac{\bar{n}_{ic_i}-1}{\bar{n}_i - 1} \end{pmatrix} \approx \begin{pmatrix} 1 + \theta_i \\ 1 \end{pmatrix} \triangleq \boldsymbol{\mu}_i^*, \quad \text{where } \theta_{ic_i}^\delta = \frac{1}{c_i} \sum_{j=1}^{c_i} n_{ij} \frac{\delta_{ij}^2}{\sigma_i^2}, \quad \text{and}$$

$$\begin{aligned} & c_i \cdot Cov(\bar{\mathsf{V}}_{ic}^\delta) \\ &= \begin{pmatrix} 2 + 4\theta_{ic}^\delta & 0 \\ 0 & \frac{2}{\bar{n}_i - 1}\frac{\bar{n}_{ic_i}-1}{\bar{n}_i-1} \end{pmatrix} + \frac{\kappa_i - 3}{(\bar{n}_i - 1)^2} \begin{pmatrix} (\bar{n}_i - 1)^2 \underline{n}_{ic_i} & (\bar{n}_i - 1)(1 - \underline{n}_{ic_i}) \\ (\bar{n}_i - 1)(1 - \underline{n}_{ic_i}) & \bar{n}_{ic_i} + \underline{n}_{ic_i} - 2 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 2 + 4\theta_i & 0 \\ 0 & \frac{2}{\bar{n}_i - 1} \end{pmatrix} + \frac{\kappa_i - 3}{(\bar{n}_i - 1)^2} \begin{pmatrix} (\bar{n}_i - 1)^2 \underline{n}_i & (\bar{n}_i - 1)(1 - \underline{n}_i) \\ (\bar{n}_i - 1)(1 - \underline{n}_i) & \bar{n}_i + \underline{n}_i - 2 \end{pmatrix} \triangleq \Sigma_i^*. \end{aligned}$$

By the independence among $\bar{\mathsf{V}}_{ic}^\delta$ and the assumption on sample sizes and subclass levels (specified as the relation (9) in the paper), it can be shown that

$$\sqrt{C}(\bar{\mathsf{V}}_C^\delta - \boldsymbol{\mu}^*) \xrightarrow{d} N_2(\mathbf{0}, \sum_{i=1}^{r} \sigma_i^4 \lambda_i \Sigma_i^*), \quad \text{where } \boldsymbol{\mu}^* = \begin{pmatrix} \beta + \theta^\sigma \\ \beta \end{pmatrix},$$

where β and θ^σ are as defined in Theorem 4.1. Because $\bar{\mathsf{V}}_C^\delta$ and M_C^* are asymptotically equivalent, as shown in (9), we then have

$$\sqrt{C}(\mathsf{M}_C^* - \boldsymbol{\mu}^*) \xrightarrow{d} N_2(\mathbf{0}, \sum_{i=1}^{r} \sigma_i^4 \lambda_i \Sigma_i^*), \text{ as } \min(c_i) \to \infty.$$

Finally, by the Δ-method with $\mathsf{s}^{*\prime} = (1, -(1+\theta^*))/\beta$, where $\theta^* = \theta^\sigma/\beta$, it can be easily verified that, as $\min(c_i) \to \infty$,

$$\sqrt{C}\left(F_C^* - (1 + \theta^*)\right) \xrightarrow{d} N\left(0, \sum_{i=1}^{r} \sigma_i^4 \lambda_i \mathsf{s}^{*\prime} \Sigma_i^* \mathsf{s}^*\right) = N\left(0,\, \Sigma_s^*\right),$$

where Σ_s^* is as defined in Theorem 4.1.

Proof of Corollary 4.2

The fact that, when the design is balanced, the unweighted statistic F_C^* equals the classical F-statistic is clear. Next, the asymptotic null distribution of Corollary 4.2 (shown as the relation (15) in the paper) follows directly from Theorem 4.1. Finally, the fact that the classical F-test procedure is not valid follows by comparing the relation (7) above and the relation (15) in the paper.

Proof of Theorem 5.1

Define $\mathsf{V}_{ij}^\delta = (U_{ij}^\delta,\ W_{ij})'$, $\bar{\mathsf{V}}_{ic}^\delta = (\bar{U}_{ic}^\delta,\ \bar{W}_{ic})'$, and $\bar{\mathsf{V}}_C^\delta = (\bar{U}_C^\delta,\ \bar{W}_C)'$, where

$$\begin{aligned} U_{ij}^\delta &= \sigma_{ij}^2 n_{ij}(\bar{e}_{ij\cdot} + \frac{\delta_{ij}}{\sigma_{ij}})^2, \quad \bar{U}_{ic}^\delta = \frac{1}{c_i}\sum_{j=1}^{c_i} U_{ij}^\delta, \quad \bar{U}_C^\delta = \frac{1}{C}\sum_{i=1}^{r} c_i \bar{U}_{ic}^\delta, \\ W_{ij} &= \frac{\sigma_{ij}^2}{n_{ij}-1}\sum_{k=1}^{n_{ij}}(e_{ijk} - \bar{e}_{ij\cdot})^2, \quad \bar{W}_{ic} = \frac{1}{c_i}\sum_{j=1}^{c_i} W_{ij}, \quad \bar{W}_C = \frac{1}{C}\sum_{i=1}^{r} c_i \bar{W}_{ic}. \end{aligned}$$

Note that

$$\begin{aligned} \bar{U}_C^\delta &= MS\delta + \left[\frac{1}{C-r}\sum_{i=1}^{r}\frac{1}{n_{i\cdot}}\left(\sum_{j=1}^{c_i}\sigma_{ij}n_{ij}\bar{e}_{ij\cdot}\right)^2 - \frac{r}{C-r}\bar{U}_C^\delta\right], \quad \text{and} \\ \bar{W}_C &= MSE^{**} - \frac{r}{C-r}\sum_{i=1}^{r}\frac{1}{C}\sum_{j=1}^{c_i} S_{ij}^2 + \frac{1}{C-r}\sum_{i=1}^{r}\frac{1}{n_{i\cdot}}\sum_{j=1}^{c_i} n_{ij}S_{ij}^2. \end{aligned}$$

Under the assumptions specified in Theorem 5.1, it can be easily verified that, as $\min(c_i) \to \infty$,

$$\sqrt{C}\ \frac{1}{C-r}\sum_{i=1}^{r}\frac{1}{n_{i\cdot}}\left(\sum_{j=1}^{c_i}\sigma_{ij}n_{ij}\bar{e}_{ij\cdot}\right)^2 \xrightarrow{P} 0, \quad \sqrt{C}\ \frac{r}{C-r}\bar{U}_C^\delta \xrightarrow{P} 0,$$
$$\sqrt{C}\frac{r}{C-r}\sum_{i=1}^{r}\frac{1}{C}\sum_{j=1}^{c_i} S_{ij}^2 \xrightarrow{P} 0, \quad \sqrt{C}\ \frac{1}{C-r}\sum_{i=1}^{r}\frac{1}{n_{i\cdot}}\sum_{j=1}^{c_i} n_{ij}S_{ij}^2 \xrightarrow{P} 0.$$

Combining the above we have that, as $\min(c_i) \to \infty$ and $r,\ n_{ij}$ remain fixed,

$$\bar{\mathsf{V}}_C^\delta \approx \mathsf{M}_C^{**} \equiv \begin{pmatrix} MS\delta \\ MSE^{**} \end{pmatrix}. \tag{11}$$

Following the same derivation in the proof of Theorem 3.1, one can easily get the asymptotic distribution of $\bar{\mathsf{V}}_{ic}^\delta$ as

$$\sqrt{c_i}(\bar{\mathsf{V}}_{ic}^\delta - \boldsymbol{\mu}_i^{**}) \xrightarrow{d} N_2(\mathsf{0}, \Sigma_i^{**}),$$

where $\boldsymbol{\mu}_i^{**}$ and Σ_i^{**} are

$$E(\bar{\mathsf{V}}_{ic}^{\delta}) = \begin{pmatrix} \frac{1}{c_i}\sum_j \sigma_{ij}^2 + \frac{1}{c_i}\sum_j n_{ij}\delta_{ij}^2 \\ \frac{1}{c_i}\sum_j \sigma_{ij}^2 \end{pmatrix} \approx \begin{pmatrix} a_{1i} + \theta_{1i} \\ a_{1i} \end{pmatrix} \triangleq \boldsymbol{\mu}_i^{**}, \quad \text{and}$$

$$\begin{aligned} & c_i \cdot Cov(\bar{\mathsf{V}}_{ic}^{\delta}) \\ &= \begin{pmatrix} 2\frac{1}{c_i}\sum_j \sigma_{ij}^4 + 4\frac{1}{c_i}\sum_j n_{ij}\sigma_{ij}^2\delta_{ij}^2 & 0 \\ 0 & 2\frac{1}{c_i}\sum_j \frac{\sigma_{ij}^4}{n_{ij}-1} \end{pmatrix} + \frac{1}{c_i}\sum_j \frac{\sigma_{ij}^4(\kappa_{ij}-3)}{n_{ij}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &\longrightarrow \begin{pmatrix} 2b_{1i} + 4\theta_{2i} & 0 \\ 0 & 2b_{2i} \end{pmatrix} + b_{3i} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \triangleq \Sigma_i^{**}. \end{aligned}$$

By the independence among $\bar{\mathsf{V}}_{ic}^{\delta}$, the assumptions in Theorem 5.1 and the asymptotic equivalence shown in (11), we then have

$$\sqrt{C}\left(\mathsf{M}_C^{**} - \begin{pmatrix} a_1 + \theta_1 \\ a_1 \end{pmatrix}\right) \xrightarrow{d} N_2\left(\mathsf{0}, \sum_{i=1}^{r} \lambda_i \Sigma_i^{**}\right),$$

where a_1 and θ_1 are as defied in the theorem above. Finally, using the Δ-method with $\mathsf{s}^{**\prime} = (1, -(1+\theta^{**}))/a_1$, $\theta^{**} = \theta_1/a_1$, one can easily get the limiting distribution of F_C^{**} as shown in Theorem 5.1 and complete the proof.

Proof of Corollary 5.2

The proof follows directly from Theorem 5.1.